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The Weak Descending Chain Condition on Right Ideals for Nearrings

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Abstract: The purpose of this paper is to introduce a weaker form of the descending chain condition on right ideals than the standard one. We shall see that fundamental results about socle and Frattini series which hold for the standard descending chain condition carry over to the weaker one. An example with this weaker chain condition that does not have the standard one will be given. Related to this example, we shall obtain a theorem concerning the transferability of this weaker chain condition from a smaller tame nearring to a larger one.

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1. Introduction

Throughout this paper, N denotes a left 0-symmetric nearring with identity and $J(N)$ denotes the J_2-radical of N. Also, all N-groups (or N-modules) will be assumed to be unitary. Numerous results have been obtained in nearring theory under the assumption that N has the descending chain condition on right ideals ($DCCR$). Sometimes, however, this assumption is stronger than necessary. For instance, [11, Theorem 5.4] and [12, Theorem 1] illustrate results where it suffices to merely assume that $N/J(N)$ has $DCCR$. Six more instances of this will occur later in this paper. Another assumption weaker than N having $DCCR$ but stronger than only $N/J(N)$ having $DCCR$ that is emerging to be of growing importance is when $N/J(N)$ has $DCCR$ and $J(N)$ is nilpotent which we will call the *weak descending chain condition on right ideals* and denote as $wDCCR$. The main purpose of this paper is to begin to develop a number of results under the assumption that N has $wDCCR$ that will form the foundation for future papers. In section 6, an example will be given of a nearring N that has $wDCCR$, but not $DCCR$, so that $wDCCR$ is indeed a weaker condition than $DCCR$.

To give a bit of an overview of the issues we will be dealing with, let us first recall the concepts of tameness for nearrings and their modules. As first defined in [10], an N-module or N-group V of our nearring N is *tame* if each N-subgroup of V is an N-ideal or submodule of V. Further, N is called a *tame nearring* if N has a faithful tame module V. Significant parts of sections 2 and 3 will be devoted to extending results known to hold about tame, socle, and Frattini series of tame modules of nearrings with $DCCR$ to the case where the nearring has $wDCCR$. (For readers who have forgotten or are unfamiliar with these series, we will review them as they arise.)

Besides issues involving the aforementioned series, another place where the $wDCCR$ condition is coming into play is in studying when properties of a smaller tame nearring transfer to a larger one. To make this more precise, we now introduce the notion of a *tame triple* by which we mean a triple (N_1, N_2, V) where $N_1 \leq N_2$ are nearrings, V is a faithful tame nearring module for both N_1 and N_2 (so that $N_1 \leq N_2 \leq M_0(V)$), and the N_1- and N_2-submodules of V coincide. As an example of a tame triple, consider any nearring N_1 with a faithful tame module V and let N_2 be the set of coset preserving functions $C_0(V)$ of V,

$$C_0(V) = \{\alpha \in M_0(V) : (v+U)\alpha \subseteq v\alpha + U \text{ for all } v \in V \text{ and for all } N_1\text{-submodules } U \text{ of } V\},$$

which is the same as the set of congruence preserving functions of V in $M_0(V)$ [6, Proposition 2.2]. Then V is a faithful tame $C_0(V)$-module (in fact, V has the stronger property of being a compatible $C_0(V)$-module) for which $(N_1, C_0(V), V)$ is a tame triple. Further, since the elements of N_2 in a tame triple (N_1, N_2, V) are coset preserving, $C_0(V)$ is the largest subnearring N_2 of $M_0(V)$ for which (N_1, N_2, V) is a tame triple. An example of a natural transferability question to ask about a tame triple (N_1, N_2, V) is does N_2 have $DCCR$ if N_1 does? In section 6 we will see that the answer is no. But also we will see that if $DCCR$ is replaced by $wDCCR$ the question of transferability becomes a more meaningful one. Indeed, we will obtain a result (Theorem 6.2) giving us necessary and sufficient conditions for this to occur. The transferability of $wDCCR$ from N_1 to N_2 for a tame triple (N_1, N_2, V) is one of several transferability questions that the authors intend to explore in future papers. In the development of section 6, we shall need some results concerning whether certain N_1-isomorphisms within V of a tame triple (N_1, N_2, V) are also N_2-isomorphisms that will be obtained in section 4. We further shall need a result involving a special type of submodule of a nearring module to be called an isolated submodule in section 5. The results of sections 4 and 5 are also of independent interest.

2. Tame and socle series

Throughout this section V will always denote a tame N-group. If there is a series of submodules

$$\{0\} = U_0 \leq U_1 \leq \cdots \leq U_n = V \qquad (2.1)$$

of V such that each factor U_{i+1}/U_i is a sum of minimal submodules of V/U_i (or equivalently, is a direct sum of minimal submodules of V/U_i by [14, Theorem 8.3]), then this series is called a *tame series* [14] of V. The *socle series* [3, 4] of V is formed by first letting the *socle* of V, denoted $soc(V)$, be the sum of the minimal submodules of V or $\{0\}$ if V has no minimal submodules. The socle series,

$$soc_0(V) \leq soc_1(V) \leq soc_2(V) \leq \cdots ,$$

is then obtained inductively by letting $soc_0(V) = \{0\}$ and $soc_i(V)$ be the submodule of V such that

$$soc_i(V)/soc_{i-1}(V) = soc(V/soc_{i-1}(V))$$

for $i \geq 1$. It is easy to see that V has a tame series as in (2.1) if and only if $soc_k(V) = V$ for some positive integer k. In this case, the smallest positive integer m such that $soc_m(V) = V$ will be called the *length of the socle series*. Further note that if V has a tame series as in (2.1), $U_i \leq soc_i(V)$ for each i, $m \leq n$ where m is the length of the socle series, and the socle series of V is a tame series of V.

We next record four basic facts involving radicals. The first of our facts about radicals is the following easily proven result. In the statement of this result, the *nilpotency degree* of $J(N)$ is the smallest positive integer n such that $(J(N))^n = \{0\}$ when $J(N)$ is nilpotent.

Proposition 2.1. If V is a faithful tame N-group and V has a tame series, then $J(N)$ is nilpotent and coincides with the intersection $\cap(0 : H_1/H_2)$ over all minimal factors H_1/H_2 of V. Further, the nilpotency degree of $J(N)$ is at most the length of the socle series of V.

The second fact involves $J(N/A)$ when A is an ideal of N. From [10, Proposition 5.2] we have that $J(N/A) = (J(N) + A)/A$ when N has $DCCR$. In fact, the argument given there only requires that $N/J(N)$ have $DCCR$ which we record as our next result. To keep the presentation self-contained, we include its proof.

Proposition 2.2. If N is a nearring, $N/J(N)$ has $DCCR$ and A is an ideal of N, then $J(N/A) = (J(N) + A)/A$.

Proof. By [9, Proposition 5.15], $J(N/A) \geq (J(N) + A)/A$ without the $DCCR$ assumption. With the $DCCR$ assumption, the opposite inclusion follows from [9, Theorem 5.32]. ♢

Our third and fourth facts involving radicals also deal with results that have previously appeared for nearrings N with $DCCR$, but hold under the weaker assumption of $N/J(N)$ having $DCCR$. These will be generalizations of [3, Lemma 3.9 and Theorem 3.11]. Once again we will include proofs of these results to keep the presentation self-contained.

Lemma 2.3. If V is a tame N-group, $N/J(N)$ has $DCCR$ and H is a subset of V such that $HJ(N) = \{0\}$, then $H \leq soc(V)$.

Proof. Let $h \in H$. Since $hNJ(N) = hJ(N) = \{0\}$, hN is a sum of minimal submodules by [9, Theorem 5.34]. Thus $hN \leq soc(V)$ and hence $H \leq soc(V)$. ◊

Proposition 2.4. If V is a tame N-group, $N/J(N)$ has $DCCR$ and H is a subset of V such that $H(J(N))^n = \{0\}$ where n is a positive integer, then $H \leq soc_n(V)$.

Proof. We use induction on n. The result holds for $n = 1$ by 2.3. Suppose it holds for n and $H(J(N))^{n+1} = \{0\}$. As $HJ(N)(J(N))^n = \{0\}$, $HJ(N) \leq soc_n(V)$ by the induction hypotheses. Thus $((H + soc_n(V))/soc_n(V))J(N) = soc_n(V)/soc_n(V)$. Since

$$J(N/(soc_n(V) : V)) = (J(N) + (soc_n(V) : V))/(soc_n(V) : V)$$

by 2.2, $(H + soc_n(V))/soc_n(V) \leq soc(V/soc_n(V))$ by 2.3 and hence $H \leq soc_{n+1}(V)$. ◊

The remainder of this section deals with existence of tame series and consequences of their existence. If a nearring N has $DCCR$ and V is a tame N-group, then we are assured that V has a tame series [14, Theorem 8.5] (or equivalently, that the socle series of V terminates at V after a finite number of terms [3, Theorem 3.13]). However, the following result tells us that N need only satisfy the $wDCCR$ condition for this to occur. It further includes a weakening of the $DCCR$ assumption in [3, Corollary 3.14] which deals with the nilpotency degree of $J(N)$ to $wDCCR$.

Theorem 2.5. If V is a tame N-group and N has $wDCCR$, then V has a tame series. In addition, if V is a faithful N-group, then the nilpotency degree of $J(N)$ is the length of the socle series of V.

Proof. Let n be the nilpotency degree of $J(N)$. Since $V(J(N))^n = \{0\}$, 2.4 gives us $V \leq soc_n(V)$. Hence the socle series of V terminates at V after at most $n + 1$ terms and V has a tame series. That n is the length of the socle series of V when V is a faithful N-group now follows from 2.1. ◊

As an immediate consequence of 2.1 and 2.5 we have:

Corollary 2.6. If V is a faithful tame N-group and N has $wDCCR$, then $J(N)$ coincides with the intersection $\cap(0 : H_1/H_2)$ over all minimal factors of H_1/H_2 of V.

As another consequence of 2.1 and 2.5, we obtain the following fact about radicals of nearrings in a tame triple.

Proposition 2.7. If (N_1, N_2, V) is a tame triple where N_1 has $wDCCR$, then $J(N_i)$, $i = 1, 2$, are both nilpotent and $J(N_1) = J(N_2) \cap N_1$.

Proof. From 2.5, V as an N_1-group has a tame series which is then also a tame series of V for N_2. Thus the $J(N_i)$, $i = 1, 2$, are nilpotent and the intersection $\cap(0 : H_1/H_2)_{N_i}$ over all minimal factors H_1/H_2 of V. Because $(0 : H_1/H_2)_{N_2} \cap N_1 = (0 : H_1/H_2)_{N_1}$, the proposition follows. ♢

We will call a tame N-group V *minimally finite* if the number of N-isomorphism types of minimal factors of V is finite and *minimally complete* if every minimal N-group is N-isomorphic to a minimal factor of V. Since a minimal N-group is a minimal $N/J(N)$-group it follows that:

Proposition 2.8. If V is a tame N-group and N has $wDCCR$, then V is minimally finite.

When the V of 2.8 is faithful the words minimally finite can be replaced by minimally complete.

Proposition 2.9. If V is a faithful tame N-group and N has $wDCCR$, then V is minimally complete.

Proof. By 2.6, $J(N)$ is the intersection $\cap(0 : H_1/H_2)$ over all minimal factors H_1/H_2 of V. Now $N/J(N)$ is a finite direct sum $A_1 \oplus \cdots \oplus A_n$ of minimal ideals and each A_i is a direct sum $R_{i1} \oplus \cdots \oplus R_{ik_i}$ of minimal right ideals that are isomorphic minimal N-groups. Further, if M is a minimal N-group, then, for some j, M is isomorphic to every R_{jl}. However as the intersection $\cap[(0 : H_1/H_2)/J(N)]$ over all H_1/H_2 is $J(N)/J(N)$, at least one H_1/H_2 is such that $A_j \not\leq (0 : H_1/H_2)/J(N)$. Consequently $(H_1/H_2)A_j \neq \{0\}$. Thus for some l, $(H_1/H_2)R_{jl} \neq \{0\}$. Choosing $h \in H_1/H_2$ such that $hR_{jl} \neq \{0\}$, we have

$$H_1/H_2 \simeq hR_{jl} \simeq R_{jl} \simeq M$$

which completes our proof. ♢

Let N have $wDCCR$ and V be a faithful tame N-group. Implications of 2.5, 2.8 and 2.9 are that V has a tame series, is minimally finite and is minimally complete. We conclude this section with an example illustrating that these three consequences of $wDCCR$ are not enough to yield $wDCCR$.

Let X be an infinite dimensional vector space over a field F which we will express in the form $X = \oplus_{i\in I} A_i$ where A_i consists of the elements of X that have an element of F in the ith component and 0 in all other components. In the ring of all linear transformations from X to X, $End_F(X)$, let S be the set of all scalar linear transformations, A be the set of all elements of $End_F(X)$ whose ranges are finite dimensional and $R = S + A$. It is easy to check that R is a subring of $End_F(X)$ and A is an ideal of R. Also since for each $0 \neq x_1 \in X$ and $x_2 \in X$ there is an element $\beta \in A$ such that $x_1\beta = x_2$, A and hence R are primitive on X.

Let H_k, $k \in K$, denote the maximal subspaces of X. We claim that:

(i) each $(0 : H_k)$ is a minimal right ideal of R that is R-isomorphic to X and

(ii) $A = \sum_{k\in K}(0 : H_k)$.

To get (i), first note that there is an A_j such that $X = H_k \oplus A_j$. Let b be a nonzero element of A_j. If $\lambda \in (0 : H_k)$ such that $b\lambda = 0$, then the linear transformation λ is the zero map on $H_k \oplus A_j = X$. Thus the map taking λ in $(0 : H_k)$ to $b\lambda$ is an R-isomorphism and (i) will follow if $b(0 : H_k) = X$. To get this, let λ_i be a linear transformation taking H_k to $\{0\}$ and A_j onto A_i. Since $\lambda_i \in (0 : H_k)$, $A_i \leq b(0 : H_k)$ for each i. Hence $X \leq b(0 : H_k)$ and we have (i). To get (ii), let $\beta \in A$ and let $H' = ker(\beta)$. As X/H' is finite dimensional, there are 1-dimensional subspaces $B_1, \ldots, B_n$ of X such that $X = H' \oplus B_1 \oplus \cdots \oplus B_n$. Each $K_i = H' \oplus B_1 \oplus \cdots B_{i-1} \oplus B_{i+1} \oplus B_n$ is a maximal subspace of X. Letting β_i be the linear transformation such that $\beta_i \in (0 : K_i)$ and $d\beta_i = d\beta$ for all $d \in B_i$, we have $\beta = \beta_1 + \cdots + \beta_n$ and hence $A = \sum_{k\in K}(0 : H_k)$.

To see that R has a faithful tame R-group V that has a tame series and is both minimally finite and complete, we set $V = R$. Since $R/A \simeq F$, (i) and (ii) give us that $\{0\} < A < R$ is a tame series for V and V is minimally finite. To get V is minimally complete, let M be a minimal R-group. If $MA \neq \{0\}$, then for some $m \in M$ and $k \in K$ we have $m(0 : H_k) \neq \{0\}$. Thus $m(0 : H_k) = M$ and consequently $M \simeq (0 : H_k)$ since $(0 : H_k)$ is a minimal right ideal. If $MA = \{0\}$, then $M \simeq R/A$ which completes our argument that V is minimally complete.

Finally, we show R does not have $wDCCR$. If it did, it would then have $DCCR$ since R being primitive on X implies $J(R) = 0$. Now, R being a primitive ring with $DCCR$ then tells us R must be a simple ring which is impossible since X and R/A are minimal R-modules that are not R-isomorphic. Thus R does not have $wDCCR$.

3. Frattini series

A discussion of the Frattini series for a nearring module appears in the last section of [7] some of the details of which we briefly review for the convenience of the reader. Generalized from group theory, the *Frattini N-subgroup* of an N-module V is the intersection of the maximal N-subgroups of V, or is V when V has no maximal N-subgroups. The *Frattini series* of V is the series

$$V = \Phi_0(V) \geq \Phi_1(V) \geq \Phi_2(V) \geq \dots$$

where

$$\Phi_i(V) = \Phi(\Phi_{i-1}(V))$$

for $i > 0$. To the knowledge of these authors, the first appearance of this series for nearring modules appears in [3, Corollary 3.18] where its terms are denoted by L_i. If V has a tame series as in (2.1), it is easy to verify that $\Phi_i(V) \leq U_{n-i}$ for each $i = 0, \dots, n$. When N has $DCCR$ and V is a faithful tame N-group, the Frattini series, as noted in [7], is a dual series to the socle series with the properties: (i) it terminates in $\{0\}$ after a finite number of steps which is the same as the nilpotency degree of $J(N)$; (ii) each Frattini factor $\Phi_{i-1}(V)/\Phi_i(V)$ is a direct sum of minimal N-modules; (iii) the annihilator of the Frattini series (that is, the intersection of the annihilators of its factors) is $J(N)$ [3, Corollaries 3.15 and 3.18]; (iv) each minimal N-module is isomorphic to a summand of some Frattini factor $\Phi_{i-1}(V)/\Phi_i(V)$. In this section, we shall see that these properties in fact hold when N has $wDCCR$. We begin by proving the following proposition.

Proposition 3.1. If V is a tame N-group and $N/J(N)$ has $DCCR$, then $V/\Phi(V)$ is completely reducible.

Proof. If $\Phi(V) = V$, the result is trivial so that we may assume V has maximal submodules. As $VJ(N) \leq M$ for each maximal submodule M of

V, it follows that $VJ(N) \leq \Phi(V)$. Since $J(N/(0 : V/\Phi(V))) = (J(N) + (0 : V/\Phi(V)))/(0 : V/\Phi(V))$ by 2.2, we must have $soc(V/\Phi(V)) = V/\Phi(V)$ by 2.3 which in turn tells us that $V/\Phi(V)$ is completely reducible. $\diamondsuit$

As a corollary to 2.2 and 3.1, we have:

Corollary 3.2. If U is a submodule of a tame N-group V where $N/J(N)$ has $DCCR$, then $U/\Phi(U)$ is completely reducible.

We are now ready to extend the four properties of the Frattini series given earlier when N has $DCCR$ to the $wDCCR$ setting.

Theorem 3.3. If V is a faithful tame N-group and N has $wDCCR$, then:

(i) The Frattini series of V terminates in $\{0\}$ after a finite number of steps which is the same as the nilpotency degree of $J(N)$.

(ii) Each Frattini factor $\Phi_{i-1}(V)/\Phi_i(V)$ is a direct sum of minimal N-modules.

(iii) The annihilator of the Frattini series is $J(N)$.

(iv) Each minimal N-module is isomorphic to a summand of some Frattini factor $\Phi_{i-1}(V)/\Phi_i(V)$ for some i.

Proof. We have (ii) by 3.2. Suppose that the socle series of V has length m. Since $\Phi_i(V) \leq soc_{m-i}(V)$ for each i, the Frattini series has length at most m and hence is a tame series for V by (ii). Thus (iv) now follows from 2.9. Since the annihilator of all the factors $\Phi_{i-1}(V)/\Phi_i(V)$ is a nilpotent ideal of N, (iv) yields (iii). Finally, as the length of the Frattini series is at most m, we only need that its length cannot be less than m to obtain (i). But this follows from 2.5 and (iii). $\diamondsuit$

As a consequence of 2.2 and part (i) of 3.3, we have:

Corollary 3.4. If V is a tame N-group and N has $wDCCR$, then the lengths of the Frattini and socle series of V are the same.

Initially, the authors thought it might be possible to obtain 3.1 whenever V has a tame series. However, this is not possible. To see why, consider any primitive nearring N with 1 that has a tame series when viewed as a right module over itself but does not have $DCCR$. The example given at the end of the previous section is such a nearring. Since N is primitive, $\Phi(N) = J(N) = \{0\}$. Suppose $N/\Phi(N) = N$ is completely reducible. Then $N = \oplus_{i\in I} R_i$ where I is an index set and each R_i is a minimal right ideal of N. Because all elements of N are finite sums of elements from the R_i, in particular 1 is a finite sum of elements from the R_i. Consequently it follows that I is a finite set and hence N has $DCCR$, which is a contradiction. Thus N cannot be completely reducible. Further, for the particular example with the ring R in the previous section, the length of the Frattini series of R is 1 while the length of its socle series is 2. Thus, in comparison with 3.4, the Frattini series may be shorter than the socle series when the nearring does not have $wDCCR$. Also, note that it is possible for the nilpotency degree of $J(N)$ in 2.1 to be less than the length of the socle series.

4. Homomorphisms within direct sums

Suppose that (N_1, N_2, V) is a tame triple and V_1 and V_2 are disjoint submodules of V (in which case $V_1 + V_2 = V_1 \oplus V_2$) and μ is an N_1-homomorphism from V_1 to V_2. In the first result of this section, we show that μ is also an N_2-homomorphism.

Proposition 4.1. If (N_1, N_2, V) is a tame triple, V_1 and V_2 are disjoint N_1-submodules of V and $\mu\colon V_1 \to V_2$ is an N_1-homomorphism, then μ is an N_2-homomorphism.

Proof. Since the sum V_1+V_2 is direct and μ is an N_1-homomorphism of V_1 into V_2, the set S of all $v_1+v_1\mu$, $v_1 \in V_1$, is an N_1-submodule of V. Thus $v_1\alpha+v_1\mu\alpha$, $\alpha \in N_2$, is in S. As $v_1\alpha+v_1\alpha\mu$ is in S, $-v_1\alpha\mu+v_1\mu\alpha$, which lies in V_2, is in S. Since the only element of V_2 in S is 0, $v_1\mu\alpha = v_1\alpha\mu$ and 4.1 is proved. ◊

A special case of 4.1 that comes up frequently in practice is when μ is an N_1-isomorphism where we have the following as an immediate corollary to 4.1.

Corollary 4.2. If (N_1, N_2, V) is a tame triple and V_1 and V_2 are disjoint N_1-isomorphic submodules of V, then V_1 and V_2 are N_2-isomorphic.

As a consequence of 4.2, we have the following result we shall need in section 6 involving the transferability of minimal finiteness in a tame triple.

Theorem 4.3. If (N_1, N_2, V) is a tame triple and N_1 has $wDCCR$, then V is a minimally finite N_2-group.

Proof. Consider a factor of socle series terms $soc_{i+1}(V)/soc_i(V)$ of V. Since V is a minimally finite N_1-group by 2.8, $soc_{i+1}(V)/soc_i(V)$ has finitely many N_1-isomorphism classes of minimal submodules which are the same as the N_2-isomorphism classes of minimal submodules of this factor by 4.2. As $soc_n(V) = V$ for some nonnegative integer n by 2.5, the result now follows. ◊

In the setting of 4.2, the question arises as to whether N_1-automorphisms of V_1 or V_2 are also N_2-automorphisms. This holds when the N_1-automorphism is fixed point free.

Proposition 4.4. If (N_1, N_2, V) is a tame triple and V_1 and V_2 are disjoint N_1-isomorphic submodules of V, then a fixed point free N_1-automorphism δ of V_2 is an N_2-automorphism.

Proof. Let μ be an N_1-isomorphism of V_1 onto V_2. Clearly $\mu\delta$ is also an N_1-isomorphism of V_1 onto V_2. Thus $X_1 = \{v_1 + v_1\mu : v_1 \in V_1\}$ and $X_2 = \{v_1 + v_1\mu\delta : v_1 \in V_1\}$ are both N_1-submodules of V. They are also N_2-submodules. The map λ of X_1 onto X_2 taking $v_1 + v_1\mu$ in X_1 to $v_1 + v_1\mu\delta$ in X_2 is readily seen to be an N_1-homomorphism of X_1 onto X_2. However $X_1 \cap X_2 = \{0\}$ since if $u_1 + u_1\mu = u_2 + u_2\mu\delta$ for u_1 and u_2 in V_1, we must have $u_1 = u_2$ and $u_1\mu = u_1\mu\delta$. Thus $u_1\mu\delta = 0$ since δ is fixed point free. As this means $u_1 = 0$, $X_1 \cap X_2 = \{0\}$.

It follows from 4.1 that λ is an N_2-homomorphism of X_1 onto X_2. Thus for α in N_2 and v_1 in V_1,

$$(v_1 + v_1\mu)\alpha\lambda = (v_1 + v_1\mu)\lambda\alpha = (v_1 + v_1\mu\delta)\alpha = v_1\alpha + v_1\mu\delta\alpha.$$

However, since μ is an N_2-isomorphism of V_1 onto V_2 by 4.1,

$$(v_1 + v_1\mu)\alpha\lambda = (v_1\alpha + v_1\alpha\mu)\lambda = v_1\alpha + v_1\alpha\mu\delta = v_1\alpha + v_1\mu\alpha\delta.$$

This can only mean $v_1\mu\delta\alpha = v_1\mu\alpha\delta$. Hence δ is an N_2-automorphism on $V_1\mu = V_2$ and 4.4 is proved. ◊

We finish this section with a more specialized proposition we shall need in section 6 whose proof depends on 4.4.

Proposition 4.5. Suppose that (N_1, N_2, V) is a tame triple and V_1 and V_2 are disjoint N_1-isomorphic minimal submodules of V. If $N_1/J(N_1)$ has $DCCR$, then V_1 and V_2 are N_2-isomorphic and $N_1 + (0 : V_1)_{N_2} = N_1 + (0 : V_2)_{N_2} = N_2$.

Proof. We already know V_1 and V_2 are N_2-isomorphic by 4.2. Let μ be an N_2-isomorphism from V_1 onto V_2. The existence of the diagonal N_i-subgroup $\{v_1 + v_1\mu : v_1 \in V_1\}$ of $V_1 + V_2 = V_1 \oplus V_2$ yields that $V_1 + V_2$ is an N_i-ring module (that is, $N_i/(0 : V_1 + V_2)_{N_i}$ is a ring and $V_1 + V_2$ is a ring module of this ring) by [14, Proposition 6.4]. Take N_3 as the nearring $[N_1 + (0 : V_2)_{N_2}]/(0 : V_2)_{N_2}$ and N_4 as the nearring $N_2/(0 : V_2)_{N_2}$. Clearly $N_3 \leq N_4$ and both are subnearrings of $M_0(V_2)$ that are tame on V_2. It is easy enough to see 4.5 will be proved if it is shown $N_3 = N_4$. Let D be the division ring consisting of 0 and all N_3-automorphisms of V_2. The N_3-automorphisms are N_1-automorphisms and, apart from 1, are fixed point free. It follows from 4.4 that D is the division ring consisting of 0 and all N_4-automorphisms of V_2. However, as N_3 is primitive on V_2 and has $DCCR$, N_3 is just the ring $End_D(V_2)$ of all D-endomorphisms of V_2. Now N_4 consists of D-endomorphisms of V_2 so that $N_4 \leq End_D(V_2) = N_3$. Because $N_3 \leq N_4$, the proposition holds. $\Diamond$

5. Isolated submodules

We shall say that a submodule U of an N-group V is *isolated* if for each submodule H of V either $H \leq U$ or $U \leq H$. In [1], an element β of a bounded lattice L is said to *cut* the lattice L if for each element α of L either $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus saying that a submodule U of an N-group V is isolated is the same as saying U cuts the submodule lattice of V. Of course, V always has two trivial isolated submodules, namely, V and $\{0\}$. In this section we shall develop some results about $C_0(V)$ involving isolated submodules of V. We begin with a technical lemma for producing elements of $C_0(V)$ via an isolated submodule U of V. Before stating this lemma, we note that if β is an element of $C_0(V)/(0 : U)_{C_0(V)}$, then any two coset representatives α_1 and α_2 of β in $C_0(V)$ have the same action on U; that is, if $\beta = \alpha_1 + (0 : U)_{C_0(V)} = \alpha_2 + (0 : U)_{C_0(V)}$, then

$u\alpha_1 = u\alpha_2$ for all $u \in U$. In particular, we may then view β as defining a map on U by setting $u\beta = u\alpha_1$ for $u \in U$.

Lemma 5.1. Suppose that V is a faithful N-group, U is an isolated submodule of V and v_i, $i \in I$, is a transversal of U in V where 0 is in I and $v_0 = 0$. If β is an element of $C_0(V)/(0 : U)_{C_0(V)}$ and S is the set of maps γ in $M_0(V)$ defined on the elements of the cosets $v_i + U$ by $(v_i + u)\gamma = u\gamma_i$ where $\gamma_i \in \{0, \beta\}$, then $S \subseteq C_0(V)$.

Proof. Suppose v is in V, W is a submodule of V and $\gamma \in S$. We prove this lemma by showing $(v + W)\gamma \subseteq v\gamma + W$. We have $v = v_i + \overline{u}$ where $i \in I$ and $\overline{u} \in U$. If v is in $V\backslash U$ and $W \leq U$, then for w in W, $(v + w)\gamma = (\overline{u} + w)\gamma_i = \overline{u}\gamma_i + w'$ with $w' \in W$. As $\overline{u}\gamma_i = v\gamma$, it follows in this case that $(v + W)\gamma \subseteq v\gamma + W$. If v is in $V\backslash U$ and $W > U$, then $(v + W)\gamma \subseteq U < W = v\gamma + W$. When v is in U and $W \leq U$, we have $(v+w)\gamma = (\overline{u}+w)\gamma_i = \overline{u}\gamma_i+w'$ with $w' \in W$. Since $\overline{u}\gamma_i = v\gamma$, $(v+W)\gamma \subseteq v\gamma+W$. For v in U and $W > U$, $(v+W)\gamma \subseteq U < W = v\gamma+W$. Because $(v + W)\gamma \subseteq v\gamma + W$ in all cases, our proof is complete. ◊

An element of S in 5.1 that we make special note of and denote by e occurs when β is the identity map, $\gamma_0 = \beta$ and $\gamma_i = 0$ for $i \neq 0$. This map e is the same as the map in $M_0(V)$ that takes each element of $V\backslash U$ to 0 and is the identity map on U which we record as the following corollary to 5.1.

Corollary 5.2. Suppose V is a faithful N-group and U is an isolated submodule of V. The element e of $M_0(V)$ that takes each element of $V\backslash U$ to 0 and is the identity map on U is in $C_0(V)$.

As defined in [7], two factors A/B and C/D of submodules $A \geq B$ and $C \geq D$ of an N-group V are said to be *coprime* if A/B and C/D have no common isomorphic minimal factors. If V is an N-group and U is a submodule of V, an element ε of N such that $V\varepsilon \leq U$ and $u\varepsilon = u$ for all $u \in U$ is called a *projection idempotent of* V *onto* U in [5]. Projection idempotents have played an important role in the study of units of compatible nearrings [2, 7, 8]. Whenever a projection idempotent ε from V onto U exists, V/U and U must be coprime since ε acts as the zero map on V/U and the identity map on U. Thus as a consequence of 5.2 we have another corollary.

Corollary 5.3. If V is a faithful N-group and U is an isolated submodule of V, then V/U and $U$are coprime $C_0(V)$-groups.

6. $DCCR$ is not transferred

At the end of section 2, we considered the question of whether a nearring N which is tame on an N-group V that has a tame series, is minimally finite and is minimally complete must have $wDCCR$ and saw an example showing the answer to this question is no. This is but one of a number of questions involving finiteness conditions that arise in the study of tame nearrings. Another is: if (N_1, N_2, V) is a tame triple and N_1 has $DCCR$, does N_2 have $DCCR$? Here it is quite easy to produce examples showing the answer to this latter question is no. Consider a finite dimensional vector space over an infinite field F and let N_1 be the ring of linear transformations of V and $N_2 = C_0(V)$. Then N_1 has $DCCR$ while N_2 does not since $C_0(V) = M_0(V)$ and $M_0(V)$ has $DCCR$ if and only if V is a finite group. But what about if we impose a further restriction on V as an N_2-group such as it be a soluble N_2-group by which we mean (see [13] or [14]) V has a series $\{0\} = V_0 \leq V_1 \leq \cdots \leq V_r = V$ of N_2-submodules of V such that each factor V_{i+1}/V_i is an N_2 ring module? While initially it might seem that this question has a positive answer, we now give an example to show this in not the case.

To begin the construction of our example, let F be an infinite field and V the vector space $F \oplus F \oplus F \oplus F$ of dimension four over F. It will be convenient to denote each successive copy of F by F_i, $i = 1, \ldots, 4$. For N_1 we will use the ring generated by the set of scalar linear transformations of V, which we will denote by $d(V)$, and the linear transformations μ_{13}, μ_{14}, μ_{23} and μ_{24} that respectively take a vector (a_1, a_2, a_3, a_4) in V to $(0, 0, a_1, 0)$, $(0, 0, 0, a_1)$, $(0, 0, a_2, 0)$ and $(0, 0, 0, a_2)$. For N_2 we will use $C_0(V)$ of the N_1-group V.

Of course, we may identify $d(V)$ with F making $F \leq N_1$. Setting

$$K = \{\mu_{13}a_1 + \mu_{14}a_2 + \mu_{23}a_3 + \mu_{24}a_4 : a_1, \ldots, a_4 \in F\},$$

it is easy to see that $N_1 = F + K$, $F \cap K = \{0\}$, K is a nilpotent ideal of N_1 of nilpotency degree 2, N_1/K isomorphic to F, each $\mu_{ij}F$ is minimal right ideal of N_1 and K is the sum of these minimal right ideals of N_1. The final three of these observations give us that N_1 has $DCCR$.

We next show that V is a soluble N_2-group. Observe that $F_3 \oplus F_4$ is the direct sum of the two N_1-isomorphic N_1-submodules F_3 and F_4 of V. Also $V/(F_3 \oplus F_4)$ has the properties that $(V/(F_3 \oplus F_4))K = \{0\}$, $V/(F_3 \oplus F_4)$ is the direct sum of the two N_1-isomorphic N_1-submodules $(F_1 \oplus F_3 \oplus F_4)/(F_3 \oplus F_4)$ and $(F_2 \oplus F_3 \oplus F_4)/(F_3 \oplus F_4)$ of $V/(F_3 \oplus F_4)$

and $K = (0 : F_3 \oplus F_4) = (0 : V/(F_3 \oplus F_4))$. Now, by 4.5 the action of N_2 on $F_3 \oplus F_4$ is the same as that of N_1. The same applies to $V/(F_3 \oplus F_4)$. Thus since $F_3 \oplus F_4$ and $V/(F_3 \oplus F_4)$ are ring modules as an N_1-group, V is N_2-soluble.

Finally, we show that N_2 does not have $DCCR$. To do so, we first show that $F_3 \oplus F_4$ is isolated in V. To see this, note that if v is in $V \backslash (F_3 \oplus F_4)$, then there exists an element of $\{\mu_{13}, \mu_{14}, \mu_{23}, \mu_{24}\}$ taking v to a nonzero element of F_3 and another taking v to a nonzero element of F_4. This means $vR \geq F_3 \oplus F_4$ and hence $F_3 \oplus F_4$ is isolated. Now, to obtain that N_2 does not have $DCCR$, let e be the projection idempotent of 5.2 in $C_0(V) = N_2$ from V onto $U = F_3 \oplus F_4$ and let $v_1, v_2, \ldots$ be an infinite sequence of distinct elements of $V \backslash (F_3 \oplus F_4)$. The maps β_i, $i = 1, 2, \ldots$, of N_2 given by $(-v_i + w)e - (-v_i)e = (-v_i + w)e = w\beta_i$ for all w in V are nonzero elements of $R_i = (0 : V \backslash (v_i + F_3 \oplus F_4))_{N_2}$. It now follows readily that $R_1, R_1 + R_2, R_1 + R_2 + R_3, \ldots$ is a properly ascending chain of right ideals of $C_0(V)$. Indeed, if for some integer $m \geq 1$, $R_1 + \cdots + R_m \geq R_{m+1}$ then, as all of the R_i, $i = 1, \ldots, m$, annihilate $v_{m+1} + F_3 \oplus F_4$, R_{m+1} annihilates $v_{m+1} + F_3 \oplus F_4$ which is impossible since β_{m+1} does not do this. Thus N_2 does not have $DCCR$ since, if it did, it must also have the ascending chain condition on right ideals by [10, Theorem 5.7].

While N_2 does not have $DCCR$, it does happen to have $wDCCR$. To see this, we next prove a preliminary lemma followed by a theorem involving transferability of $wDCCR$.

Lemma 6.1. Suppose V is an N-group and V_i, $i = 1, \ldots, k$, are submodules of V. If each V/V_i has the descending chain condition (ascending chain condition) on submodules, then so does $V/(V_1 \cap \cdots \cap V_k)$.

Proof. As the proof for the ascending chain condition is similar to that for the descending chain condition (DCC), we deal only with the latter. The lemma is immediate for $k = 1$. If the lemma is proved for $k = 2$, it will follow for $k \geq 3$ by induction because $\cap_{i=1}^{k} V_i = (V_1 \cap V_2) \cap [\cap_{i=3}^{k} V_i]$.

For $k = 2$, not only do submodules of V between V_1 and V have DCC, but so do submodules of V between $V_1 \cap V_2$ and V_1. Indeed, for a descending chain $H_1 \geq H_2 \geq \ldots$ of such submodules, there exists a positive integer n such that $V_2 + H_i = V_2 + H_{i+1}$ for all $i \geq n$. Using the modular law, $H_i = [V_2 + H_i] \cap V_1 = [V_2 + H_{i+1}] \cap V_1 = H_{i+1}$ for all $i \geq n$ which gives us the required DCC condition between $V_1 \cap V_2$ and V_1.

Now suppose $X_1 \geq X_2 \geq \ldots$ is a descending chain of submodules of V between $V_1 \cap V_2$ and V. We have that there exists a positive integer r such that $V_1 + X_i = V_1 + X_{i+1}$ for all $i \geq r$ and, from what has just been proved, there exists a positive integer s such that $V_1 \cap X_i = V_1 \cap X_{i+1}$ for all $i \geq s$. Another use of the modular law gives us $X_i = [V_1 + X_{i+1}] \cap X_i = V_1 \cap X_i + X_{i+1} = X_{i+1}$ for all $i \geq max\{r, s\}$ which completes our proof. ♢

Theorem 6.2. Suppose that (N_1, N_2, V) is a tame triple and N_1 has $wDCCR$. Then N_2 has $wDCCR$ if and only if $N_2/(0 : H_1/H_2)_{N_2}$ has $DCCR$ for each minimal factor H_1/H_2 of V.

Proof. As the only if part is trivial, we need only give a proof of the if part. Suppose that $N_2/(0 : H_1/H_2)_{N_2}$ has $DCCR$ for each minimal factor H_1/H_2 of V. Since V has a tame series as an N_1-group by 2.5 which is then a tame series for V as an N_2-group, $J(N_2)$ is nilpotent by 2.1. To complete this proof, we must show that $N_2/J(N_2)$ has $DCCR$. We know that V is a minimally finite N_2-group by 4.3. Let $L_1/K_1, \ldots, L_n/K_n$ be minimal factors of V that serve as representatives for the N_2-isomorphism classes of minimal factors of V. We have that $J(N_2) \leq \cap_i(0 : L_i/K_i)_{N_2}$. Conversely since $\cap_i(0 : L_i/K_i)_{N_2}$ annihilates each factor in any tame series of V, $\cap_i(0 : L_i/K_i)_{N_2}$ is nilpotent which tells us that $\cap_i(0 : L_i/K_i)_{N_2} \leq J(N_2)$. Thus $\cap_i(0 : L_i/K_i)_{N_2} = J(N_2)$. This in turn gives us $\cap_i((0 : L_i/K_i)_{N_2} + J(N_2))$ is the zero ideal in $N_2/J(N_2)$. Applying 6.1 with $V = N_2/J(N_2)$ and $V_i = (0 : L_i/K_i)_{N_2} + J(N_2)$ then gives us the required condition that $N_2/J(N_2)$ has $DCCR$. ♢

Now, let us return to our example. We have seen that the action of N_2 on each of $F_3 \oplus F_4$ and $V/(F_3 \oplus F_4)$ is the same as that of N_1 when showing that V is N_2-soluble. As each minimal factor H_1/H_2 of V is N_2-isomorphic to one of $F_3 \oplus F_4$ or $V/(F_3 \oplus F_4)$ and $N_1/(0 : H_1/H_2)_{N_1}$ has $DCCR$, we must have each $N_2/(0 : H_1/H_2)_{N_2}$ has $DCCR$. Hence N_2 has $wDCCR$ by 6.2. In particular, we have an example of a nearring with $wDCCR$, but not $DCCR$ as promised in the introduction. We shall leave the further study of transferability of $wDCCR$ from N_1 to N_2 in a tame triple (N_1, N_2, V) for future investigations.

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