

The behavior of solutions of the system of two first order linear ordinary differential equations. Part I

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Abstract: The Riccati equation method is used for study the behavior of solutions of the system of two linear first order ordinary differential equations. All types of oscillation and regularity of this system are revealed. A generalization of Leighton's theorem is obtained. Three new principles for the second order linear differential equation are derived. Stability and non conjugation criteria are proved for the mentioned system, as well as estimates are obtained for the solutions of the last one.

1. Introduction

Let $a_{jk}(t)$ ($j, k = 1, 2$) be real valued continuous functions on $[t_0; +\infty)$. Consider the system of equations

$$\begin{cases} \phi' = a_{11}(t)\phi + a_{12}(t)\psi; \\ \psi' = a_{21}(t)\phi + a_{22}(t)\psi, \quad t \geq t_0. \end{cases} \quad (1.1)$$

Study of the questions of the asymptotic behavior (oscillation, non oscillation, non conjugation, rate of growth) of the solutions and stability of the linear system of ordinary differential equations, in

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particular of the system (1.1) is an important problem of the qualitative theory of differential equations and many works are devoted to them (see [1] and cited works therein, [2], [3], [4], [5], [6], [7]). Let $p(t)$, $q(t)$ and $r(t)$ be real valued continuous functions on $[t_0; +\infty)$, and let $p(t) > 0$, $t \geq t_0$. Along with the system (1.1) consider the equation

$$(p(t)\phi')' + q(t)\phi' + r(t)\phi = 0, \quad t \geq t_0. \quad (1.2)$$

The substitution

$$p(t)\phi' = \psi \quad (1.3)$$

in this equation reduces it to the system

$$\begin{cases} \phi' = \frac{1}{p(t)}\psi; \\ \psi' = -r(t)\phi - \frac{q(t)}{p(t)}\psi, \quad t \geq t_0, \end{cases} \quad (1.4)$$

which is a particular case of the system (1.1). For $p(t) \equiv 1$, $q(t) \equiv 0$ Eq. (1.2) takes the forme

$$\phi'' + r(t)\phi = 0, \quad t \geq t_0. \quad (1.5)$$

It is well known (see for example [8]), that by using different transformations Eq. (1.2) can be reduced to the Eq. (1.5). One can show that the system (1.1) can be reduced to Eq. (1.5), if (for example) $a_{12}(t) \neq 0$, $t \geq t_0$. There exist also other conditions for which the system (1.1) can be reduced to Eq. (1.5). Of course the reduction of the system (1.1) to Eq. (1.5), if it is possible to carry it out (until now, it is not known whether this can always be done), can be very useful for study of different qualitative characteristics of the system (1.1). However this method not always can help to solve the assigned problem. One of effective methods of qualitative investigation of Eq. (1.5), as well as of the system (1.1) is the Riccati equation method. In this work we use this method for the study of the behavior of solutions of the system of two linear first order ordinary differential equations. We reveal all types of oscillation and regularity of this system. We obtain a generalization of Leighton's oscillation theorem. We derive three new principles for the second order linear ordinary differential equation. We prove some stability and non conjugation criteria for the mentioned system. We also obtain estimates for the solutions of the last one. Due to large number of sheets of this article we represent here the first part of obtained results of this work. The second part of it we will represent for publication later.

2. Auxiliary propositions

Let $a(t)$, $b(t)$ and $c(t)$ be real valued continuous function on $[t_0; +\infty)$. Consider the Riccati equation

$$x' + a(t)x^2 + b(t)x + c(t) = 0, \quad t \geq t_0. \quad (2.1)$$

In this paragraph we study some important properties of global solutions (existing on $[t_1; +\infty)$ for some $t_1 \geq t_0$) of this equation which will be used further for the study of asymptotic properties of solutions of the system (1.1). Along with Eq. (2.1) consider the system of equations

$$\begin{cases} \phi' = a(t)\psi; \\ \psi' = -c(t)\phi - b(t)\psi, \end{cases} \quad t \geq t_0. \quad (2.2)$$

The solutions $x(t)$ of Eq. (2.1), existing on some interval $[t_1; t_2)$ ($t_0 \leq t_1 < t_2 \leq +\infty$), are connected with the solutions $(\phi(t), \psi(t))$ of the system (2.2) by the equalities (see [9], pp. 153 – 154)

$$\phi(t) = \phi(t_1) \exp\left\{\int_{t_1}^t a(\tau)x(\tau)d\tau\right\}, \quad \phi(t_1) \neq 0, \quad \psi(t) = x(t)\phi(t). \quad (2.3)$$

In this paragraph we will take that all solutions of equations and systems of equations are real valued. For brevity we introduce the denotations:

$$J_u(t_1; t) \equiv \exp\left\{\int_{t_1}^t a(\tau)u(\tau)d\tau\right\}, \quad J_u(t) \equiv J_u(t_0; t),$$

$$I_{u,v}^+(t_1; t) \equiv \int_{t_1}^t u(\tau)J_{-v}(t_1; \tau)d\tau, \quad I_{u,v}^+(t_1) \equiv \int_{t_1}^{+\infty} u(\tau)J_{-v}(t_1; \tau)d\tau,$$

$$I_{u,v}^-(t_1; t) \equiv \int_{t_1}^t J_{-u}(\tau; t)v(\tau)d\tau, \quad t_1, t \geq t_0,$$

where $u(t)$ and $v(t)$ be arbitrary continuous functions on $[t_0; +\infty)$. Rewrite Eq. (2.1) in the form:

$$x' + h_x(t)x + b(t) = 0, \quad t \geq t_0,$$

where $h_x(t) \equiv a(t)x + b(t)$, $t \geq t_0$. The by virtue of the Cauchy formula Eq. (2.1) is equivalent to the following integral equation

$$x = J_{-h_x}(t_1; t) \left[x(t_1) - \int_{t_1}^t J_b(t; \tau) c(\tau) \phi_x(t_1; \tau) d\tau \right], \quad t \geq t_0, \quad (2.4)$$

where $\phi_x(t_1; t) \equiv \exp \left\{ \int_{t_1}^t a(\tau) x(\tau) d\tau \right\}$, $t_1, t \geq t_0$. Let $a_1(t)$, $b_1(t)$ and $c_1(t)$ be real valued continuous function on $[t_0; +\infty)$. Along with Eq. (2.1) consider the equation

$$x' + a_1(t)x^2 + b_1(t)x + c_1(t) = 0, \quad t \geq t_0, \quad (2.5)$$

and differential inequalities

$$\eta' + a(t)\eta^2 + b(t)\eta + c(t) \geq 0, \quad t \geq t_0. \quad (2.6)$$

Note that for $a(t) \geq 0$, $t \geq t_0$ each solution of the linear equation

$$\eta' + b(t)\eta + c(t) = 0, \quad t \geq t_0,$$

is a solution of (2.6). Therefore for each initial condition $\eta_{(0)}$ inequality (2.6) has a solution $\eta_0(t)$ on $[t_0; +\infty)$ with $\eta_0(t_0) = \eta_{(0)}$.

Theorem 2.1. *Let Eq. (2.5) has a solution $x_1(t)$ on $[t_0; \tau_0)$ ($\tau_0 \leq +\infty$) and let the following condition be satisfied:*

$$\begin{aligned} a(t) \geq 0, \quad & \int_{t_0}^t \exp \left\{ \int_{t_0}^{\xi} [a(\xi)(\eta_0(\xi) + x_1(\xi)) + b(\xi)] d\xi \right\} \times \\ & \times [(a_1(\tau) - a(\tau))x_1^2(\tau) + (b_1(\tau) - b(\tau))x_1(\tau) + c_1(\tau) - c(\tau)] d\tau \geq 0, \\ & t \in [t_0; \tau_0), \end{aligned}$$

where $\eta_0(t)$ is a solution of (2.6) on $[t_0; \tau_0)$ with $\eta_0(t_0) \geq x_1(t_0)$. Then for each $x_{(0)} \geq x_1(t_0)$ Eq. (2.1) has a solution $x_0(t)$ on $[t_0; \tau_0)$, and $x_0(t) \geq x_1(t)$, $t \in [t_0; \tau_0)$.

See proof in [10]. Let $t_1 \geq t_0$.

Definition 2.1. *A solution of Eq. (2.1) is called t_1 -regular, if it exists on $[t_1; +\infty)$. Eq. (2.1) is called regular if it has a t_1 -regular solution for some $t_1 \geq t_0$.*

Definition 2.2. A t_1 -regular solution $x(t)$ of Eq. (2.1) is called t_1 -normal, if there exists a neighborhood $U_x(t_1)$ of the point $x(t_1)$ such that each solution $\tilde{x}(t)$ of Eq. (2.1) with $\tilde{x}(t_1) \in U_x(t_1)$ is t_1 -regular. Otherwise $x(t)$ is called t_1 -extremal.

Remark 2.1. From the results of work [11] it follows that for some $t_1 \geq t_0$ the regular equation (2.1) can have: the unique t_1 -regular solution (then it is t_1 -extremal); no t_1 -extremal solution (then its all t_1 -regular solutions are t_1 -normal); the unique t_1 -extremal solution (and all other t_1 -regular solutions are t_1 -normal); two t_1 -extremal solutions (all other t_1 -regular solutions are t_1 -normal).

In what follow we will assume that the functions $a(t)$ and $c(t)$ have unbounded supports (the case when one of these functions has a bounded support is trivial). For arbitrary continuous function $u(t)$ on $[t_0; +\infty)$ denote:

$$\mu_u(t_1; t) \equiv \int_{t_1}^t a(\tau) \exp \left\{ - \int_{t_1}^{\tau} [2a(\xi)u(\xi) + b(\xi)] d\xi \right\} d\tau,$$

$$\nu_u(t) \equiv \int_t^{+\infty} a(\tau) \exp \left\{ - \int_t^{\tau} [2a(\xi)u(\xi) + b(\xi)] d\xi \right\} d\tau, \quad t_1, t \geq t_0.$$

Theorem 2.2. Let for some t_1 -regular solution $x_0(t)$ of Eq. (2.1) the integral $\nu_{x_0}(t_1)$ be convergent. Then the following assertions are valid.

A) For each $t \geq t_1$ and for all t_1 -normal solutions $x(t)$ of Eq. (2.1) and only for them the integrals $\nu_x(t)$ converge.

B) In order that Eq. (2.1) have t_1 -extremal solution it is necessary and sufficient that

$$\nu_{x_0}(t) \neq 0, \quad t \geq t_1.$$

Under this condition the unique t_1 -extremal solution $x_*(t)$ is defined by formula

$$x_*(t) = x_0(t) - \frac{1}{\nu_{x_0}(t)}, \quad t \geq t_1, \quad (2.7)$$

and

$$\nu_{x_*}(t) = +\infty, \quad t \geq t_1, \quad \text{or} \quad \nu_{x_*}(t) = -\infty, \quad t \geq t_1, \quad (2.8)$$

$$\int_t^{+\infty} a(\tau)[x_1(\tau) - x_2(\tau)]d\tau = \ln \left[\frac{x_*(t) - x_1(t)}{x_*(t) - x_2(t)} \right], \quad t \geq t_1, \quad (2.9)$$

$$\int_{t_1}^{+\infty} a(\tau)[x_1(\tau) - x_2(\tau)]d\tau = -\infty. \quad (2.10)$$

Proof. All assertions of this theorem except (2.8) and (2.10), are proved in [11]. Let us prove (2.8). We will use the equalities (see [11]):

$$\mu_{x_*}(t_2; t) = \frac{\mu_{x_0}(t_2; t)}{1 + \lambda_*(t_2)\mu_{x_0}(t_2; t)}, \quad (2.11)$$

$$x_1(t) = x_2(t) + \frac{\lambda_{12}(t_2) \exp \left\{ - \int_{t_1}^t [2a(\tau)x_2(\tau) + b(\tau)]d\tau \right\}}{1 + \lambda_{12}(t_2)\mu_{x_2}(t_2; t)}, \quad t \geq t_2 \geq t_1, \quad (2.12)$$

where $\lambda_*(t_2) \equiv x_*(t_2) - x_0(t_2)$, $\lambda_{12}(t_2) \equiv x_1(t_2) - x_2(t_2)$, $t \geq t_2 \geq t_1$, $x_1(t)$ and $x_2(t)$ be arbitrary t_1 -regular solutions of Eq. (2.1). From (2.12) it follows that $\mu_{x_0}(t_2; t)$ is bounded by t on $[t_2; +\infty)$. Then since obviously

$$\nu_{x_0}(t_2) = \lim_{t \rightarrow +\infty} \mu_{x_0}(t_2; t) \neq 0, \quad t_2 \geq t_1, \quad (2.13)$$

necessarily

$$\lim_{t \rightarrow +\infty} [1 + \lambda_*(t_2)\mu_{x_0}(t_2; t)] = 0, \quad t_2 \geq t_1. \quad (2.14)$$

From here from (2.11) and (2.13) it follows (2.8). Let us prove (2.10). By (2.12) we have:

$$x_*(t) - x_0(t) = \frac{\lambda_*(t_1) \exp \left\{ - \int_{t_1}^t [2a(\tau)x_0(\tau) + b(\tau)]d\tau \right\}}{1 + \lambda_*(t_1)\mu_{x_0}(t_1; t)}, \quad t \geq t_1.$$

Multiplying both sides of this equality on $a(t)$ and integrating from t_1 to t we will get

$$\int_{t_1}^t a(\tau)[x_*(\tau) - x_0(\tau)]d\tau = \ln [1 + \lambda_*(t_1)\mu_{x_0}(t_1; t)], \quad t \geq t_1.$$

Passing to the limit in this equality when $t \rightarrow +\infty$ and taking into account (2.14) we will come to (2.10). The theorem is proved.

Corollary 2.1. *If for some t_1 -regular solution $x_*(t)$ the equality $\nu_{x_*}(t_1) = \pm\infty$ is fulfilled, then $x_*(t)$ is the unique t_1 -extremal solution of Eq. (2.1), and Eq. (2.1) has t_1 -normal solutions, and for each $t \geq t_1$ and for all t_1 -normal solutions $x(t)$ of Eq. (2.1) the integrals $\nu_x(t)$ converge; for every normal solutions $x_0(t)$, $x_1(t)$, $x_2(t)$ of Eq. (2.1) and for $x_*(t)$ the correlations (2.7) - (2.10) are satisfied.*

Proof. Let $\nu_{x_*}(t_1) = +\infty$ (the proof in the case $\nu_{x_*}(t_1) = -\infty$ by analogy). Then

$$\mu_{x_*}(t_1; t) > 1, \quad t \geq T, \quad (2.15)$$

for some $T > t_1$. Let $\bar{\mu}_{x_*} \equiv \max_{t \in [t_0; T]} |\mu_{x_*}(t_1; t)|$, and let $x_0(t)$ be a solution to Eq. (2.1) with $x_*(t_1) < x_0(t_1) < x_*(t_1) + \frac{1}{\max\{1, \bar{\mu}_{x_*}\}}$. Then taking into account (2.15) we will have:

$$1 + \lambda_*(t_1)\mu_{x_*}(t_1; t) > 0, \quad t \geq t_1,$$

where $\lambda_*(t_1) \equiv x_0(t_1) - x_*(t_1) > 0$. Hence (see [11]) by (2.11) $x_0(t)$ is a t_1 -regular solution of Eq. (2.1). Show that the integrals $\nu_{x_*}(t)$ converge for all $t \geq t_1$ and $\nu_{x_*}(t) \neq 0$. We use the equality (see [11])

$$\mu_{x_0}(t_2; t) = \frac{\mu_{x_*}(t_2; t)}{1 + \lambda_*(t_2)\mu_{x_*}(t_2; t)}, \quad t \geq t_2 \geq t_1,$$

where $\lambda_*(t_2) \equiv x_0(t_2) - x_*(t_2) \neq 0$. For enough large values of $t > t_2$ we have $\mu_{x_*}(t_2; t) > 0$. Therefore,

$$\nu_{x_0}(t_2) = \lim_{t \rightarrow +\infty} \mu_{x_0}(t_2; t) = \lim_{t \rightarrow +\infty} \frac{1}{\lambda_*(t_2) + \frac{1}{\mu_{x_*}(t_2; t)}} = \frac{1}{\lambda_*(t_2)} \neq 0.$$

So for $x_0(t)$ all conditions of Theorem 2.2 are fulfilled. Therefore Eq. (2.1) has t_1 -normal solutions and for every t_1 -normal solutions $x(t)$ of Eq. (2.1) and for all $t \geq t_1$ the integrals $\nu_x(t)$ converge; also for every t_1 -normal solutions $x_0(t)$, $x_1(t)$, $x_2(t)$ of Eq. (2.1) and for $x_*(t)$ the correlations (2.7) - (2.10) are satisfied. The corollary is proved.

Denote by $reg(t_1)$ the set of values $x(0) \in R$, for which the solution $x(t)$ of Eq. (2.1) with $x(t_1) = x(0)$ is t_1 -regular.

Lemma 2.1. Let $a(t) \geq 0$, $t \geq t_0$, and let Eq. (2.1) has t_1 -regular solution. Then it has the unique t_1 -extremal solution $x_*(t)$, and $\text{reg}(t_1) = [x_*(t_1); +\infty)$.

See proof in [2].

Lemma 2.2. let $a(t) \geq 0$, $t \geq t_0$; $t_0 \leq t_1 < t_2$, and let $(t_1; t_2)$ be the maximal existence interval for the solution $x(t)$ of Eq. (2.1). Then $\lim_{t \rightarrow t_1+0} x(t) = +\infty$.

See proof in [10].

Lemma 2.3. Let $a(t) \geq 0$, $t \geq t_0$, $x_0(t)$ be a t_0 -normal solution of Eq. (2.1), $x_0(t) \neq 0$, $t \geq t_0$. Then for its unique t_0 -extremal solution $x_*(t)$ the equality

$$\int_{t_0}^t a(\tau)x_*(\tau)d\tau = -\ln \nu_{x_0}(t_0) + \ln \left[\exp \left\{ \int_{t_0}^t a(\xi)x_0(\xi)d\xi \right\} \times \right. \\ \left. \times \int_t^{+\infty} \frac{a(s)x_0(s)}{x_0(t_0)} \exp \left\{ \int_{t_0}^s \left[\frac{c(\xi)}{x_0(\xi)} - a(\xi)x_0(\xi) \right] d\xi \right\} ds \right], \quad t \geq t_0. \quad (2.16)$$

holds.

Proof. By Lemma 2.1 Eq. (2.1) has a t_0 -normal solution $x_0(t)$. Then since $a(t) \geq 0$, $t \geq t_0$ and has unbounded support, the integral $\nu_{x_0}(t)$ converges for all $t \geq t_0$ and $\nu_{x_0}(t) \neq 0$, $t \geq t_0$. By virtue of Theorem 2.2 from here it follows that Eq. (2.1) has the unique t_0 -extremal solution $x_*(t)$, satisfying the equality $x_*(t) = x_0(t) - \frac{1}{\nu_{x_0}(t)}$, $t \geq t_0$. From here it follows:

$$\int_{t_0}^t a(\tau)x_*(\tau)d\tau = \int_{t_0}^t a(\tau)x_0(\tau)d\tau - \int_{t_0}^t \frac{a(\tau)}{\nu_{x_0}(\tau)}d\tau = \\ = \ln \left[\exp \left\{ \int_{t_0}^t a(\tau)x_0(\tau)d\tau \right\} \right] - \ln \nu_{x_0}(t_0) + \\ + \int_{t_0}^t d \left(\ln \left[\int_{\tau}^{+\infty} a(s) \exp \left\{ - \int_{t_0}^s [2a(\xi)x_0(\xi) + b(\xi)] d\xi \right\} ds \right] \right), \quad t \geq t_0. \quad (2.17)$$

On the strength of (2.1) from the condition $x_0(t) \neq 0$, $t \geq t_0$, it follows:

$$2a(\xi)x_0(\xi) + b(\xi) = -\frac{x'_0(\xi) - a(\xi)x_0^2(\xi) + c(\xi)}{x_0(\xi)}, \quad \xi \geq t_0.$$

From here and from (2.17) it follows (2.16). The lemma is proved.

Lemma 2.4. *Let $a(t) \geq 0$, $c(t) \geq 0$, $t \geq t_0$, $I_{a,b}^+(t_0) = +\infty$ and let Eq. (2.1) has a solution on $[t_1; +\infty)$ for some $t_1 \geq t_0$. Then Eq. (2.1) has a positive solution on $[t_1; +\infty)$.*

See proof in [5].

Theorem 2.3. *Let $a(t) \geq 0$, $c(t) \leq 0$, $t \geq t_0$. Then the following assertions are valid.*

P). For each $x_{(0)} \geq \frac{-1}{I_{a,b}^+(t_0)}$ (for $I_{a,b}^+(t_0) = +\infty$ we take that $\frac{1}{I_{a,b}^+(t_0)} = 0$) Eq. (2.1) has a t_0 -regular solution $x_0(t)$ with $x_0(t_0) = x_{(0)}$, and

$$\frac{x_{(0)}J_{-b}(t)}{1 + x_{(0)}I_{a,b}^+(t_0;t)} \leq x_0(t) \leq x_{(0)}J_{-b}(t) - I_{a,b}^-(t_0;t), \quad t \geq t_0, \quad (2.18)$$

moreover if $x_{(0)} = 0$, then there exists $t_1 \geq t_0$ such that $x_0(t) = 0$, $t \in [t_0; t_1]$, $x_0(t) > 0$, $t > t_1$. If $x_{(0)} > 0$ then $x_0(t) > 0$, $t \geq t_0$.

IP). The unique t_0 -extremal solution $x_(t)$ of Eq. (2.1) is negative.*

III^o). If $I_{a,b}^+(t_0) = +\infty$ or $I_{c,-b}^+(t_0) = -\infty$, then for each solution $x(t)$ of Eq. (2.1) with $x(t_0) \in (x_(t_0); 0)$ there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) < 0$, $t \in [t_0; t_1]$, $x(t) = 0$, $t \in [t_1; t_2]$ and $x(t) > 0$, $t > t_2$.*

IV^o). If $I_{a,b}^+(t_0) = +\infty$, then for each t_0 -normal solution $x_N(t)$ of Eq. (2.1) the equality $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = +\infty$. is fulfilled.

V^o). If $I_{c,-b}^+(t_0) = -\infty$, then $\int_{t_0}^{+\infty} a(\tau)x_(\tau)d\tau = -\infty$, where $x_*(t)$ is the unique t_0 -extremal solution of Eq. (2.1).*

VI^o). If $I_{a,b}^+(t_0) < +\infty$ and $I_{-c,-b}^+(t_0) < +\infty$, then Eq. (2.1) has a negative t_0 -normal solution $x_N^-(t)$ such that for each solution $x(t)$ of Eq. (2.1) with $x_0(t_0) \in (x_N^-(t_0); 0)$ there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) < 0$, $t \in [t_0; t_1]$, $x(t) = 0$, $t \in [t_1; t_2]$, $x(t) > 0$, $t > t_2$.

VII^o). If

$$I_{a,b}^+(t_0) = +\infty, \quad \int_{t_0}^{+\infty} |c(\tau)| I_{-b,a}^-(t_0; \tau) d\tau < +\infty, \quad (2.19)$$

then $\int_{t_0}^{+\infty} a(\tau) x_*(\tau) d\tau > -\infty$, $\int_{t_0}^{+\infty} c(\tau) u_*(\tau) d\tau = +\infty$, where $u_*(t)$ is the unique t_0 -extremal solution of the equation

$$u' - c(t)u^2 - b(t)u - a(t) = 0, \quad t \geq t_0.$$

Proof. Set $a_1(t) = a(t)$, $b_1(t) = b(t)$, $t \geq t_0$, $c_1(t) \equiv 0$. Then for each $x_{(0)} \geq \frac{-1}{I_{a,b}^+(t_0)} \stackrel{\text{def}}{=} A$ the function $x_1(t) \equiv \frac{x_{(0)} J_{-b}(t)}{1 + x_{(0)} I_{a,b}^+(t_0; t)}$ is a t_0 -regular solution of Eq. (2.5), and the conditions of Theorem 2.1 are fulfilled. Therefore for each $x_{(0)} \geq A$ Eq. (2.1) has a t_0 -regular solution $x_0(t)$ with $x_0(t_0) = x_{(0)}$ and the first of conditions of inequalities (2.18) is satisfied. Set $a_1(t) = a(t)$, $b_1(t) = b(t)$, $c_1(t) = c(t)$, $t \geq t_0$. Then by already proven Eq. (2.5) will have t_0 -regular solutions, coinciding with the t_0 -regular solutions of Eq. (2.1). In the Eq. (2.1) set: $a(t) \equiv 0$. Then $x_2(t) \equiv x_{(0)} J_{-b}(t_0; t) - I_{b,c}^-(t_0; t)$ is a t_0 -regular solution of Eq. (2.1). Obviously in this case the conditions of Theorem 2.1. are satisfied. Therefore the second of the inequalities (2.18) is fulfilled. Let $x_{(0)} = 0$. Then since $c(t)$ has unbounded support by virtue of (2.17) from the inequality $c(t) \leq 0$, $t \geq t_0$, it follows existence of $t_1 > t_0$ such that $x(t) = 0$, $t \in [t_0; t_1]$ and $x(t) > 0$, $t > t_1$. The assertion I^o is proved. Prove II^o. Let $x_0(t)$ be a solution of Eq. (2.1) with $x_0(t_0) > 0$. By virtue of I^o $x_0(t)$ is t_0 -normal and positive. Therefore from Theorem 2.2 it follows that for each $t \geq t_0$ the integral $\nu_{x_0}(t)$ converges. Obviously $\nu_{x_0}(t) > 0$, $t \geq t_0$. Then by virtue of the same Theorem 2.2 $x_*(t) \equiv x_0(t) - \frac{1}{\nu_{x_0}(t)}$ is the unique t_0 -extremal solution of Eq. (2.1). Show that $x_*(t) \leq 0$, $t \geq t_0$. By virtue of the first of inequalities (2.18) we have:

$$\frac{x_0(t) J_{-b}(t; s)}{1 + x_0(t) I_{a,b}(t; s)} \leq x_0(s), \quad t_0 \leq t \leq s.$$

Multiplying both sides of this inequality on $a(s)$ and integrating by s from t to τ . we will get: $\ln[1 + x_0(t) I_{a,b}(t; \tau)] \leq \int_{t_0}^t a(s) x_0(s) ds$, $t_0 \leq t \leq s$.

Then

$$\begin{aligned} \nu_{x_0}(t) &\leq \int_t^{+\infty} \frac{a(\tau)J_{-b}(t;\tau)d\tau}{1+x_0(t)I_{a,b}^+(t;\tau)} = -\frac{1}{x_0(t)} \int_t^{+\infty} d\left(\frac{1}{1+x_0(t)I_{a,b}^+(t;\tau)}\right) = \\ &= \frac{1}{x_0(t)} \left[1 - \frac{1}{1+x_0(t)I_{a,b}^+(t;+\infty)}\right] \leq \frac{1}{x_0(t)}, \quad t \geq t_0. \end{aligned}$$

From here it follows that $x_*(t) \leq 0, t \geq t_0$. Show that

$$x_*(t) < 0, \quad t \geq t_0. \tag{2.20}$$

Suppose that it is not true. Then since $x_*(t) \leq 0, t \geq t_0$, there exists $t_1 \geq t_0$ such that $x_*(t_1) = 0$. By the first of the inequalities (2.18) from here it follows that $x_*(t) \geq 0, t \geq t_1$. Hence $x_*(t) \equiv 0$ on $[t_1; +\infty)$, which is impossible (since on $[t_1; +\infty)$ $c(t) \not\equiv 0$.) The obtained contradiction proves (2.20), and therefore the assertion II°. Prove III°. Let $x_0(t)$ and $x_1(t)$ be solutions of Eq. (2.1) with the initial conditions $x_0(t_0) > 0, x_1(t_0) \in (x_*(t_0); 0)$. By virtue of Lemma 2.1 $x_0(t)$ and $x_1(t)$ are t_0 -normal. Therefore by (2.8) we have

$$\int_{t_0}^{+\infty} a(\tau)[x_0(\tau) - x_1(\tau)]d\tau < +\infty. \tag{2.21}$$

Let $I_{a,b}^+(t_0) = +\infty$. Then from the first of inequalities (2.18) it follow:

$$\int_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau \geq \ln[1+x_0(t_1)I_{a,b}^+(t_0)] = +\infty, \tag{2.22}$$

Show that there exists $\tilde{t}_1 \geq t_0$ such that $x_1(\tilde{t}_1) = 0$. Suppose that it is not true. Then $x_1(t) < 0, t \geq t_0$. Taking into account (2.18) from here we will get:

$$\int_{t_0}^{+\infty} a(\tau)(x_0(\tau) - x_1(\tau))d\tau \geq \int_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau = +\infty, \tag{2.23}$$

which contradicts (2.21). The obtained contradiction shows that $x_1(\tilde{t}_1) = 0$ for some $\tilde{t}_1 > t_0$. Since $c(t)$ has unbounded support by virtue of (2.4) from here and from non positivity of $c(t)$ it follows that $x_1(t) < 0$, $t \in [t_0; t_1)$, $x_1(t) = 0$, $t \in [t_1; t_2]$ and $x_1(t) > 0$, $t > t_2$, for some $t_2 \geq t_1 > t_0$. Let $I_{c,-b}^+(t_0) = -\infty$. Consider the equation

$$u' - c(t)u^2 - b(t)u - a(t) = 0, \quad t \geq t_0. \quad (2.24)$$

By II $^\circ$ the unique t_0 -extremal solution $u_*(t)$ of this equation is negative. Therefore $\tilde{x}_*(t) \equiv \frac{1}{u_*(t)}$ is a t_0 -regular solution of Eq. (2.1). By already proven, from here and from the equality $I_{c,-b}^+(t_0) = -\infty$ it follows that each solution $u(t)$ of Eq. (2.24) with $u(t_0) \in (u_*(t_0); 0)$ vanishes on $[t_0; +\infty)$. Therefore each solution $x(t)$ of Eq. (2.1) with $x(t_0) < \tilde{x}(t_0)$ is not t_0 -regular. By virtue of Lemma 2.1 from here it follows that

$$u_*(t) = \frac{1}{x_*(t)}, \quad t \geq t_0. \quad (2.25)$$

Suppose that some solution $\tilde{x}(t)$ of Eq. (2.1) with $\tilde{x}(t_0) \in (x_*(t_0); 0)$ is negative. Then $\tilde{u}(t) \equiv \frac{1}{\tilde{x}(t)}$, $t \geq t_0$, is a t_0 -regular solution of Eq. (2.24), and $\tilde{u}(t_0) = \frac{1}{\tilde{x}(t_0)} < \frac{1}{x_*(t_0)}$. From here and from (2.25) it follows that $\tilde{u}(t_0) < u_*(t_0)$, which contradicts Lemma 2.1. The obtained contradiction shows that for each solution $x(t)$ of Eq. (2.1) with $x(t_0) \in (x_*(t_0); 0)$ there exists $t_1 = t_1(x) > t_0$ such that $x(t_1) = 0$, $x(t) < 0$, $t \in [t_0; t_1)$. Since $c(t)$ has unbounded support by (2.4) from here and from non negativity of $c(t)$ it follows that there exists $t_2 = t_2(x) \geq t_1$ such that $x(t) = 0$, $t \in [t_1; t_2]$, $x(t) > 0$, $t > t_2$. The assertion III $^\circ$ is proved. Prove IV $^\circ$. Let $x_+(t)$ be a solution of Eq. (2.1) with $x_+(t_0) = 1$. On the strength of Lemma 2.1 from the assertion I $^\circ$ it follows that $x_+(t)$ is t_0 -normal. By the first of the inequalities (2.18) we have $x_+(t) \geq J_{-b}(t)$, $t \geq t_0$. Let $I_{a,b}^+(t_0) = +\infty$. Then from the last inequality it follows that

$$\int_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau \geq I_{a,b}^+(t_0) = +\infty. \quad (2.26)$$

Let $x_N(t)$ be an arbitrary t_0 -normal solution of Eq. (2.1). By (2.9) we have: $\int_{t_0}^{+\infty} |x_+(\tau) - x_N(\tau)|d\tau < +\infty$.

From here and from (2.26) we will get:

$$\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int_{t_0}^{+\infty} a(\tau)(x_N(\tau) - x_+(\tau))d\tau + \int_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau = +\infty.$$

The assertion IV^o is proved. Prove V^o. Since on the strength of II^o $I_*(t) \equiv \int_{t_0}^t a(\tau)x_*(\tau)d\tau$ is a monotonically non increasing function on $[t_0; +\infty)$, from Lemma 2.3 it follows (after differentiation (2.10)):

$$\begin{aligned} a(t)x_0(t) \exp\left\{\int_{t_0}^t a(\xi)x_0(\xi)d\xi\right\} \cdot \\ \cdot \int_t^{+\infty} \frac{a(s)x_0(s)}{x_0(t_0)} \exp\left\{\int_{t_0}^s \left[\frac{c(\xi)}{x_0(\xi)} - a(\xi)x_0(\xi)\right]d\xi\right\}ds \leq \\ \leq \frac{a(t)x_0(t)}{x_0(t_0)} \exp\left\{\int_{t_0}^t \frac{c(\xi)}{x_0(\xi)}d\xi\right\}, \quad t \geq t_0. \end{aligned} \quad (2.27)$$

where $x_0(t)(> 0, t \geq t_0)$ is a t_0 -normal solution of Eq. (2.1) (by virtue of Lemma 2.1 from I^o it follows the existence of $x_0(t)$). Since $u_0(t) \equiv \frac{1}{x_0(t)}$ is a t_0 -normal solution of Eq. (2.24) and $I_{-c,-b}^+(t_0) = +\infty$, by IV^o we have:

$$\int_{t_0}^{+\infty} c(\tau)u_0(\tau)d\tau = \int_{t_0}^{+\infty} \frac{c(\tau)}{x_0(\tau)}d\tau = -\infty. \quad (2.28)$$

Since $a(t)$ has unbounded support there exists infinitely large sequence $t_0 < t_1 < \dots < t_m < \dots$ such that $a(t_m) > 0, m = 1, 2, \dots$. Then from (2.27) it follows

$$\begin{aligned} \exp\left\{\int_{t_0}^{t_m} a(\xi)x_0(\xi)d\xi\right\} \int_{t_m}^{+\infty} \frac{a(s)x_0(s)}{x_0(t_0)} \exp\left\{\int_{t_0}^s \left[\frac{c(\xi)}{x_0(\xi)} - a(\xi)x_0(\xi)\right]d\xi\right\}ds \leq \\ \leq \exp\left\{\int_{t_0}^{t_m} \frac{c(\xi)}{x_0(\xi)}d\xi\right\}, \end{aligned}$$

$m = 1, 2, \dots$. Due to Lemma 2.3 From here and from (2.28) it folloes that $I_*(t_m) \rightarrow -\infty$ for $m \rightarrow +\infty$. Hence, $\int_{t_0}^{+\infty} a(\tau)x(\tau)d\tau = -\infty$. The assertion V^o is proved. Prove VI^o. Show that Eq. (2.1) has a t_0 -normal negative solution. In Eq. (2.1) make the change: $x = J_{-b}(t_0; t)X$, $t \geq t_0$. We will come to the equation

$$X' + a(t)J_{-b}(t)X^2 + c(t)J_b(t) = 0, \quad t \geq t_0. \quad (2.29)$$

Due to conditions of VI^o chose $t_1 (> t_0)$ so large that

$$\left[\int_{t_1}^{+\infty} a(\tau)J_{-b}(\tau)d\tau \right]^{-1} > - \int_{t_1}^{+\infty} c(\tau)J_b(\tau)d\tau.$$

Then

$$- [I_{a,b}^+(t_0)]^{-1} < I_{c,-b}^+(t_0) < 0. \quad (2.30)$$

Let then $X_-(t)$ be a solution to Eq. (2.29) with

$$X_-(t_1) \in (- [I_{a,b}^+(t_0)]^{-1}; I_{c,-b}^+(t_0)). \quad (2.31)$$

By (2.18) the inequalities

$$\frac{X_-(t_1)}{1 + X_-(t_1)I_{a,b}^+(t_1; t)} \leq X_-(t) \leq X_-(t_1) - I_{c,-b}^+(t_1; t), \quad t \geq t_1, \quad (2.32)$$

are fulfilled.

From here and from (2.30) and (2.31) it follows that $X_-(t)$ is defined on $[t_1; +\infty)$ negative t_1 -normal solution of Eq. (2.29). Then $x_-(t) \equiv X_-(t)J_{-b}(t_1; t)$ is defined on $[t_1; +\infty)$ negative t_1 -normal solution to Eq. (2.1). Show that $x_-(t)$ is continuable on $[t_0; +\infty)$ as a solution to Eq. (2.1). Suppose $x_-(t)$ can not be continued on $[t_0; +\infty)$ as a solution of Eq. (2.1). Let then $(t_2; +\infty)$ be the maximum existence interval for $x_-(t)$, where $t_2 \geq t_0$. By Lemma 2.2 there exists $t_3 > t_2$ such that $x_-(t_3) > 0$. On the strength of the first of the inequalities (2.18) from here it follows that $x_-(t) > 0$, $t \geq t_3$. The obtained contradiction shows that $x_-(t)$ is continuable on $[t_0; +\infty)$. By virtue of the first of the inequalities (2.18) the supposition that $x_-(t_4) \geq 0$ for some $t_4 \geq t_0$ also leads to the contradiction.

So, $x_-(t) < 0$, $t \geq t_0$. Since $x_-(t)$ is t_1 -normal, by continuable dependence of solutions of Eq. (2.1) from their initial values, the solution $x_-(t)$ also is t_0 -normal. According to I° the solution $x_0(t)$ of Eq. (2.1) with $x_0(t_0) = 0$ starting with some $t_1 = t_1(x_0) \geq t_6$ becomes positive. Then by continuable dependence of solutions of Eq. (2.1) from their initial values, all initial values $x_{(0)}$, for which the solutions $x(t)$ of Eq. (2.1) with $x(t_0) = x_{(0)}$ eventually become positive, form an open set. From here from Lemma 2.1 and from the fact that $x_-(t)$ is negative it follows that there exists a negative t_0 -normal solution $X_N^-(t)$ of Eq. (2.1) such that each solution $x(t)$ of Eq. (2.1) with $x(t_0) > X_N^-(t_0)$ eventually becomes positive.

By (2.4) from here it follows that for each solution $x(t)$ of Eq. (2.1) with $x(t_0) \in (x_N^-(t_0); 0)$ there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) < 0$, $t \in [t_0; t_1)$, $x(t) = 0$, $t \in [t_1; t_2]$, $x(t) > 0$, $t > t_2$. By (2.8) $\int_{t_0}^{+\infty} a(\tau)[x_*(\tau) - x_N(\tau)]d\tau = -\infty$.

Then $\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau \leq \int_{t_0}^{+\infty} a(\tau)[x_*(\tau) - x_N(\tau)]d\tau = -\infty$. Let $x_+(t)$ is a solution to Eq. (2.1) with $x_+(t_0) = 1$. On the strength of Lemma 2.1 from I° it follows that $x_+(t)$ is t_0 -normal. Then by (2.9) we have:

$$0 < \int_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau \leq \int_{t_0}^{+\infty} a(\tau)[x_+(\tau) - x_N^-(\tau)]d\tau < +\infty. \quad (2.33)$$

Let $x_N(t)$ be an arbitrary t_0 -normal solution to Eq. (2.1). Then since $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int_{t_0}^{+\infty} [x_N(\tau) - x_+(\tau)]d\tau + \int_{t_0}^{+\infty} a(\tau)x_+(\tau)d\tau$ and by virtue of (2.9) $\int_{t_0}^{+\infty} a(\tau)|x_N(\tau) - x_+(\tau)|d\tau < +\infty$, from (2.33) it follows that the integral $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau$ converges. The assertion VI° is proved. Prove VII°. Due to the second of inequalities (2.18) taking into account

the inequality $c(t) \leq 0$, $t \geq t_0$ we have:

$$\int_{t_0}^t \frac{c(\tau)}{x_0(\tau)} d\tau \geq \frac{1}{x_0(t_0)} \int_{t_0}^t c(\tau) J_b(t) \left[1 + x_0(t_0) I_{a,b}^+(t_0; \tau) \right] d\tau, \quad t \geq t_0, \quad (2.34)$$

where $x_0(t)$ is a positive t_0 -normal solution of Eq. (2.1), existence of which follows from Lemma 2.1 and from I° . By virtue of Fubini's theorem from the second relation of (2.19) it follows that $I_{-b,-c}^+(t_0; +\infty) < +\infty$. From here and from the second relation of (2.19) and from (2.34) it follows that

$$\int_{t_0}^{+\infty} \frac{c(\tau)}{x_0(\tau)} d\tau > -\infty. \quad (3.35)$$

From the first relations of (2.18) and (2.19) it follows that

$$\int_{t_0}^{+\infty} a(\tau) x_0(\tau) d\tau = +\infty. \quad (2.36)$$

Let $g(t) \equiv \exp \left\{ - \int_{t_0}^t a(\xi) x_0(\xi) d\xi \right\}$. Obviously the inverse function $g^{-1}(t)$ of $g(t)$ exists on $\text{supp } a(t)$. Denote:

$$g_1(t) \equiv \begin{cases} g^{-1}(t), & t \in \text{supp } a(t); \\ t_0, & t \notin \text{supp } a(t), \quad t \geq t_0. \end{cases}$$

Then taking into account (2.36) the equality (2.16) can be rewritten in the form

$$\int_{t_0}^t a(\tau) x_*(\tau) d\tau = -\ln \nu_{x_0}(t_0) + \ln \left[\frac{\int_0^{g(t)} \exp \left\{ \int_{t_0}^{g^{-1}(\zeta)} \frac{c(\xi)}{x_0(\xi)} d\xi \right\} d\zeta}{x_0(t_0) g(t)} \right] \geq 1, \quad t \geq t_0.$$

From here and from (2.35) it follows that

$$\int_{t_0}^{+\infty} a(\tau) x_*(\tau) d\tau \geq -\ln[x_0(t_0) \nu_{x_0}(t_0)] + \int_{t_0}^{+\infty} \frac{c(\xi)}{x_0(\xi)} d\xi > -\infty.$$

By virtue of Lemma 2.1 $u_0(t) \equiv \frac{1}{x_0(t)}$ is a t_0 -normal solution of Eq. (2.24) (since for $x_1(t_0) > x_0(t_0)$ the function $u_1(t) \equiv \frac{1}{x_1(t)}$ is a t_0 -regular solution of Eq. (2.24), where $x_1(t)$ is a t_0 -regular solution to Eq. (2.1)). Then by (2.8) taking into account (2.35) we will have:

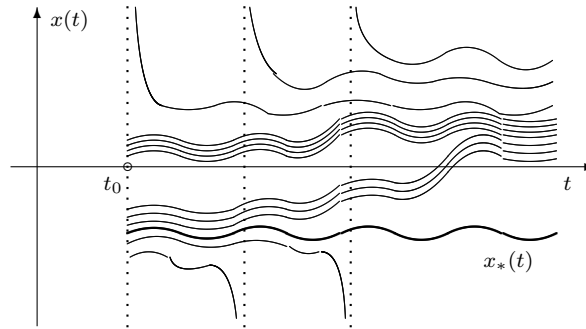
$$\int_{t_0}^{+\infty} c(\tau)u_*(\tau)d\tau = \int_{t_0}^{+\infty} c(\tau)(u_*(\tau) - u_0(\tau))d\tau + \int_{t_0}^{+\infty} \frac{c(\tau)}{x_0(\tau)}d\tau = +\infty.$$

The assertion VII° is proved. The theorem is proved.

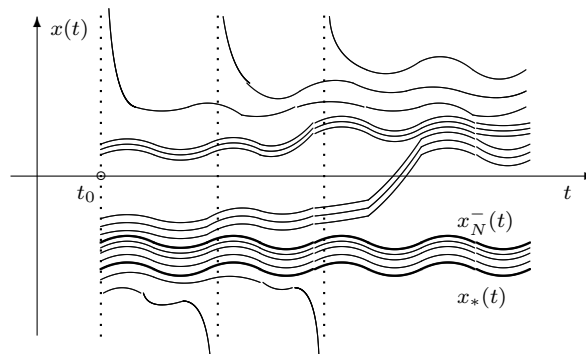
On the basis of Theorem 2.3 it can be make phase portrait of solutions of Eq. (2.1) in the case $a(t) \geq 0, c(t) \leq 0, t \geq t_0$, for the following two possible restrictions:

- a) $I_{a,b}^+(t_0) = +\infty$ or $I_{c,-b}^+(t_0) = -\infty$ (see pict. 1) ;
- b) $I_{a,b}^+(t_0) < +\infty$ and $I_{-c,-b}^+(t_0) < +\infty$ (see pict. 2).

In pict 1. we see only one negative global solution of Eq (2.1), meanwhile in pict. 2 we see whole slice of negative global solutions of Eq. (2.1).



pict.1. $I_{a,b}^+(t_0)=+\infty$ or $I_{-c,-b}^+(t_0)=+\infty$



pict.2. $I_{a,b}^+(t_0)<+\infty$ and $I_{-c,-b}^+(t_0)<+\infty$

Theorem 2.4. Let $a(t) \geq 0$, $c(t) \geq 0$, $t \geq t_0$, and let Eq. (2.1) has a t_0 -regular solution. Then the following assertions are valid.

*I** If $I_{a,b}^+(t_0) = +\infty$, then each t_0 -regular solution of Eq. (2.1) is positive

and for its each t_0 -normal solution $x_N(t)$ the equality $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau =$

$+\infty$ is fulfilled. Moreover if in addition $\int_{t_0}^{+\infty} c(\tau)I_{a,-b}^+(t_0;\tau)d\tau = +\infty$,

then also $\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = +\infty$, where $x_*(t)$ is the unique t_0 -extremal

solution of Eq. (2.1).

*II** If $I_{c,-b}^+(t_0) = +\infty$, then for each t_0 -regular solution $x(t)$ of Eq.

(2.1) with $x(t_0) > 0$ there exist $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such

that $x(t) > 0$, $t \in [t_0; t_1)$, $x(t) = 0$, $t \in [t_1; t_2]$, $x(t) < 0$, $t > t_2$,

and if $x(t_0) = 0 (< 0)$, there exists $t_1 = t_1(x) \geq t_0$ such that $x(t) = 0$,

$t \in [t_0; t_1]$, $x(t) < 0$, $t > t_1$ (then $x(t) < 0$, $t \geq t_0$), and

$\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty$. Moreover if in addition $\int_{t_0}^{+\infty} a(\tau)I_{b,c}^-(t_0;\tau)d\tau =$

$+\infty$, then for each t_0 -normal solution $x_N(t)$ of Eq. (2.1) the equality

$\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = -\infty$ is fulfilled.

*III** If $I_{a,b}^+(t_0) < +\infty$ and $I_{c,-b}^+(t_0) < +\infty$, then there exist $t_1 \geq t_0$ such

that $x_*(t) < 0$, $t \geq t_1$; the solutions $x(t)$ of Eq. (2.1) with

$x(t_1) \in (x_*(t_1); 0)$ are t_0 -regular and $x(t) < 0$, $t \geq t_1$; there exists a

t_1 -normal positive on $[t_1; +\infty)$ solution $x_N^+(t)$ of Eq. (2.1) such that

for each solution $x(t)$ of Eq. (2.1) with $x(t_1) \in (0; x_N^+(t_1))$ there exist

$t_3 = t_3(x) \geq t_2 = t_2(x) > t_1$ such that $x(t) > 0$, $t \in [t_1; t_2)$,

$x(t) = 0$, $t \in [t_2; t_3]$, $x(t) < 0$, $t > t_3$, and if $x(t_1) = 0$ ($x(t_1) < 0$), there

exists $t_2 = t_2(x) \geq t_1$ such that $x(t) = 0$, $t \in [t_1; t_2]$, $x(t) < 0$, $t > t_2$

(then $x(t) < 0$, $t \geq t_1$); for each t_0 -normal solution $x_N(t)$ of Eq. (2.1)

the integral $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau$ converges and $\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty$.

Proof. Let us prove *I**. Let $x_*(t)$ be the t_0 -extremal solution of Eq. (2.1). Show that $x_*(t) > 0$, $t \geq t_0$. Suppose for some $t_1 > t_0$ the inequality

$x_*(t_1) < 0$ is satisfied. Let then $x(t)$ be a solution to Eq. (2.1) with

$x(t_1) \in (x_*(t_1); 0)$. By virtue of Lemma 2.1 $x(t)$ is t_1 -normal. Since

$x(t_1) < 0$ and $c(t) \geq 0$, $t \geq t_0$, by (2.4) we have $x(t) < 0$, $t \geq t_1$.

From here it follows that

$$\nu_x(t_1) \geq I_{a,b}^+(t_1). \tag{2.37}$$

Let $I_{a,b}^+(t_0) = +\infty$. Then from the easily verifiable equality

$$I_{a,b}^+(t_0) = I_{a,b}^+(t_0; t) + J_b(t_1)I_{a,b}^+(t_1) \tag{2.38}$$

and from (2.37) it follows that $\nu_x(t_1) = +\infty$. But on the other hand since $x(t)$ is t_1 -normal by virtue of Theorem 2.2 we have $\nu_x(t_1), +\infty$. The obtained contradiction shows that $x_*(t) \geq 0, t \geq t_0$. Show that the equality $x_*(t) = 0$ impossible for all $t \geq t_0$. Suppose for some $t_2 \geq t_0$ the equality $x_*(t_2) = 0$ is satisfied. Then by (2.4) from the inequality $c(t) \geq 0, t \geq t_0$ it follows that $x_*(t) \leq 0, t \geq t_2$. Hence, $x_*(t) \equiv 0$ on $[t_2; +\infty)$, which is impossible (since on $[t_2; +\infty)$ we have $c(t) \not\equiv 0$). On the strength of Lemma 2.1 from here it follows that each t_0 -regular solution of Eq. (2.1) is positive. Let $x_N(t)$ be a t_0 -normal solution of Eq. (2.1).

Then since $x_*(t) > 0, t \geq t_0$, by (2.8) we have: $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int_{t_0}^{+\infty} a(\tau)[x_N(\tau) - x_*(\tau)]d\tau + \int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau \geq \int_{t_0}^{+\infty} a(\tau)[x_N(\tau) - x_*(\tau)]d\tau = +\infty$. Let

$$\int_{t_0}^{+\infty} c(\tau)I_{a,b}(t_0; \tau)d\tau = +\infty. \tag{2.39}$$

Suppose $\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau < +\infty$. Show that then

$$x_*(t) = \int_t^{+\infty} J_b(t; \tau)c(\tau)\phi_*(t; \tau)d\tau, \quad t \geq t_0, \tag{2.40}$$

where $\phi_*(t; \tau) \equiv \exp\left\{\int_t^\tau a(s)x_*(s)ds\right\}, \tau \geq t \geq t_0$. By (2.4) we have:

$$x_*(t) = J_{-h_*}(t_1; t) \left[x_*(t_1) - \int_{t_1}^t J_b(t_1; \tau)c(\tau)\phi_*(t; \tau)d\tau \right], \quad t \geq t_1 \geq t_0, \tag{2.41}$$

where $h_*(t) \equiv a(t)x_*(t) + b(t)$, $t \geq t_0$. From here and from the positivity of $x_*(t)$ it follows that

$$x_*(t) \geq \int_{t_1}^{+\infty} j_b(t_1; \tau) c(\tau) \phi_*(t_1; \tau) d\tau, \quad t_1 \geq t_0.$$

Show that the strict inequality

$$x_*(t) > \int_{t_1}^{+\infty} J_b(t_1; \tau) c(\tau) \phi_*(t_1; \tau) d\tau \quad (2.42)$$

impossible for all $t_1 \geq t_0$. Multiplying both sides of (2.41) on $a(t)\phi_*(t_1; t)$ and integration from t_1 to $+\infty$ we will get:

$$\begin{aligned} \exp \left\{ \int_{t_1}^{+\infty} a(\tau) x_*(\tau) d\tau \right\} &= \\ &= 1 + \int_{t_1}^{+\infty} J_{-b}(t_1; \tau) \left[x_*(t_1) - \int_{t_1}^{\tau} J_b(t_1; \tau) c(\tau) \phi_*(t_1; \tau) d\tau \right] d\tau \geq \\ &\geq 1 + I_{a,b}^+(t_1) \left[x_*(t_1) - \int_{t_1}^{+\infty} J_b(t_1; \tau) c(\tau) \phi_*(t_1; \tau) d\tau \right]. \end{aligned} \quad (2.43)$$

Suppose for some $t_1 \geq t_0$ the inequality (2.42) is satisfied. Then by (2.38) from the equality $I_{a,b}^+(t_0) = +\infty$ and from (2.43) it follows that

$\int_{t_1}^{+\infty} a(\tau) x_*(\tau) d\tau = +\infty$. The obtained contradiction proves (2.40).

Multiplying both sides of (2.40) on $a(t)\phi_*(t_0; t)$ and integrating from t_0 to $+\infty$ we will get:

$$\begin{aligned} \exp \left\{ \int_{t_0}^{+\infty} a(\tau) x_*(\tau) d\tau \right\} &= \\ &= 1 + \int_{t_0}^{+\infty} a(\tau) \phi_*(t_0; \tau) d\tau \int_t^{+\infty} J_b(t; \tau) c(\tau) \phi_*(t; \tau) d\tau \geq \\ &\geq 1 + \int_{t_0}^{+\infty} a(t) dt \int_t^{+\infty} J_b(t; \tau) c(\tau) d\tau. \end{aligned}$$

By virtue of Fubini's theorem from here and from (2.39) it follows that $\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = +\infty$. We came to the contradiction. The assertion I* is proved. Let us prove II*. Let $x(t)$ be a t_0 -regular solution of Eq. (2.1). If $x(t_0) < 0$, then by (2.4) from inequality $c(t) \geq 0, t \geq t_0$ it follows that $x(t) < 0, t \geq t_0$. Let $x(t_0) \geq 0$. Show that in this case impossible that

$$x(t) \geq 0, \quad t \geq t_0. \tag{2.44}$$

Suppose that this relation takes place. Then since $a(t) \geq 0, c(t) \geq 0, t \geq t_0$, we have

$$\int_{t_0}^t J_b(\tau)c(\tau) \exp\left\{\int_{t_0}^{\tau} a(s)x(s)ds\right\}d\tau \geq I_{c,-b}^+(t_0;t), \quad t \geq t_0.$$

By (2.4) from here and from equality $I_{c,-b}^+(t_0) = +\infty$ it follows that $x(t_1) < 0$ for some $t_1 > t_0$. We came to the contradiction. Hence by (2.4) if $x(t_0) > 0$, then there exists $t_2 = t_2(x) \geq t_1 = t_1(x) > t_0$ such that $x(t) > 0, t \in [t_0;t_1), x(t) = 0, t \in [t_1;t_2], x(t) < 0, t > t_2$, and if $x(t_0) = 0$, then there exists $t_1 = t_1(x) \geq t_0$ such that $x(t) = 0, t \in [t_0;t_1],$ and $x(t) < 0, t \geq t_1$. Let $x_0(t)$ be a t_0 -normal solution of Eq. (2.1) with $x_0(t_0) \geq 0$, and let $t_1 = t_1(x) \geq t_0$ such that $x_0(t_1) = 0, x_0(t) < 0, t > t_1$. Then since by (2.8) $\int_{t_1}^{+\infty} a(\tau)[x_*(\tau) - x_0(\tau)]d\tau = -\infty$, we have

$$\begin{aligned} & \int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = \\ & = \int_{t_0}^{t_1} a(\tau)x_*(\tau)d\tau + \int_{t_1}^{+\infty} a(\tau)[x_*(\tau) - x_0(\tau)]d\tau + \int_{t_1}^{+\infty} a(\tau)x_0(\tau)d\tau \leq \\ & \leq \int_{t_0}^t a(\tau)x_*(\tau)d\tau + \int_{t_1}^{+\infty} a(\tau)[x_*(\tau) - x_0(\tau)]d\tau = -\infty. \end{aligned}$$

Using Theorem 2.1 by analogy of the second of inequalities (2.18) can be obtained the estimation

$$x_0(t) \leq x_0(t_1)J_{-b}(t_1;t) - I_{b,c}^-(t_1;t) = -I_{b,c}^-(t_1;t), \quad t \geq t_1.$$

Then

$$\int_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau \leq \int_{t_0}^{t_1} a(\tau)x_0(\tau)d\tau - \int_{t_1}^{+\infty} a(\tau)I_{a,b}^-(t_1; \tau)d\tau. \quad (2.45)$$

Since $I_{b,c}^-(t_0; t) = I_{b,c}^-(t_0; t_1)J_{-b}(t_1; t) + I_{b,c}^-(t_1; t)$, we have

$$\begin{aligned} & \int_{t_0}^{+\infty} a(\tau)I_{b,c}^-(t_0; \tau)d\tau = \\ & = \int_{t_0}^{t_1} I_{b,c}^-(t_1; \tau)d\tau + I_{b,c}^-(t_0; t_1)I_{a,b}^+(t_1; +\infty) + \int_{t_1}^{+\infty} a(\tau)I_{b,c}^-(t_1; \tau)d\tau. \end{aligned} \quad (2.46)$$

Since $x_0(t) < 0$, $t > t_1$, by virtue of I^* we will get

$$I_{a,b}^+(t_1) < +\infty. \quad (2.47)$$

Let $\int_{t_0}^{+\infty} a(\tau)I_{b,c}^-(t_0; \tau)d\tau = +\infty$. Then from (2.45) - (2.47) it follows that

$$\int_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau = -\infty. \quad (2.48)$$

Let $x_N(t)$ be an arbitrary t_0 -normal solution of Eq. (2.1). Then since

by (2.9) $\int_{t_0}^{+\infty} a(\tau)|x_N(\tau) - x_0(\tau)|d\tau < +\infty$, taking into account (2.48) we

will have: $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau = \int_{t_0}^{+\infty} a(\tau)[x_N(\tau) - x_0(\tau)]d\tau + \int_{t_0}^{+\infty} a(\tau)x_0(\tau)d\tau = -\infty$. The assertion II^* is proved. Let us prove III^* . Show that

$$x_*(t_1) < 0 \quad (2.49)$$

for some $t_1 \geq t_0$. Suppose that it is not true. Then $x_*(t) \geq 0$, $t \geq t_0$ and therefore $\nu_{x_*}(t_0) \leq I_{a,b}^+(t_0) < +\infty$. But on the other hand by (2.7) we have $\nu_{x_*}(t_0) = +\infty$. The obtained contradiction proves (2.49). By (2.4) from (2.49) and from non negativity of $c(t)$ it follows that

$$x_*(t) < 0, \quad t \geq t_1. \quad (2.50)$$

Hence $v_*(t) \equiv \frac{1}{x_*(t)}$, $t \geq t_1$ is a t_1 -regular solution of the equation

$$v' + c(t)v^2 - b(t)v + a(t) = 0, \quad t \geq t_0. \quad (2.51)$$

Let $I_{c,-b}^+(t_0) < +\infty$. By (2.38) from here it follows that $I_{c,-b}^+(t_1) < +\infty$. Then by already proven $v_*(t) < 0$, $t \geq t_2$ for some $t_2 \geq t_1$, where $v_*(t)$ is the t_1 -extremal solution of Eq. (2.51). From here it follows that $x_1(t) \equiv -\frac{1}{v_*(t)}$ is an positive solution of Eq. (2.1) defined on $[t_2; +\infty)$. Since according to (2.50) $x_*(t_2) < 0$, by virtue of Lemma 2.1 $x_1(t)$ is t_2 -normal. By virtue of continuously dependence of solutions of Eq. (2.1) from their initial values from here from (2.4) and (2.50) it follows that there exists t_1 -normal positive solution $x_N^+(t)$ of Eq. (2.1) on $[t_2; +\infty)$ having the property: for each solution $x(t)$ of Eq. (2.1) with $x(t_2) \in (0; x_N^+(t_2))$ there exists $t_4 = t_4(x) \geq t_3 = t_3(x) > t_2$ such that $x(t) > 0$, $t \in [t_2; t_3)$, $x(t) = 0$, $t \in [t_3; t_4]$, $x(t) < 0$, $t > t_4$; if $x(t_2) = 0$, then there exists $t_3 = t_3(x) \geq t_2$ such that $x(t) = 0$, $t \in [t_2; t_3]$, $x(t) < 0$, $t > t_3$; if $x(t_2) \in (x_*(t_2); 0)$, then $x(t) < 0$, $t \geq t_2$. Let $x_-(t)$ be a solution of Eq. (2.1) with $x_-(t_2) \in (x_*(t_2); 0)$. Then by virtue of Lemma 2.1 $x_-(t)$ is t_2 -normal and, as it was already proved, $x_-(t) < 0$, $t \geq t_2$. Therefore taking into account (2.9) we will have:

$$0 < \int_{t_0}^{+\infty} a(\tau)x_N^+(\tau)d\tau \leq \int_{t_0}^{t_2} a(\tau)x_N^+(\tau)d\tau + \int_{t_2}^{+\infty} a(\tau)[x_N^+(\tau) - x_-(\tau)]d\tau < +\infty. \quad (2.52)$$

Let $x_N(t)$ be an arbitrary t_0 -normal solution of Eq. (2.1). Then

$$\begin{aligned} \int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau &= \int_{t_0}^{t_2} a(\tau)x_N(\tau)d\tau + \int_{t_2}^{+\infty} a(\tau)[x_N(\tau) - x_N^+(\tau)]d\tau + \\ &+ \int_{t_2}^{+\infty} a(\tau)x_N^+(\tau)d\tau. \end{aligned}$$

Since by (2.9) $\int_{t_2}^{+\infty} a(\tau)|x_N(\tau) - x_N^+(\tau)|d\tau < +\infty$, from the last equality and from (2.52) it follows convergence of the integral $\int_{t_0}^{+\infty} a(\tau)x_N(\tau)d\tau$. Since $x_N^+(t) > 0$, $t \geq t_2$; $\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = \int_{t_0}^{t_2} a(\tau)x_*(\tau)d\tau + \int_{t_2}^{+\infty} a(\tau)[x_*(\tau) - x_N^+(\tau)]d\tau + \int_{t_2}^{+\infty} a(\tau)x_N^+(\tau)d\tau$. and by (2.8)

$$\int_{t_2}^{+\infty} a(\tau)[x_*(\tau) - x_N^+(\tau)]d\tau = -\infty,$$

taking into account (2.52) we will have:

$$\int_{t_0}^{+\infty} a(\tau)x_*(\tau)d\tau = -\infty.$$

The theorem is proved.

Remark 2.2. Existence criteria of t_1 -regular solutions of Eq. (2.1) are proved in [5] and [12].

Remark 2.3. If $a(t) > 0, t \geq t_0$, then existence of t_1 -regular solutions of Eq. (2.1) is equivalent to the non oscillation of the equation

$$\left(\frac{\phi'}{a(t)}\right)' - a(t)b(t)\phi' - c(t)\phi = 0, \quad t \geq t_0.$$

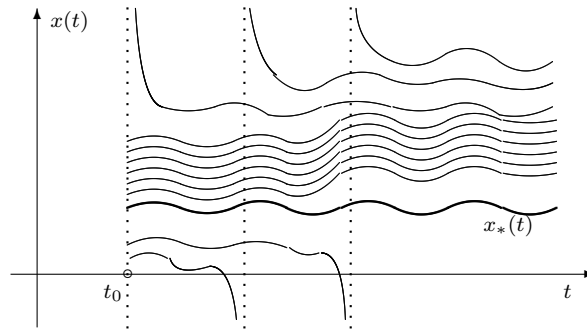
Non oscillatory criteria for the last equation is proved in [13].

Corollary 2.2. Let $a(t) \geq 0, c(t) \geq 0, t \geq t_0, I_{a,b}^+(t_0) = I_{c,-b}(t_0) = +\infty$. Then Eq. (2.1) has no t_1 -regular solutions for all $t_1 \geq t_0$.

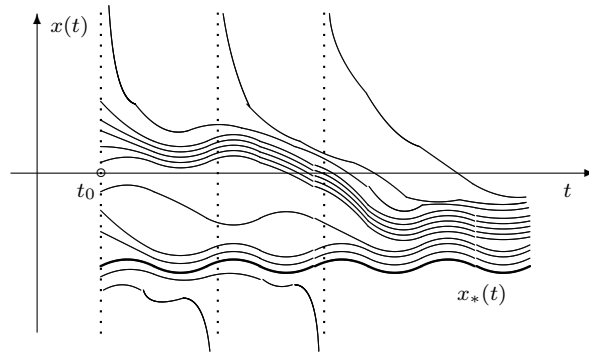
Proof. Suppose that for some $t_1 \geq t_0$ Eq. (2.1) has t_1 -regular solution $x(t)$. Then by virtue of Theorem 2.4 from the equality $I_{a,b}^+(t_1) = I_{a,b}^+(t_0)$ (see (2.38)) it follows that $x(t) > 0, t \geq t_1$. Therefore $v(t) \equiv -\frac{1}{x(t)}, t \geq t_1$, is a negative t_1 -regular solution of Eq. (2.51). But on the other hand by virtue of Theorem 2.4 I* from the equality $I_{c,-b}^+(t_1) = I_{c,-b}^+(t_0) = +\infty$ it follows that $v(t) > 0, t \geq t_1$. We came to the contradiction. The corollary is proved.

On the basis of Theorem 2.1 and corollary 2.2 we can make the phase portrait of solutions of Eq. (2.1) if $a(t) \geq 0, c(t) \geq 0, t \geq t_0$ for the following four cases:

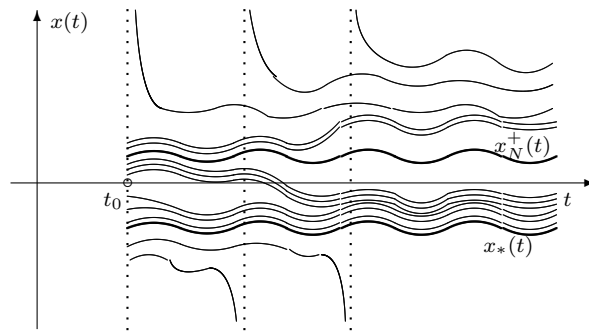
$\alpha)$ $I_{a,b}^+(t_0) = +\infty$; and Eq. (2.1) has a t_1 -regular solution for some $t_1 \geq t_0$ (see pict. 3);



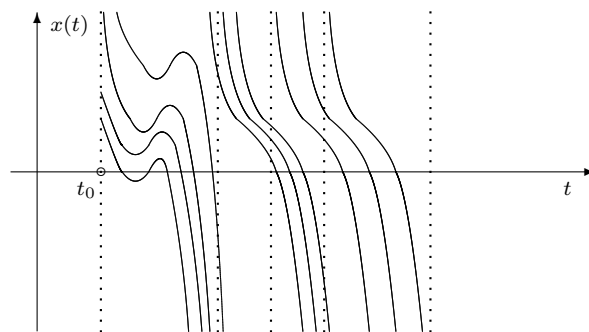
pict.3. $I_{a,b}^+(t_0) = +\infty$



pict.4. $I_{c,-b}^+(t_0) = +\infty$



pict.5. $I_{a,b}^+(t_0) < +\infty$ and $I_{c,-b}^+(t_0) < +\infty$



pict.6. $I_{a,b}^+(t_0) = I_{c,-b}^+(t_0) = +\infty$

β) $I_{c,-b}^+(t_0) = +\infty$; and Eq. (2.1) has a t_1 -regular solution for some $t_1 \geq t_0$ (see pict. 4);

γ) $I_{a,b}^+(t_0) < +\infty$, $I_{c,-b}^+(t_0) < +\infty$; and Eq. (2.1) has a t_1 -regular solution for some $t_1 \geq t_0$ (see pict. 5);

δ) $I_{a,b}^+(t_0) = I_{c,-b}^+(t_0) = +\infty$ (see pict. 6).

Let $a(t) > 0$, $t \geq t_0$, and let $x_0(t)$ is a solution of Eq. (2.1) with $x_0(t_0) = 0$. Then by virtue of Theorem 2.3. $x_0(t)$ is t_0 -normal and non negative. Obviously

$$x_0(t) + \int_{t_0}^t a(\tau) \left(x_0(\tau) + \frac{b(\tau)}{2a(\tau)} \right)^2 d\tau = \int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{4a(\tau)} d\tau, \quad t \geq t_0.$$

Then since $x_0(t) \geq 0$, $t \geq t_0$, we have

$$\int_{t_0}^t a(\tau) \left(x_0(\tau) + \frac{b(\tau)}{2a(\tau)} \right)^2 d\tau \leq \int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{4a(\tau)} d\tau, \quad t \geq t_0. \quad (2.53)$$

According to Cauchy - Schwarz inequality we have:

$$\int_{t_0}^t a(\tau) \left(x_0(\tau) + \frac{b(\tau)}{2a(\tau)} \right) d\tau \leq \sqrt{\int_{t_0}^t a(\tau) d\tau} \sqrt{\int_{t_0}^t a(\tau) \left(x_0(\tau) + \frac{b(\tau)}{2a(\tau)} \right)^2 d\tau}$$

From here and from (2.53) we get:

$$\int_{t_0}^t a(\tau) x_0(\tau) d\tau \leq -\frac{1}{2} \int_{t_0}^t b(\tau) d\tau + \frac{1}{2} \sqrt{\int_{t_0}^t a(\tau) d\tau \left[\int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{a(\tau)} d\tau \right]}, \quad (2.54)$$

$t \geq t_0$. Let $x_*(t)$ be a t_0 -extremal solution to Eq. (2.1). Then (see [11])

$$a(t)x_*(t) = \frac{\nu_{x_0}(t)}{\nu_{x_0}(t_0)} - a(t)x_0(t) - b(t), \quad t \geq t_0.$$

From here and from (2.54) we carry out:

$$\int_{t_0}^t a(\tau) x_*(\tau) d\tau \geq -\frac{1}{2} \int_{t_0}^t b(\tau) d\tau - \frac{1}{2} \sqrt{\int_{t_0}^t a(\tau) d\tau \left[\int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{a(\tau)} d\tau \right]} + \ln \frac{\nu_{x_0}(t)}{\nu_{x_0}(t_0)}, \quad t \geq t_0. \quad (2.55)$$

Remark 2.2 The estimates (2.54) and (2.55) are sharp in the sense that for $a(t) = \text{const}$, $b(t) = \text{const}$, $c(t) = \text{const}$ the estimate (2.54) becomes an equality up to constant summand and the inequality (2.55) becomes an equality.

Let $I_{a,b}^+(t_0) < +\infty$, $I_{-c,-b}^+(t_0) < +\infty$. Then due to Theorem 2.3. VI° Eq. (2.1) has a negative t_0 -normal solution. Therefore,

$$\begin{aligned} \nu_{x_0}(t) &= \int_t^{+\infty} a(\tau) \exp \left\{ - \int_t^\tau [2a(s)(x_0(\xi) - x_N^-(\xi)) + 2a(\xi)x_N^-(\xi) + b(\xi)] d\xi \right\} d\tau \geq \\ &\geq \exp \left\{ - \int_t^{+\infty} 2a(s)(x_0(\xi) - x_N^-(\xi)) d\xi \right\} I_{a,b}^+(t, +\infty), \quad t \geq t_0. \end{aligned}$$

From here and from (2.55) we get:

$$\begin{aligned} \int_{t_0}^t a(\tau)x_*(\tau)d\tau &\geq -\frac{1}{2} \int_{t_0}^t b(\tau)d\tau - \\ &- \frac{1}{2} \sqrt{\int_{t_0}^t a(\tau)d\tau \left[\int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{a(\tau)} d\tau \right]} + \ln I_{a,b}^+(t) + c, \quad t \geq t_0. \end{aligned} \tag{2.56}$$

where

$$c \equiv - \int_{t_0}^{+\infty} a(\tau)(x_0(\tau) - x_N^-(\tau))d\tau - \ln \nu_{x_0}(t_0). \tag{2.57}$$

Let $a(t) > 0$, $c(t) \geq 0$, $t \geq t_0$, $I_{a,b}^+(t_0) = +\infty$, and let Eq. (2.1) has a t_0 -regular solution. Obviously

$$x_1(t) + \int_{t_0}^t a(\tau) \left(x_1(\tau) + \frac{b(\tau)}{2a(\tau)} \right)^2 d\tau = x_1(t_0) + \int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{4a(\tau)} d\tau,$$

$t \geq t_0$. Then since by Theorem 2.4. I* $x_1(t) > 0$, $t \geq t_0$, we have

$$\int_{t_0}^t a(\tau) \left(x_1(\tau) + \frac{b(\tau)}{2a(\tau)} \right)^2 d\tau \leq x_1(t_0) + \int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{4a(\tau)} d\tau, \quad t \geq t_0.$$

From here using Cauchy - Schwarz inequality by analogy of (2.54) we get:

$$\int_{t_0}^t a(\tau)x_0(\tau)d\tau \leq -\frac{1}{2} \int_{t_0}^t b(\tau)d\tau +$$

$$+ \frac{1}{2} \sqrt{\int_{t_0}^t a(\tau)d\tau \left[4x_1(t_0) + \int_{t_0}^t \frac{b^2(\tau) - 4a(\tau)c(\tau)}{a(\tau)} d\tau \right]}, \quad t \geq t_0. \quad (2.58)$$

3. The behavior of solutions of the system (1.1)

Definition 3.1. The function $u(t)$ is called oscillatory if it has arbitrary large zeroes, otherwise $u(t)$ is called non oscillatory.

Definition 3.2. The system (1.1) is called oscillatory (non oscillatory), if for its each non trivial solution $(\phi(t), \psi(t))$ the functions $\phi(t)$ and $\psi(t)$ are oscillatory (non oscillatory).

Remark 3.1. Some oscillatory and non oscillatory criteria are proved in [6] (see also [5]).

Definition 3.3. The system (1.1) is called weak oscillatory (weak non oscillatory), if for its each non trivial solution $(\phi(t), \psi(t))$ at least one of the functions $\phi(t)$ and $\psi(t)$ is oscillatory (non oscillatory) and there exist two solutions $(\phi_j(t), \psi_j(t)), j = 1, 2$, such that $\phi_1(t)$ and $\psi_1(t)$ are oscillatory (non oscillatory), and at least one of the functions $\phi_2(t)$ and $\psi_2(t)$ is non oscillatory (oscillatory).

Definition 3.4. The system (1.1) is called half oscillatory if for its each non trivial solution $(\phi(t), \psi(t))$ one of the functions $\phi(t)$, $\psi(t)$ is oscillatory and other is non oscillatory.

Definition 3.5. The system (1.1) is called singular, if it has two non trivial solutions $(\phi_j(t), \psi_j(t)), j = 1, 2$, such that $\phi_1(t)$ and $\psi_1(t)$ are oscillatory, and $\phi_2(t)$ and $\psi_2(t)$ are non oscillatory.

Remark 3.2. It is evident that each system (1.1) is or else oscillatory or else non oscillatory or else weak oscillatory or else weak non oscillatory or else half oscillatory or else singular.

Example 3.1. Consider the system

$$\begin{cases} \phi' = \cos(\lambda t)\psi; \\ \psi' = -\cos(\lambda t)\phi, \quad t \geq t_0 \end{cases} \quad (3.8)$$

where $\lambda = \text{const} > 0$ is a parameter. The general solution $(\phi(t), \psi(t))$ to this system is given by formulas:

$$\phi(t) = c_1 \sin\left(\frac{1}{\lambda} \sin \lambda t + c_2\right), \quad \psi(t) = c_1 \cos\left(\frac{1}{\lambda} \sin \lambda t + c_2\right),$$

where c_1 and c_2 are arbitrary constants. It is not difficult to verify that if:

- 1) $0 < \lambda \leq \frac{2}{\pi}$, then the system (3.8) is oscillatory;
- 2) $\frac{2}{\pi} < \lambda \leq \frac{4}{\pi}$, then the system (3.8) is weak oscillatory;
- 3) $\lambda > \frac{4}{\pi}$, then the system (3.8) is weak non oscillatory.

Example 3.2. Consider the system

$$\begin{cases} \phi' = \psi; \\ \psi' = (-\cos^2 t - \sin t)\phi, \quad t \geq t_0. \end{cases} \quad (3.9)$$

The general solution $(\phi(t), \psi(t))$ of this system is given by formulas:

$$\begin{aligned} \phi(t) &= e^{\sin t} \left(c_1 + c_2 \int_{t_0}^t e^{-2 \sin \tau} d\tau \right), \quad \psi(t) = \\ &= e^{\sin t} \left[c_1 \cos t + c_2 \left\{ \cos t \int_{t_0}^t e^{-2 \sin \tau} d\tau + e^{-2 \sin t} \right\} \right], \end{aligned}$$

where c_1 and c_2 are arbitrary constants. Obviously $\phi(t)$ is non oscillatory and $\psi(t)$ is oscillatory. Hence the system (3.9) is half oscillatory.

Example 3.3. Consider the system

$$\begin{cases} \phi' = 3 \cos t \phi - 2 \cos t \psi; \\ \psi' = 4 \cos t \phi - 3 \cos t \psi, \quad t \geq t_0. \end{cases} \quad (3.10)$$

It has the solutions

$$(e^{\sin t}, e^{\sin t}), \quad (e^{\sin t} - e^{-\sin t}, e^{\sin t} - 2e^{-\sin t}), \quad t \geq t_0.$$

Obviously the components of the first solution are non oscillatory; the first component of the second solution vanishes in the points $\pi k \geq t_0$, $k = 0, \pm 1, \pm 2, \dots$, and the nulls of the second component of the second solution are all solutions of the equation $\sin t = \ln \sqrt{2}$ on $[t_0; +\infty)$. Therefore the system (3.10) is singular.

Definition 3.6 *The system (1.1) is called stable by Lyapunov (asymptotically), if its all solutions are bounded on $[t_0; +\infty)$ (vanish on $+\infty$).*

Theorem 3.1. *Let for each solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\phi(t)$ is bounded. Then there exists a solution $(\phi_0(t), \psi_0(t))$ of the system (1.1) such that $J_{-S/2}(t)\psi_0(t) \not\rightarrow 0$ for $t \rightarrow +\infty$. Moreover if in addition $a_{12}(t)$ does not change sign and $\int_{t_0}^{+\infty} |a_{12}(\tau)| d\tau = +\infty$, then the system (1.1) is oscillatory and for each nontrivial solution $(\phi(t), \psi(t))$ of the system (1.1) $J_{-S/2}(t)\psi(t) \not\rightarrow 0$ for $t \rightarrow +\infty$.*

Proof. By (3.3) from the conditions of the theorem it follows that

$$y_0(t) \geq \varepsilon, \quad t \geq t_0, \quad (3.11)$$

for some $\varepsilon > 0$. Suppose for each solution $(\phi(t), \psi(t))$ of the system (1.1) $J_{-S/2}(t)\psi(t) \rightarrow 0$ for $t \rightarrow +\infty$. Then according to (3.4) we have $y_0(t) \rightarrow 0$ for $t \rightarrow +\infty$, which contradicts (3.11). The obtained contradiction shows the existence of a solution $(\phi_0(t), \psi_0(t))$ of the system (2.1) with $J_{-S/2}(t)\psi_0(t) \not\rightarrow 0$ for $t \rightarrow +\infty$. If in addition $a_{12}(t)$ does not change sign and $\int_{t_0}^{+\infty} |a_{12}(\tau)| d\tau = +\infty$, then from (3.11) it

follows that $|\int_{t_0}^{+\infty} a_{12}(\tau)y_0(\tau)d\tau| = +\infty$. From here and from (3.6) it

follows oscillation of the system (1.1). From the last equality from (3.6) and (3.11) it follows that for each solution $(\phi(t), \psi(t))$ for the system (1.1) the relation $J_{-S/2}(t)\psi(t) \not\rightarrow 0$ for $t \rightarrow +\infty$ is fulfilled. The theorem is proved.

Theorem 3.2. *Let for each solution $(\phi(t), \psi(t))$ of the system (1.1) the relation $J_{-S/2}(t)\phi(t) \rightarrow 0$ for $t \rightarrow +\infty$ be satisfied. Then there exists a solution $(\phi_0(t), \psi_0(t))$ of the system (1.1) such that $J_{-S/2}(t)\psi_0(t)$ is unbounded. Moreover if in addition $a_{12}(t)$ does not change sign and*

$\int_{t_0}^{+\infty} |a_{12}(\tau)|d\tau = +\infty$, then the system (1.1) is oscillatory, and for each nontrivial solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi(t)$ is unbounded.

Proof. By (3.3) from the condition of the theorem it follows that

$$y_0(t) \rightarrow +\infty \text{ for } t \rightarrow +\infty. \tag{3.12}$$

Suppose for each solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi(t)$ is bounded. Then from (3.4) it follows that $y_0(t)$ is bounded, which contradicts (3.12). Hence for at least one solution $(\phi_0(t), \psi_0(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi_0(t)$ is unbounded. Let $a_{12}(t)$ does not change sign and let $\int_{t_0}^{+\infty} |a_{12}(\tau)|d\tau = +\infty$. Then from (3.6) and (3.12) it follows that the system (1.1) is oscillatory and by virtue of the second of equalities (3.6) from (3.12) it follows that for each nontrivial solution $(\phi(t), \psi(t))$ of the system (1.1) the function $J_{-S/2}(t)\psi(t)$ is unbounded. The theorem is proved.

Theorem 3.3 (about rings). *Suppose for each solution $\Phi(t) \equiv (\phi(t), \psi(t))$ of the system (1.1) there exists $R_\Phi > 0$ such that $\|\Phi(t)\| \leq R_\Phi J_{S/2}(t)$, $t \geq t_0$. Then for each nontrivial solution $\Phi(t)$ of the system (1.1) there exists r_Φ such that*

$$\|\Phi(t)\| \geq r_\Phi J_{S/2}(t), \quad t \geq t_0. \tag{3.13}$$

Proof. By (3.3) - (3.5) from the conditions of the theorem it follows that

$$\sqrt{y_0(t)} \leq M, \quad \frac{1}{\sqrt{y_0(t)}} \leq M, \quad \frac{x_0(t)}{y_0(t)} \leq M, \quad t \geq t_0. \tag{3.14}$$

for some $M = const > 0$. Suppose for some solution $\Phi_0(t) \equiv (\phi_0(t), \psi_0(t))$ of the system (1.1) the relation (3.13) does not fulfill. Then there exists infinitely large sequence $\{t_n\}_{n=1}^{+\infty}$ such that

$$J_{-S/2}(t_n)\phi_0(t_n) \rightarrow 0, \quad J_{-S/2}(t_n)\psi_0(t_n) \rightarrow 0 \text{ for } n \rightarrow +\infty. \tag{3.15}$$

By (3.6) we have:

$$J_{-S/2}(t_n)\phi_0(t_n) = \frac{\mu_0}{\sqrt{y_0(t_n)}} \sin(\gamma_n);$$

$$J_{-S/2}(t_n)\psi_0(t_n) = \mu_0 \sqrt{1 + \frac{x_0(t_0)}{y_0(t_0)}} \sqrt{y_0(t_n)} \cos(\gamma_n - \alpha_0(t_n)),$$

where $\gamma_n \equiv \int_{t_0}^{t_n} y_0(\tau) d\tau + \nu_0$, $n = 1, 2, \dots$; ν_0 and μ_0 are some constants.

From here from (3.14) and (3.15) it follows:

$$\sin(\gamma_n) \rightarrow 0, \text{ for } n \rightarrow +\infty; \quad (3.16)$$

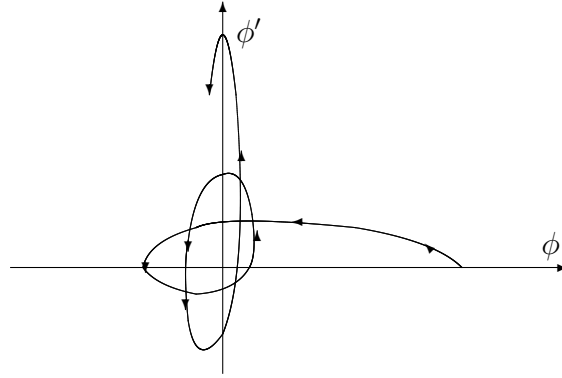
$$\cos(\gamma_n - \alpha_0(t_n)) \rightarrow 0, \text{ for } n \rightarrow +\infty; \quad (3.17)$$

From (3.14) and from (3.7) it follows that there exists $\delta > 0$ such that $|\cos(\alpha_0(t_n))| > \delta, n = 1, 2, \dots$. From here and from (3.16) it follows that $|\cos(\gamma_n - \alpha_0(t_n))| = 1 \pm \sqrt{1 - \sin^2 \gamma_n} \cos(\alpha_0(t_n)) + \sin \gamma_n \sin \alpha_0(t_n) \geq \geq \delta/2$ for all enough large values of n , which contradicts (3.17). The obtained contradiction proves (3.13). The theorem is proved.

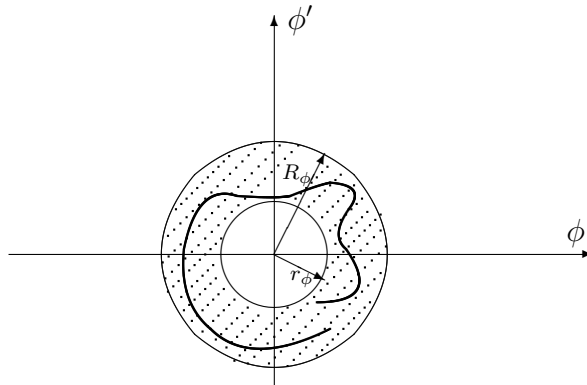
Remark 3.3. *The geometrical meaning of Theorem 3.4 is that if for all solutions $\Phi(t) \equiv (\phi(t), \psi(t))$ of the system (1.1) the vector functions $J_{-S/2}(t)\Phi(t)$ are bounded then each of the last ones lies in some ring of radiuses $0 < r_\Phi < R_\Phi$.*

By correlation (1.3) between Eq. (1.2), Eq. (1.5) and the system (1.4) From Theorems 3.1 – 3.4 we deduce the following **three principles for Eq. (1.5)**:

- A) *If all solutions of eq. (1.5) are bounded then it is oscillatory and for its each nontrivial solution $\phi(t)$ the relation $\phi'(t) \not\rightarrow 0$ for $t \rightarrow +\infty$ is fulfilled.*
- B) *If all solutions of Eq. (1.5) vanish on $+\infty$, then the derivative of its each nontrivial solution is unbounded.*
- C) *If Eq. (1.5) is stable by Lyapunov then for its each nontrivial solution $\phi(t)$ there exist positive numbers $r_\phi < R_\phi$ such that $r_\phi \leq \sqrt{\phi^2(t) + \phi'(t)^2} \leq R_\phi, t \geq t_0$.*



Pict.7. An illustration to the principle B)



Pict.8. An illustration to the principle C)

Let us compare these principles with the following assertion proved in [1] (see [1], p. 222, Corollary 6.2.4).

Proposition 3.1. *Let $|r(t)| \leq M, t \geq t_0$. Then if all solutions of Eq. (1.5) vanish on $+\infty$, then Eq. (1.5) is asymptotically stable.*

Obviously from any of principles A) - C) it follows that the equations (1.5) satisfying the conditions of Proposition 3.1, form an empty set.

Example 3.4. Consider the Mathieu equation (see [14])

$$\phi'' + (\delta + \varepsilon \cos t)\phi = 0, \quad t \geq t_0, \quad \delta, \varepsilon \in R.$$

From the principle A) it follows that this equation for all pairs (δ, ε) of zone of stability is oscillatory, and from the principle C) it follows that (for this restriction) for its each nontrivial solution $\phi(t)$ there exist $R_\phi > r_\phi > 0$ such that $r_\phi \leq \phi^2(t) + \phi'(t)^2 \leq R_\phi, t \geq t_0$ (see Pict. 8),

which agrees quite well with the Floquet's theory. Note that some part of mentioned above zone of stability relates to the extremal case of Eq. (1.5), when $\int_{t_0}^{+\infty} r(t)dt = -\infty$.

Example 3.5. Consider the Airy's equation

$$\phi'' + t\phi = 0, \quad t \geq t_0.$$

By virtue of L. A. Gusarov's theorem (see [15], Theorem 1) all solutions of this equation vanish on $+\infty$. From the principles A) and B) it follows that this equation is oscillatory and for its each nontrivial solution $\phi(t)$ the function $\phi'(t)$ is unbounded (see Pict. 7).

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