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# Second order parallel tensors and Ricci solitons on Lorentzian Para $r$-Sasakian manifold with a coefficient $\alpha$

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**Abstract:** The geometry of Lorentzian para $r$-Sasakian manifold is developed by Takahashi [15] and Matsumoto [10]. The present paper deals with the study of second order parallel tensor in an LP $r$-Sasakian manifold with a coefficient $\alpha$. It is proved that a second order parallel symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$, is a constant multiple of the metric tensor, where as the second order parallel skew-symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$ does not exist.

## 1. Introduction

In 1923, Eisenhart [5] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. Then Levy [8] had obtained the necessary and sufficient conditions for the existence of such tensors. Sharma

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[13] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor of an n-dimensional ($n > 2$) space of constant curvature is a constant metric tensor. Then Li [9] studied second order parallel tensors on $P$-Sasakian manifold with a coefficient $k$. Also in [14], Singh et al. studied second order parallel tensors on LP-Sasakian manifolds. Recently Das ([2],[3]) has proved that on a Para $r$-Sasakian manifold with a coefficient $\alpha$, a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor. In this paper we have extended these ideas further and we have defined Lorentzian para $r$-Sasakian manifold with a coefficient $\alpha$ (non-zero scalar function) and it is proved that a second order parallel symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$, is a constant multiple of the metric tensor. However, it is proved that there do not exist second order parallel skew-symmetric tensors on an LP $r$-Sasakian manifold with a coefficient $\alpha$.

In 1982, Hamilton [6] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman [11] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(M, g)$ is a generelization of Einstein metric such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $S$ is the Ricci tensor and $\pounds_V$ is the Lie derivative along the vector field $V$ on $M$ and $\lambda$ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively. Recently Chandra et al. [1] studied second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds.

In this paper we prove that if the tensor field $\pounds_V g + 2S$ on an LP $r$-Sasakian manifold with a coefficient $\alpha$ is parallel then $(g, V, \lambda)$ is a Ricci soliton.

## 2. Preliminaries

Let $M^{2n+r}$ be an (2n+r)-dimensional smooth manifold equipped with the ring of real valued differentiable functions $C^{\infty}(M)$ and the module of derivation $\chi(M)$ and an (1,1)tensor field $\phi$ as a linear map such that such that $\phi : \chi(M) \to \chi(M)$. Let there be a r $C^{\infty}$-contravariant vector fields $\xi_1, \xi_2, \cdots, \xi_r$ satisfying the following condition:

$$\eta_p(\xi^p) = \epsilon\delta^p_q, \qquad p, q = 1, 2, \cdots r \tag{2.1}$$

$$\phi(\xi^p) = 0, \qquad p = 1, 2, \cdots r \tag{2.2}$$

$$\eta_p(\phi X) = 0, \qquad p = 1, 2, \cdots r \tag{2.3}$$

$$\phi^2 X = X - \epsilon\eta_p(X)\xi^p, \qquad p = 1, 2, \cdots r \tag{2.4}$$

for any vector field $X \in \chi(M)$. Here the summation convention is employed on repeated indices for $p = 1, 2, \cdots, r$ and

$$\delta^p_q = 1, \quad p = q$$
$$= 0, \quad p \neq q.$$

If moreover $M^{2n+r}$ admits an indefinite metric $g$ such that

$$g(\xi^p, \xi^p) = \epsilon \tag{2.5}$$

$$\eta_p(X) = g(X, \xi^P) \tag{2.6}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta_p(X)\eta_p(Y) \tag{2.7}$$

for any vector field $X$ and $Y \in \chi(M)$, where $\epsilon$ is 1 or -1 according as $\xi$ is a spacelike or timelike vector field, then a manifold satisfying condition (2.1)-(2.7) is called a Lorentzian Para r-Sasakian manifold (briefly LP $r$-Sasakian manifold).
In an LP r-Sasakian manifold $M^{2n+r}$, the following relations hold:

$$\Phi(X, Y) = g(X, \phi Y) = g(Y, \phi X) = \Phi(Y, X), \tag{2.8}$$

$$\Phi(X, \xi^p) = 0. \tag{2.9}$$

**Definition 2.1.** If in an LP r-Sasakian manifold $M^{2n+r}$, the following relations

$$\phi X = \frac{1}{\alpha}\nabla_X \xi^p, \quad \Phi(X,Y) = \frac{1}{\alpha}(\nabla_X \eta_p)(Y), \tag{2.10}$$

$$\alpha(X) = \nabla_X \alpha = g(X, \overline{\alpha}), \tag{2.11}$$

$$\begin{aligned}(\nabla_X \phi)(Y,Z) &= \alpha[\{g(X,Y) - \epsilon\eta_p(X)\eta_p(Y)\}\eta_p(Z) \\ &+ \{g(X,Z) - \epsilon\eta_p(X)\eta_p(Z)\}\eta_p(Y)]\end{aligned} \tag{2.12}$$

hold for arbitrary smooth vector fields $X$, $Y$, $Z \in \chi(M)$, where $\nabla$ denotes the Riemannian coefficient of the metric tensor g, then $M^{2n+r}$ is called a $\epsilon$-Lorentzian Para $r$-Sasakian manifold with a coefficient $\alpha$.

In an LP $r$-Sasakian manifold with a coefficient $\alpha$,the following relations hold:

$$\begin{aligned}\eta_p(R(X,Y)Z) =&\alpha^2[g(Y,Z)\eta_p(X) + g(X,Z)\eta_p(Y)] \\ &- [\alpha(X)\Phi(Y,Z) - \alpha(Y)\Phi(X,Z)],\end{aligned} \tag{2.13}$$

$$\begin{aligned}R(\xi^p, X)Y =&\alpha^2[\epsilon g(X,Y)\xi^p - \eta_p(Y)X] \\ &+ \alpha(Y)\phi X - \overline{\alpha}\Phi(X,Y),\end{aligned} \tag{2.14}$$

$$R(\xi^p, X)\xi^p = \beta\phi X + \alpha^2[X - \epsilon\eta_p(X)\xi^P] \tag{2.15}$$

for all vector fields $X, Y, Z \in \chi(M)$, $p = 1, 2, \cdots, r$, where $\alpha(\xi^p) = \beta$.

## 3. second order parallel symmetric tensors and Ricci solitons

Let $J$ be a symmetric (0,2) tensor field on an LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$ such that $\nabla J = 0$. Then we have

$$J\big(R(W,X)Y,Z\big) + J\big(Y,R(W,X)Z\big) = 0 \tag{3.1}$$

for arbitrary vector fields $X$, $Y$, $Z$, $W$ on $M^{2n+r}$.
Putting $W = Y = Z = \xi^p$ in (3.1), we get

$$J(\xi^p, R(\xi^p, X)\xi^p) = 0. \tag{3.2}$$

In view of (2.15) and (2.9) it follows from (3.2) that

$$\alpha^2[J(X,\xi^p) - \epsilon\eta_p(X)J(\xi^P,\xi^P)] = 0. \tag{3.3}$$

Since $\alpha^2 \neq 0$ and $\epsilon$ is either 1 or -1, we have from (3.3) that

$$J(X,\xi^p) - \epsilon\eta_p(X)J(\xi^P,\xi^P) = 0. \tag{3.4}$$

Differentiating (3.4) covariantly along Y, we get

$$\begin{aligned} &-\epsilon g(\nabla_Y X,\xi^p)J(\xi^p,\xi^p) - \epsilon g(X,\nabla_Y\xi^p)J(\xi^p,\xi^p) \\ &-2\epsilon g(X,\xi^p)J(\nabla_Y\xi^p,\xi^p) + J(\nabla_Y X,\xi^p) + J(X,\nabla_Y\xi^p) = 0. \end{aligned} \tag{3.5}$$

Putting $X = \nabla_Y X$ in (3.4) we obtain

$$J(\nabla_Y X,\xi^p) - \epsilon g(\nabla_Y X,\xi^p)J(\xi^p,\xi^p) = 0. \tag{3.6}$$

In view of (3.6) it follows from (3.5) that

$$-\epsilon g(X,\nabla_Y\xi^p)J(\xi^p,\xi^p) - 2\epsilon g(X,\xi^p)J(\nabla_Y\xi^p,\xi^p) + J(X,\nabla_Y\xi^p) = 0. \tag{3.7}$$

Using (2.10) in (3.7) we get

$$-\epsilon g(X,\phi Y)J(\xi^p,\xi^p) - 2\epsilon\eta_p(X)J(\phi Y,\xi^p) + J(X,\phi Y) = 0, \quad \text{since} \quad \alpha \neq 0. \tag{3.8}$$

Replacing $Y$ by $\phi Y$ in (3.8) and then using (2.4) and (3.4) we obtain

$$J(X,Y) = \epsilon J(\xi^p,\xi^p)g(X,Y). \tag{3.9}$$

Differentiating (3.9) covariantly along any vector field on $M^{2n+r}$, it can be easily shown that $J(\xi^p,\xi^p)$ is constant. This leads to the following:

**Theorem 3.1.** A second order parallel symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$, is a constant multiple of the associated metric tensor.

**Corollary 3.2.** [14] A second order parallel symmetric tensor on an LP-Sasakian manifold is a constant multiple of the associated metric tensor.

Suppose that the (0, 2) type symmetric tensor field $\pounds_V g + 2S$ is parallel for any vector field $V$ on an LP $r$-Sasakian manifold $M^{2n+r}$. Then by Theorem 3.1, it follows that $\pounds_V g + 2S$ is a constant multiple of the metric tensor $g$, i.e. $\pounds_V g + 2S = -2\lambda g$ for all $X$, $Y$ on $M^{2n+r}$, where $\lambda$ is a constant. Hence the relation (1.1) holds. This implies that $(g,V,\lambda)$ yields a Ricci soliton. Thus we can state the following:

**Theorem 3.3.** If the tensor field $\pounds_V g + 2S$ on an LP $r$-Sasakian manifold with a coefficient $\alpha$, is parallel for any vector field $V$, then $(g, V, \lambda)$ is a Ricci soliton.

**Corollary 3.4.** If the tensor field $\pounds_V g + 2S$ on an LP-Sasakian manifold is parallel for any vector field then $(g, V, \lambda)$ is a Ricci solution.

Let $(g, \xi^p, \lambda)$ be a Ricci soliton on a LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$. Then we have

$$(\pounds_{\xi^p} g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0 \tag{3.10}$$

where $\pounds_{\xi^p}$ is the Lie derivative along the vector field $\xi^p$ on $M^{2n+r}$. From (2.10), we have

$$\begin{aligned} (\pounds_{\xi}^{p} g)(Y, Z) &= g(\nabla_Y \xi^p, Z) + g(Y, \nabla_Z \xi^p) \\ &= \alpha[g(\phi Y, Z) + g(Y, \phi Z)] \\ &= 2\alpha\Phi(Y, Z). \end{aligned} \tag{3.11}$$

Using (3.11) in (3.10) we get

$$S(Y, Z) = -\lambda g(Y, Z) - \alpha\Phi(Y, Z),$$

which implies that the manifold under consideration is nearly quasi-Einstein manifold [4]. This leads the following:

**Theorem 3.5.** If $(g, \xi^p, \lambda)$ is a Ricci soliton on an LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$, then $M^{2n+r}$ is nearly quasi-Einstein manifold.

**Corollary 3.6.** If $(g, \xi, \lambda)$ is a Ricci soliton on an LP-Sasakian manifold $M$ then $M$ is nearly quasi-Einstein manifold.

If possible, let $J$ be a second order skew symmetric parallel tensor field on an LP $r$-Sasakian manifold $M^{2n+r}$ with a coefficient $\alpha$. Then we have the relation (3.1). Putting $W = Y = \xi^p$ in (3.1) we get

$$J(R(\xi^p, X)\xi^p, Z) + J(\xi^p, R(\xi^p, X)Z) = 0. \tag{3.12}$$

Using (2.14) and (2.15) in (3.12) and by straightforward calculation, we obtain that $\alpha = 0$ , which is a contradiction. Thus we can state the following:

**Theorem 3.7.** There do not exist second order parallel skew-symmetric tensor on an LP $r$-Sasakian manifold with a coefficient $\alpha$.

**Corollary 3.8.** There do not exist second order parallel skew-symmetric tensor on an LP-Sasakian manifold.

## References

[1] CHANDRA, S., HUI, S. K. and SHAIKH, A. A., *Second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds*, Commun. Korean Math. Soc., **30** (2015), 123–130.

[2] DAS, L. S., *Second order parallel tensor on $\alpha$-Sasakian manifold*, Acta Math. Acad. Pedag. Nyire., **23** (2007), 65–69.

[3] DAS, L. S., *Second order parallel tensor on almost $r$-contact manifolds*, Tensor (N. S.), **68** (2007), 347–353.

[4] DE, U. C. and GAJI, A. K., *On nearly quasi-Einstein manifolds*, Novi Sad J. Math., **38** (2008) 115–121.

[5] EISENHART, L. P., *Symmetric tensors of the second order whose first covariant derivatives are zero*, Trans. amer. Math. Soc., **25** (1923), 297–306.

[6] HAMILTON, R. S., *Three-manifolds with positive Ricci curvature*, J. Diff. Geom., **17** (1982), 255–306.

[7] HAMILTON, R. S., *The Ricci flow on surfaces, Mathematics and General Relativity*, Contemp. Math., **71** (1988), 237–262.

[8] LEVY, H., *Symmetric tensors of the second order whose covariant derivatives vanish*, Annals of Math., **27** (1926), 91–98.

[9] LI, Z., *Second order parallel tensor on $P$-Sasakian manifold with a coefficient $\kappa$*, Soochow J. Math., **23**(1997),97–102

[10] MATSUMOTO, K., *On Lorentzian almost paracontact manifolds*, Bull. of Yamagata Univ. Nat. Sci., **12** (1989), 151–156.

[11] PERELMAN, G., *The entropy formula for the Ricci flow and its geometric applications*, http://arXiv.org/abs/math/0211159, **2002**, 1–39.

[12] SHARMA, R., *Second order parallel tensor in real and complex space forms*, International J. Math. and Math. Sci., **12** (1989), 787–790.

[13] SHARMA, R., *Second order parallel tensor on contact manifolds*, Algebras, Groups and Geometrics, **7** (1990), 787–790.

[14] SINGH,R.N., PANDEY, S.K. and PANDEY, G., *Second order parallel tensors on LP-Sasakian manifolds*, J. Int. Academy Phy. Sciences, **13**(2009),383-388

[15] TAKAHASHI, T., *Sasakian manifold with pseudo Riemannian metric*, Tohoku Math. J., **21** (1969), 271–290.