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On Primeness and Radicals in Near-rings of Continuous Functions

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Abstract: The author has previously studied primeness in the near-ring $N_0(G)$ of continuous zero-preserving functions of an additive topological group G. In this paper, we consider primeness in the near-ring $N(G)$ of all all continuous functions of G, and characterise the various prime radicals in certain cases. We then discuss primeness in the sandwich near-ring $N(X, G, \theta)$, where X and G are a topological space and a topological group, respectively, and $\theta : G \to X$ is a continuous mapping.

1. Preliminaries

In this paper, all near-rings will be right distributive, and will only be zero-symmetric when explicitly so stated. Implicitly, this means that all functions will act from the left. For the basics on near-rings, we refer to any of the standard texts, e.g. [13] and [14]. If N is a near-ring, the notation "$I \triangleleft N$" and "$I \triangleleft_\ell N$ " will mean "I is an ideal of N" and "I is a left ideal of N", respectively. Recall that a subgroup A of the additive group of a near-ring N is called an N-*subgroup* of N if $NA \subseteq A$, Let $(G, +)$ be a (not necessarily Abelian) group, and let $M(G)$ denote the set of all self-maps of G. As is well known, $M(G)$ is a near-ring with respect to pointwise addition and composition of maps. The set $M_0(G) := \{a \in M(G) : a(0) = 0\}$ is a subnear-ring of $M(G)$ which is

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zero-symmetric, i.e. $a0 = 0$ for all $a \in M_0(G)$. $M(G)$ and $M_0(G)$ provide prototypes for all near-rings (resp. all zero-symmetric near-rings) in the sense that every near-ring (resp. zero-symmetric near-ring) is isomorphic to a subnear-ring of $M(G)$ (resp. $M_0(G)$) for a suitable choice of G (cf [14], Corollary 1.18). The following result is well known.

Proposition 1.1. [13, *Theorems 1.42 and 1.43*] *Let* $(G, +)$ *be a group. Then*:

1. $M_0(G)$ *is a simple near-ring.*
2. *If* $|G| \neq 2$, *then* $M(G)$ *is a simple near-ring.*

From the late 1960's various authors began to add a topological flavour to the study of near-rings of self-maps by requiring that the group be topological and that the self-maps be continuous. In the sequel, all topological groups will be T_0, and hence completely regular.

Definition 1.2. *Let* $(G, +)$ *be a topological group. Then*
1. $N(G) := \{a \in M(G) : a \text{ is continuous}\}$;
2. $N_0(G) := \{a \in M_0(G) : a \text{ is continuous}\}$.

It is clear that $N(G)$ and $N_0(G)$ are subnear-rings of $M(G)$ and $M_0(G)$, respectively. Moreover, if the topology on G is discrete, then $N(G) = M(G)$ and $N_0(G) = M_0(G)$. For surveys of early work done on these near-rings, we refer to [11] and [12]. For information on the theory of topological groups, any of the standard texts may be consulted, for example [9]. In order to avoid trivial cases and exceptions, all topological groups and topological spaces considered in the sequel will be assumed to contain more than one element.

Several notions of primeness for near-rings exist in the literature, We will consider some of these.

Definition 1.3. *A near-ring* N *is*

- 0-*prime if* $A, B \lhd N$, $AB = \{0\}$ *implies* $A = \{0\}$ *or* $B = \{0\}$;
- 1-*prime if* $A, B \lhd_\ell N$, $AB = \{0\}$ *implies* $A = \{0\}$ *or* $B = \{0\}$;
- *2-prime if* A, B N*-subgroups of* N, $AB = \{0\}$ *implies* $A = \{0\}$ *or* $B = \{0\}$;

- 3-*prime if* $a, b \in N,\ aNb = \{0\}$ *implies* $a = 0$ *or* $b = 0$ [7];
- *equiprime* (*e-prime*) *if* $a, x, y \in N, anx = any$ *for all* $n \in N$ *implies* $a = 0$ *or* $x = y$ [4].

We remark that an equiprime near-ring is necessarily zero-symmetric [4]. The relationships between the different definitions of primeness are as follows:

equiprime$\Rightarrow$ 3$-$prime$\Rightarrow$ 2-prime; 3-prime$\Rightarrow$ 0-prime and 1-prime$\Rightarrow$0-prime. 2-prime$\Rightarrow$1-prime holds for zero-symmetric near-rings, but not for arbitrary near-rings. An ideal P of N is called ν-prime ($\nu \in \{0, 1, 2, 3, e\}$) if the factor near-ring N/P is ν-prime. Note that if P is an e-prime ideal of N, then P is left invariant since N/P is a zero-symmetric near-ring. The *ν-prime radical* of $N, \mathcal{P}_\nu(N)$, is the intersection of the ν-prime ideals of N.

The equiprime radical $\mathcal{P}_e$ is of special interest in that it is the only known Kurosh-Amitsur prime radical for both zero-symmetric and all near-rings [4].

2. The Near-ring $N(G)$

Previously the author has studied primeness in the zero-symmetric case. See for example [1], [2], [3] and [5]. In this section we will consider primeness in $N(G)$, which is not a zero-symmetric near-ring. In the sequel, if $k \in G, \varphi_k$ will denote the self-map of G defined by $\varphi_k(g) := k$ for all $g \in G$.

Proposition 2.1. *$N(G)$ is 3-prime, and hence both 2-prime and 0-prime.*

Proof. Let $0 \neq a, b \in N(G)$. Then there exists $g \in G$ such that $a(g) \neq 0$. Then $a\varphi_g b(h) = a(g)$ for all $h \in G$. Hence $a\varphi_g b \neq 0$ and so $N(G)$ is 3-prime. $\Diamond$

Since $N(G)$ is not zero-symmetric if G has more than one element, it cannot be equiprime or $\mathcal{P}_e$-semisimple in this case. However, it may or may not be $\mathcal{P}_e$-radical, as the following examples show.

Example 2.2. *Let G be a topological division ring whose characteristic is not 2. Then $N(G)$ is simple* [10, *Theorem 2.4*]. *Since it is not $\mathcal{P}_e$-semisimple, it is $\mathcal{P}_e$-radical. Note that the proof given in* [*10*] *is not valid* *in the case that G has characteristic 2, and the result does not in fact hold in this case.*

Example 2.3. *Consider $N(\mathbb{Z}_2)$ with the discrete topology. It is well known that the constant part C of this near-ring is an ideal of $N(\mathbb{Z}_2)$ and that $N(\mathbb{Z}_2)/C$ is isomorphic to $\mathbb{Z}_2$. Hence C is an equiprime ideal of $N(\mathbb{Z}_2)$. It follows that $\mathcal{P}_e(N(\mathbb{Z}_2)) = C$.*

G is said to be *2-transitive* if $v, w, x, y \in G$, $v \neq w$ implies that there exists $a \in N(G)$ such that $a(v) = x$ and $a(w) = y$. Note that the class of 2-transitive topological groups includes the 0-dimensional as well as the arcwise connected ones.

Lemma 2.4. *Suppose that G is 2-transitive and that $0 \neq I \lhd_\ell N(G)$. Then $IG = G$.*

Proof. Let $g \in G$. We will show that there exist $b \in I$ and $h \in G$ such that $g = b(h)$. This holds trivially if $g = 0$, so suppose that $g \neq 0$. Let $0 \neq a \in I$. Then there exists $h \in G$ such that $a(h) \neq 0$. Let $0 \neq k \in G$. Then $(a + \varphi_k)(h) = a(h) + k \neq k$. Since G is 2-transitive, there exists $m \in N(G)$ such that $m(a(h) + k) = g$ and $m(k) = 0$, i.e. $m(a + \varphi_k)(h) = g$ and $m\varphi_k(h) = 0$, whence $(m(a + \varphi_k) - m\varphi_k)(h) = g$. But $m(a + \varphi_k) - m\varphi_k \in I$ since $a \in I$ and $I \lhd_\ell N(G)$, and the proof is complete. $\diamondsuit$

Proposition 2.5. *Let G be a 2-transitive topological group. Then $N(G)$ is* 1*-prime.*

Proof. Suppose that $0 \neq I, J \lhd_\ell N(G)$. Then $(IJ)G = I(JG) = IG$ (by Lemma 2.4) $= G$ (again by Lemma 2.4). Hence $IJ \neq 0$ and so $N(G)$ is 1-prime. $\diamondsuit$

Proposition 2.6. *Suppose that G is disconnected and has arcwise connected, open components, and let G_0 denote the component of G which contains* 0*. Let $I := \{a \in N(G) : a(G) \subseteq G_0\}$ and $J := \{a \in N(G) : a(G_0) = \{0\}\}$. Then $I \cap J \subseteq \mathcal{P}_1(N(G)) \subseteq I$.*

Proof. Clearly, $I \lhd N(G)$ and $J \lhd_\ell N(G)$. Hence $I \cap J \lhd_\ell N(G)$. Moreover $(I \cap J)^2 = \{0\} \subseteq \mathcal{P}_1(N(G))$. Since $\mathcal{P}_1(N(G))$ is the intersection of the 1-prime ideals of $N(G), I \cap J \subseteq \mathcal{P}_1(N(G))$.

We now show that $N(G)/I$ is isomorphic to $N(G/G_0)$, where G/G_0 has the quotient topology with respect to the canonical epimorphism of G

onto G_0. If $a \in N(G)$, let $\overline{a} : G \to G_0$ be defined by $\overline{a}(g+G_0) := a(g)+G_0$ for all $g \in G$. Then $\overline{a}$ is well defined, for suppose that $g, g' \in G_0$ are such that $g + G_0 = g' + G_0$. Then g and g' are elements of the same coset of G_0. By the continuity of a, and since the cosets of G_0 are precisely the components of $G, a(g)$ and $a(g')$ are elements of the same coset of G_0. It is easily checked that $\overline{a} + \overline{b} = \overline{a+b}$ and $\overline{a} \cdot \overline{b} = \overline{(a \cdot b)}$ for all $a, b \in N(G)$. Now let $c \in N(G/G_0)$. Let the set of cosets of G_0 in G be $\{G_k : k \in I\}$, where I is an indexing set. Choose a representative g_k from $G \backslash G_k$ for each coset G_k. Let $a : G \to G$ be defined by $a(h) := g_k$ if $h \in G_k$. Since the cosets of G_0 are open (and hence clopen, i.e both open and closed), it follows that the mapping $a \to \overline{a}$ is continuous. Moreover $\overline{a} = c$. Hence the mapping $a \to \overline{a}$ defines an epimorphism of $N(G)$ onto $N(G/G_0)$. Moreover, the kernel of this epimorphism is I. Hence $N(G/G_0)$ is isomorphic to $N(G)/I$. Since the components of G_0 are open, the topology on $N(G/G_0)$ is discrete, and hence 2-transitive. Hence $N(G/G_0)$ is a 1-prime near-ring and so I is a 1-prime ideal of $N(G)$. It follows that $\mathcal{P}_1(N(G)) \subseteq I$, as required. $\diamondsuit$

We now turn our attention to the equiprime radical of $N(G)$.

Proposition 2.7. *Let G be a disconnected topological group and let I and J be as in Proposition* 2.6. *Suppose further that the components of G are open and that the index of G_0 in G is not equal to* 2. *Then $\mathcal{P}_e(N(G)) = N(G)$.*

Proof. Let P be a proper equiprime ideal of $N(G)$. Then $JI = \{0\} \subseteq P$. Since P is an equiprime (and hence 1-prime) ideal of $N(G)$, either $I \subseteq P$ or $J \subseteq P$. Suppose that $I \subseteq P$. Since G_0 is not of index 2 in $G, G/G_0$ does not have order 2. Hence $M(G/G_0)$ is a simple near-ring. As in the proof of Proposition 2.6, $N(G/G_0)$ is isomorphic to $N(G)/I$. Since the cosets of G_0 in G are open and hence clopen the quotient topology on G/G_0 is discrete, so $N(G/G_0) = M(G/G_0)$, and hence $N(G/G_0)$ is a simple near-ring. It follows that I is a maximal ideal of $N(G)$. Hence $P = I$ or $P = N(G)$. But since P is an an equiprime ideal, it is left invariant. However, I is not left invariant. For if $g \in G \backslash G_0$ and a is an arbitrary element of I, then $\varphi_g a = \varphi_g \notin I$. Hence $P = N(G)$, contradicting our assumption that P is a proper ideal of $N(G)$. Thus $J \subseteq P$. Now define the function $a : G \to G$ by

$$a(g) := \begin{cases} 0 & \text{if } g \in G_0 \\ g & \text{otherwise} \end{cases} .$$

Since the cosets of G_0 are clopen, a is continuous. Clearly $a \in J \subseteq P$. Choose a representative g_1 from a coset $G_1 \neq G_0$. Define $m, n : G \to G$ by

$$m(g) := \begin{cases} g - g_1 & \text{if } g \in G_1 \\ 0 & \text{otherwise} \end{cases} \text{ and } n(g) := \begin{cases} g + g_1 & \text{if } g \in G_0 \\ 0 & \text{otherwise} \end{cases}.$$

The continuity of m and n again follow from the fact that the cosets of G_0 in G are clopen. It may easily be checked that $man(g) = \begin{cases} g & \text{if } g \in G_0 \\ 0 & \text{otherwise} \end{cases}$. Since P is an equiprime (and hence left invariant) ideal of $N(G), man \in P$ and so $a + man \in P$. But $a + man = i$, the identity mapping on G. Hence $P = N(G)$, contradicting our assumption that P is a proper ideal of $N(G)$. Thus $N(G)$ has no nonzero proper equiprime ideals and so $\mathcal{P}_e(N(G)) = N(G)$. $\Diamond$

Recall that $M(\mathbb{Z}_2)$ is not simple, but has three ideals: $\{0\}$, $M(\mathbb{Z}_2)$ and its constant part $(M(\mathbb{Z}_2))_c$. We will now consider the situation for $N(G)$ when G has two components.

Proposition 2.8. *Let G be a disconnected topological group, and let I and J be as in Proposition 2.6. Suppose further that the index of G_0 in G is 2. Then $\mathcal{P}_e(N(G)) = C := \{a \in N(G) :$ a maps G into a single coset of $G_0\}$.*

Proof. Let the cosets of G_0 in G be G_0 and G_1, and let P be a proper equiprime ideal of $N(G)$. As in the proof of Proposition 2.7, we have that $J \subseteq P$ or $I \subseteq P$. Suppose that $I \subseteq P$. As in the proof of Proposition 2.6,$N(G)/I$ is isomorphic to $N(G/G_0)$, which is in turn isomorphic to $N(\mathbb{Z}_2) = M(\mathbb{Z}_2)$. Let Ψ be the natural epimorphism of $N(G)$ onto $M(\mathbb{Z}_2)$ with kernel I and let $C := \Psi^{-1}((M(\mathbb{Z}_2))_c) = \{a \in N(G) : a$ maps G into a single coset of $G_0\}$. Clearly $I \subseteq C$. Then C/I is isomorphic to the constant part of $M(\mathbb{Z}_2)$, which is isomorphic to the constant near-ring on the 2-element group, and so is simple. Hence I is a maximal ideal of C. Similarly, $N(G)/C$ is isomorphic to the two-element field $\mathbb{Z}_2$, and so is simple, and so is simple, whence C is a maximal, equiprime ideal of $N(G)$. Hence P must be equal to I, C or $N(G)$. As in the proof of Proposition 2.7, I is not left invariant, and since P is a proper ideal of $N(G)$, it follows that $P = C$ in this case.

Now suppose that $J \subseteq P$. Using the argument employed in the proof of Proposition 2.7, we conclude that $P = N(G)$ in this case. It follows that the only proper equiprime ideal of $N(G)$ is C, and hence $\mathcal{P}_e(N(G)) = C$. $\Diamond$

The notion of strongly prime was defined for rings by Handelman and Lawrence [8], and extended to near-rings by Groenewald [6]. It is well known that this notion is strictly stronger than that of prime, even for non-commutative rings.

Definition 2.9. *A near-ring N is*:

1. *Strongly prime if for every $0 \neq a \in N$ there exists* a *finite subset F of N (called an insulator of a) such that $aFx = 0$ implies $x = 0$, for all $x \in N$.*
2. *Uniformly strongly prime if N is strongly prime and the insulator F is independent of the choice of a.*
3. *Completely prime if $xy = 0$ implies $x = 0$ or $y = 0$ for all $x, y \in N$.*

Proposition 2.10.

1. *$N(G)$ is strongly prime.*
2. *$N(G)$ is not completely prime.*

Proof. (1) Let $0 \neq a \in N$. Then there exists $g \in G$ such that $a(g) \neq 0$. Let $F := \{\varphi_g\}$. Then $a\varphi_g x(h) = a(g) \neq 0$ for all $x \in N(G)$ and $h \in G$. Hence $N(G)$ is strongly prime.

(2) Let $0 \neq g_0 \in G$. Let a be defined by $a(g) = g - g_0$ for all $g \in G$. If $h \in G$, then $a\varphi_{g_0}(h) = a(g_0) = g_0 - g_0 = 0$, and so $a\varphi_{g_0} = 0$. But $a, \varphi_{g_0} \neq 0$ and hence $N(G)$ is not completely prime. $\Diamond$

Proposition 2.11. *Suppose that G is infinite, and is either 0-dimensional or it contains an arc. Then $N(G)$ is not uniformly strongly prime.*

Proof. Let $F := \{f_1, \ldots, f_n\}$ be a finite subset of $N(G)$. Let $0 \neq g \in G$. For $1 \leq i \leq n$, the range of $f_i\varphi_g$ contains a single element, k_i say. Let $K := \{k_1, \ldots, k_n\}$. Let $0 \neq h \in G \backslash K$.

Suppose that G is 0-dimensional. For $1 \leq i \leq n$ there exists a clopen set U_i such that $k_i \in U_i$ and $h \notin U_i$. Let $U := \bigcup_{i=1}^{n} U_i$. Then U is clopen, $K \subseteq U$ and $h \notin U$. Define the mapping a as follows. Let

$$a(x) := \begin{cases} 0 & \text{if } x \in U \\ h & \text{otherwise} \end{cases}.$$

Then $0 \neq a \in N(G)$ and $af_i\varphi_g(x) = af_i(g) = a(k_i) = 0$ for all $x \in G$ and $1 \leq i \leq n$. Hence $af_i\varphi_g = 0$ for $1 \leq i \leq n$ and so $aF\varphi_g$. But $a, \varphi_g \neq 0$ and so $N(G)$ is not uniformly strongly prime.

Now suppose that G contains an arc. The set K is closed and $h \notin K$. Since G is completely regular, there exists a continuous mapping $\alpha : G \to [0,1]$ such that $\alpha(K) = 0$ and $\alpha(h) = 1$. Since G contains an arc, there exists a continuous mapping $\beta : [0,1] \to G$ such that $\beta(0) = g_0$ and $\beta(1) = g_1$, where $g_0 \neq g_1$. We may assume without loss of generality that $g_0 = 0$. If necessary, replace β with γ, where $\gamma(t) := \beta(t) - g_0$ for all $t \in [0,1]$. Let $a := \beta\alpha$. Then $a \neq 0$ and $af_i\varphi_g(x) = af_i(g) = a(k_i) = \beta\alpha(k_i) = \beta(0)$ (since $k_i \in U$) $= 0$. Hence $af_i\varphi_g = 0$ for $1 \leq i \leq n$ and so $aF\varphi_g = 0$. But $a, \varphi_g \neq 0$ and so $N(G)$ is not uniformly strongly prime in this case. $\Diamond$

3. Sandwich Near-rings

Let X and G be a topological space and a topological group, respectively, and let $\theta : G \to X$ be a continuous mapping. Let $N(X, G, \theta)$ denote the set of all continuous mappings from X into G. Then $N(X, G, \theta)$ is a near-ring with pointwise addition and multiplication defined by $a \cdot b := a\theta b$ for all $a, b \in N(X, G, \theta)$. In the sequel X will be completely regular, and hence Hausdorff. If $g \in G, \lambda_g$ will denote the mapping defined by $\lambda_g(x) := g$ for all $x \in X$.

Proposition 3.1. *If* $\mathrm{cl}(\theta(G)) = X$, *then* $N(X, G, \theta)$ *is 3-prime* (*and hence both 0-prime and 2-prime*), *where* $\mathrm{cl}(\theta(G))$ *denotes the closure of* $\theta(G)$ *in* X.

Proof. Suppose that $0 \neq a, b \in N(X, G, \theta)$. Let $x \in X$ be such that $a(x) \neq 0$. By continuity of a, there exists an open neighbourhood U of x such that $a(y) \neq 0$ for all $y \in U$. Since $\mathrm{cl}(\theta(G)) = X$, there exists $g \in G$ such that $\theta(g) \in U$. Hence $a\theta(g) \neq 0$. Then $\lambda_g\theta n\theta b(z) = g$ for all $z \in X$.

Thus $a\theta\lambda_g\theta n\theta b(z) = a\theta(g) \neq 0$ and so $a\theta\lambda_g\theta n\theta b \neq 0$. Hence $N(X, G, \theta)$ is 3-prime. $\Diamond$

If certain additional conditions are imposed on X or G, the converse of Proposition 3.1 holds.

Proposition 3.2. *Suppose that X is 0-dimensional or G is arcwise connected. Then the following are equivalent*:

1. $\mathrm{cl}(\theta(G)) = X$.
2. $N(X, G, \theta)$ *is 3-prime.*
3. $N(X, G, \theta)$ *is 3-semiprime.*

Proof. (1) $\Rightarrow$ (2): Follows from Proposition 3.1.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Suppose firstly that X is 0-dimensional. Let $x \in X\backslash\mathrm{cl}(\theta(G))$. Since $X\backslash\mathrm{cl}(\theta(G))$ is open and $N(X, G, \theta)$ is 0-dimensional, there exists a clopen set U such that $x \in U \subseteq X\backslash\mathrm{cl}(\theta(G))$. Let $0 \notin g \in G$. Define $a : X \to G$ by

$$a(y) := \begin{cases} g & \text{if } y \in U \\ 0 & \text{otherwise} \end{cases}.$$

Then $0 \neq a \in N(X, G, \theta)$. Since $\theta(G) \subseteq X\backslash U, a\theta(h) = 0$ for all $h \in G$. Hence $a\theta n\theta a(z) = 0$ for all $n \in N(X, G, \theta)$ and $z \in X$. Hence $a\theta n\theta a \neq 0$ and so $N(X, G, \theta)$ is not 3-semiprime.

Now let G be arcwise connected. $x \in X\backslash\mathrm{cl}(\theta(G))$. Since X is completely regular, there exists a continuous mapping $\alpha : X \to [0, 1]$ such that $\alpha(\mathrm{cl}(\theta(G)) = 0$ and $\alpha(x) = 1$. Let $0 \in g \in G$. Since G is arcwise connected, there exists a continuous mapping $\beta : [0, 1] \to G$ such that $\beta(0) = 0$ and $\beta(1) = g$. Let $a := \beta\alpha$. Then $0 \neq a \in N(X, G, \theta)$ and $a\theta(h) = 0$ for all $h \in G$. Hence $a\theta n\theta a(z) = 0$ for all $n \in N(X, G, \theta)$ and $z \in X$. Hence $a\theta n\theta a \neq 0$ and so $N(X, G, \theta)$ is not 3-semiprime in this case. $\Diamond$

$N(X, G, \theta)$ is called *2-transitive* if for all $x, y \in X$ such that $x \neq y$ and $g, h \in G$, there exists $a \in N(X, G, \theta)$ such that $a(x) = g$ and $a(y) = h$. It is easily seen that $N(X, G, \theta)$ is 2-transitive if X is 0-dimensional or G is arcwise connected, inter alia.

Lemma 3.3. *Suppose that $N(X, G, \theta)$ is 2-transitive and that there exists an element x_1 of X such that $\theta^{-1}(x_1)$ contains exactly one element. Let $0 \neq I \triangleleft_\ell N(X, G, \theta)$. Then $IX = G$.*

Proof. Let $g \in G$. We must show that there exists $x \in X$ and $b \in N(X, G, \theta)$ such that $a(x) = g$. This is trivial if $g = 0$, so suppose that $g \neq 0$. Let $0 \neq a \in N(X, G, \theta)$ and let $x \in X$ be such that $a(x) \neq 0$. Let h be the unique element of G such that $x_1 = \theta(h)$. Since $a(x) + h \neq h, \theta(a(g) + h) \neq x_1 = \theta(x)$, whence $\theta(a + \psi_h)(x) \neq \theta\psi_h(x)$. Since $N(X, G, \theta)$ is 2-transitive, there exists $m \in N(X, G, \theta)$ such that $m\theta(a + \psi_h)(x) = g$ and $m\theta\psi_h(x) = 0$, so $(m\theta(a + \psi_h) - m\theta\psi_h) = g$. Let $a := m\theta(a + \psi_h) - m\theta\psi_h$. Then $m\theta(a + \psi_h) - m\theta\psi_h \in I$ and $a(x) = g$, as required. $\diamondsuit$

Proposition 3.4. *Suppose that $N(X, G, \theta)$ is 2-transitive, that* $\mathrm{cl}(\theta(G)) =$ *X and that there exists an element x_1 of X such that $\theta^{-1}(x_1)$ contains exactly one element. Then $N(X, G, \theta)$ is* 1*-prime.*

Proof. Let $0 \neq U, V \triangleleft_\ell N(X, G, \theta)$. Then $U\theta V(X) = U\theta(G)$ by Lemma 3.3. If $u\theta(g) = 0$ for all $u \in U$ and $g \in G$, it follows from the continuity of u and the fact that $\mathrm{cl}(\theta(G)) = X$ that $u(x) = 0$ for all $x \in X$ and $u \in U$, so $U = 0$, contrary to our assumption. Hence $U\theta V \neq 0$, so $N(X, G, \theta)$ is 1-prime. $\diamondsuit$

Theorem 3.5. *Suppose that G, is arcwise connected and that there exists an element x_1 of X such that $\theta^{-1}(x_1)$ contains exactly one element. Then the following are equivalent.*

1. $\mathrm{cl}(\theta(G)) = X$.
2. *$N(X, G, \theta)$ is* 1*-prime.*
3. *$N(X, G, \theta)$ is* 1*-semiprime.*

Proof. (1) $\Rightarrow$ (2). Follows from Proposition 3.4.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Suppose that $\mathrm{cl}(\theta(G)) \neq X$ and let $x \in X \backslash \mathrm{cl}(\theta(G))$. Let $K := \{a \in N(X, G, \theta) : a(\mathrm{cl}(\theta(G)) = 0\}$ Then $K \triangleleft_\ell N(X, G, \theta)$. Since X is completely regular, there exists a continuous map $\alpha : X \to [0, 1]$ such that $\alpha(\mathrm{cl}(\theta(G)) = 0$ and $\alpha(x) = 1$. Let $0 \neq g \in G$. Since G is arcwise connected, there exists a continuous mapping $\beta : [0, 1] \to G$ such that

$\beta(0) = 0$ and $\beta(1) = g$. Let $a := \beta\alpha$. Then $0 \neq a \in K$, so $K \neq 0$. But $K\theta K(X) \subseteq K(\theta(G)) = 0$, so $K\theta K = 0$. Hence $N(X, G, \theta)$ is not 1-semiprime, and the result follows. $\diamondsuit$

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