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On right relative normalizations of ruled surfaces

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Abstract: This paper deals with skew ruled surfaces in the Euclidean space $\mathbb{E}^3$ which are right normalized, that is, they are equipped with relative normalizations, whose support function is of a specific form. This class of relatively normalized ruled surfaces contains surfaces whose relative image Φ^* is either a curve or a ruled surface the generators of which are parallel to those of Φ. Moreover we investigate various properties concerning the Tchebychev vector field and the support vector field of right normalized ruled surfaces.

1. Preliminaries

In this section we present briefly some definitions, results and formulae of relative Differential Geometry of surfaces and Differential Geometry of ruled surfaces in the Euclidean space $\mathbb{E}^3$. The reader can use [3] and [5] as general references.

In the three-dimensional Euclidean space $\mathbb{E}^3$ we denote by $\Phi = (U, \overline{x})$ a skew ruled C^r-surface (that is a surface of nonvanishing Gaussian

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curvature), $r \geq 3$, defined by an injective C^r-immersion $\overline{x} = \overline{x}(u,v)$ on a region $U := (a,b) \times \mathbb{R}$, where $(a,b) \subseteq \mathbb{R}$ is an open interval of $\mathbb{R}^2$. We denote by $\langle\, ,\rangle$ the standard scalar product in $\mathbb{E}^3$. We introduce the so-called standard parameters $u \in (a,b), v \in \mathbb{R}$ of $\varPhi$, such that

$$\overline{x}(u,v) = \overline{s}(u) + v\,\overline{e}(u), \tag{1.1}$$

and

$$|\overline{e}| = |\overline{e}'| = 1, \quad \langle \overline{s}', \overline{e}' \rangle = 0, \tag{1.2}$$

where the differentiation with respect to u is denoted by a prime. Here $\varGamma : \overline{s} = \overline{s}(u)$ is the striction curve of $\varPhi$ and the parameter u is the arc length along the spherical curve $\overline{e} = \overline{e}(u)$.

The distribution parameter $\delta(u) := (\overline{s}', \overline{e}, \overline{e}')$, the conical curvature $\kappa(u) := (\overline{e}, \overline{e}', \overline{e}'')$ and the function $\lambda(u) := \cot\sigma$, where $\sigma(u) := \sphericalangle(\overline{e}, \overline{s}')$ is the striction of $\varPhi$ $(-\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}$, $\operatorname{sign}\sigma = \operatorname{sign}\delta)$, are the fundamental invariants of $\varPhi$ and determine uniquely the ruled surface $\varPhi$ up to Euclidean rigid motions. In what follows we consider ruled surfaces for which $\kappa = 0$ for every $u \in (a,b)$ or $\kappa \neq 0$ for every $u \in (a,b)$. We also consider the moving frame $\mathcal{D} := \{\overline{e}, \overline{n}, \overline{z}\}$ of $\varPhi$, where $\overline{n}(u) := \overline{e}'$ is the central normal vector and $\overline{z}(u) := \overline{e} \times \overline{n}$ is the central tangent vector. It is well known that the following equations are valid [3, p. 280]

$$\overline{e}' = \overline{n}, \quad \overline{n}' = -\overline{e} + \kappa\,\overline{z}, \quad \overline{z}' = -\kappa\,\overline{n}. \tag{1.3}$$

Then we have

$$\overline{s}' = \delta\,\lambda\,\overline{e} + \delta\,\overline{z}. \tag{1.4}$$

We denote partial derivatives of a function (or a vector-valued function) f in the coordinates $u^1 :=$, $u^2 := v$ by $f_{/i}, f_{/ij}$ etc. Then from (1.1) and (1.4) we obtain

$$\overline{x}_{/1} = \delta\,\lambda\,\overline{e} + v\,\overline{n} + \delta\,\overline{z}, \quad \overline{x}_{/2} = \overline{e}. \tag{1.5}$$

Let $I = g_{ij}\mathrm{d}u^i\mathrm{d}u^j$ and $II = h_{ij}\mathrm{d}u^i\mathrm{d}u^j$, $i,j = 1,2$, be the first and the second fundamental form of $\varPhi$, respectively, where

$$g_{11} = \delta^2\lambda^2 + v^2 + \delta^2, \qquad g_{12} = \delta\lambda, \qquad g_{22} = 1,$$

$$h_{11} = -\frac{\kappa\, w^2 + \delta'\, v - \delta^2\, \lambda}{w}, \qquad h_{12} = \frac{\delta}{w}, \qquad h_{22} = 0 \tag{1.6}$$

and

$$w := \sqrt{\det(g_{ij})} = \sqrt{\delta^2 + v^2}. \tag{1.7}$$

The unit normal vector can be written as

$$\bar{\xi} = \frac{\delta\, \overline{n} - v\, \overline{z}}{w}.$$

The Gaussian curvature $\widetilde{K}(u,v)$ and the mean curvature $\widetilde{H}(u,v)$ of $\varPhi$ are given by (see [3])

$$\widetilde{K} = -\frac{\delta^2}{w^4}, \quad \widetilde{H} = -\frac{\kappa w^2 + \delta' v + \delta^2 \lambda}{2w^3}. \tag{1.8}$$

A C^s- *relative normalization* of $\varPhi$ is a C^s-mapping $\overline{y} = \overline{y}(u,v), 1 \leq s < r$, defined on U, such that
(1.9)

$$\operatorname{rank}(\{\overline{x}_{/1}, \overline{x}_{/2}, \overline{y}\}) = 3,\ \operatorname{rank}(\{\overline{x}_{/1}, \overline{x}_{/2}, \overline{y}_{/i}\}) = 2,\ i = 1,2,\ \forall\,(u,v) \in U.$$

The pair $(\varPhi, \overline{y})$ is called a *relatively normalized* ruled surface and the line issuing from a point $P \in \varPhi$ in the direction of $\overline{y}$ is called the *relative normal* of $\varPhi$ at P. The pair $\varPhi^* = (U, \overline{y})$ is called the *relative image* of $(\varPhi, \overline{y})$.

The support function of the relative normalization $\bar{y}$ is defined by $q(u,v) := \langle \bar{\xi}, \overline{y} \rangle$ (see [2]). Because of (1.9), q never vanishes on U. Conversely, when a support function q is given, the relative normalization $\overline{y}$ of the ruled surface $\varPhi$ is uniquely determined and can be expressed in terms of the moving frame $\mathcal{D}$ as follows [6, p. 179]:

$$\overline{y} = y_1\, \overline{e} + y_2\, \overline{n} + y_3\, \overline{z}, \tag{1.10}$$

where

$$y_1 = -w\frac{\delta q_{/1} + q_{/2}(\kappa\, w^2 + \delta' v)}{\delta^2}, \quad y_2 = \frac{\delta^2\, q - w^2\, v\, q_{/2}}{\delta w}, \quad y_3 = -\frac{v\, q + w^2\, q_{/2}}{w}. \tag{1.11}$$

The coefficients $G_{ij}(u,v)$ of the relative metric $G(u,v)$ of $(\mathit{\Phi},\overline{y})$, which is indefinite, are given by $G_{ij} = q^{-1}\, h_{ij}$.

For a function (or a vector-valued function) f we denote by $\nabla^G f$ the first Beltrami differential operator and by $\nabla^G_i f$ the covariant derivative in the direction u^i, both with respect to the relative metric. The coefficients $A_{ijk}(u,v)$ of the Darboux tensor are defined by

$$A_{ijk} \coloneqq q^{-1} \langle \overline{\xi},\, \nabla^G_k \nabla^G_j \,\overline{x}_{/i}\rangle.$$

Then, by using the relative metric tensor G_{ij} for "raising and lowering the indices", the *Pick invariant* $J(u,v)$ of $(\mathit{\Phi},\overline{y})$ is given by

$$J \coloneqq \frac{1}{2} A_{ijk}\, A^{ijk}.$$

As we showed in [8] (see equation (2.2)) the Pick invariant is calculated by

$$J = \frac{3\left(w^2 q_{/2} + v\, q\right)}{2\delta^2 w^3\, q} \cdot \cdot \left\{ w^2 \left[\kappa q v + 2\delta q_{/1} + q_{/2}\left(\kappa\, w^2 + \delta' v - \delta^2 \lambda\right)\right] - \delta^2 q\left(\lambda v - \delta'\right) \right\}. \tag{1.12}$$

The *relative shape operator* has the coefficients $B^j_i(u,v)$ defined by

$$\overline{y}_{/i} \eqqcolon -B^j_i\, \overline{x}_{/j}. \tag{1.13}$$

Then, for the *relative curvature* $K(u,v)$ and the *relative mean curvature* $H(u,v)$ of $(\mathit{\Phi},\overline{y})$ we have

$$K \coloneqq \det\left(B^j_i\right), \quad H \coloneqq \frac{B^1_1 + B^2_2}{2}. \tag{1.14}$$

We mention finally, that among the surfaces $\mathit{\Phi} \subset \mathbb{E}^3$ with negative Gaussian curvature the ruled surfaces are characterized by the relation

$$3H - J - 3S = 0 \tag{1.15}$$

(see [7]), where $S(u,v)$ is the *scalar curvature* of the relative metric G, which is defined formally as the curvature of the pseudo-Riemannian manifold $(\mathit{\Phi}, G)$.

2. Right normalizations

We focus now our investigation on the main subject of this paper, namely the right normalizations of a skew ruled surface Φ, that is, relative normalizations which are given by (1.10) and (1.11) by means of the support function

$$q = \frac{f + g\,v}{w}, \tag{2.1}$$

where f and g are arbitrary C^{s+1}-functions of u, such that $q \neq 0$. These normalizations are introduced in [8] by the authors.

When the function g vanishes in I, the relative normal at each point $P \in \Phi$ lies on the corresponding asymptotic plane $\{P; \overline{e}, \overline{n}\}$ of Φ. Normalizations of this type are called *asymptotic* and they have been studied by I. Kaffas and S. Stamatakis [6]. Another special case arises when the function f vanishes in I. Then the relative normal at each point $P \in \Phi$ lies on the corresponding central plane $\{P; \overline{e}, \overline{z}\}$ of Φ. Normalizations of this type are called *central* and they have been studied in [8]. Since both asymptotic and central normalizations belong to the right ones and they have been studied thoroughly in the above mentioned papers, we assume that in what follows none of the functions f and g is vanishing.

From (1.10), (1.11) and (2.1) it follows that a right normalization of the given ruled surface Φ is

$$\overline{y} = \frac{(\kappa\, f - \delta\, g')\, v + \delta' f - \delta\, f' - \delta^2\, \kappa\, g}{\delta^2}\, \overline{e} + \frac{f}{\delta}\, \overline{n} - g\, \overline{z}. \tag{2.2}$$

Then, by using (1.3), (1.5), (1.13) and (2.2), we obtain the coefficients B_i^j of the relative shape operator of a right normalization:

$$B_1^1 = B_2^2 = \frac{\delta g' - \kappa f}{\delta^2}, \quad B_2^1 = 0,$$
$$B_1^2 = \frac{1}{\delta^3} \Big[\big(2\kappa\delta' f - \delta\kappa f' - \delta\delta' g' - \delta\kappa' f + \delta^2 g''\big)\, v + \delta^2 f\,(\kappa\lambda + 1)$$
$$+ 2\delta'\,(\delta' f - \delta f') + \delta^3 g'\,(\kappa - \lambda) + \delta^3 \kappa' g - \delta\delta'' f + \delta^2 f'' \Big].$$

Hence, via (1.14), the relative mean curvature H and the relative curvature K are

$$H = \frac{\delta\, g' - \kappa\, f}{\delta^2}, \quad K = H^2. \tag{2.3}$$

Firstly, we observe that *all points of Φ are relative umbilics* $(H^2-K\equiv 0)$. Thus, for the relative principal curvatures k_1 and k_2, which by definition are the eigenvalues of the relative shape operator (see [5, p. 215]), $k_1=k_2=H$ holds. Then, from (1.12) we find for the Pick invariant

$$J=3g\,\frac{\kappa\,g\,v^2+2\delta\,g'v+\delta^2 g\,(\kappa-\lambda)-\delta' f+2\delta f'}{2\delta^2\,(f+g\,v)}. \tag{2.4}$$

Consequently J vanishes identically iff

$$\kappa\,g\,v^2+2\delta\,g'v+\delta^2 g\,(\kappa-\lambda)-\delta' f+2\delta f'=0,$$

or, equivalently, after successive differentiations of this last equation relative to v, iff

$$\kappa=g'=\delta^2 g\,(\kappa-\lambda)-\delta' f+2\delta f'=0,$$

from which we have $\kappa=0$, i.e., Φ is conoidal, $g=c_1\in\mathbb{R}^*$ and $f=|\delta|^{1/2}\left(\frac{c_1}{2}\int|\delta|^{1/2}\,\lambda\,\mathrm{d}u+c_2\right)$, $c_2\in\mathbb{R}$. Thus, the following has been shown

Proposition 2.1. The Pick invariant of a right normalized skew ruled surface $\Phi\subset\mathbb{E}^3$ vanishes identically iff Φ is conoidal, the function g is a nonvanishing constant c_1 and the function f is given by

$$f=|\delta|^{1/2}\left(\frac{c_1}{2}\int|\delta|^{1/2}\,\lambda\,\mathrm{d}u+c_2\right),\ c_2\in\mathbb{R}.$$

Additionally, in view of (2.3a), *a right normalized ruled surface with vanishing Pick invariant is relatively minimal.*

By using (1.15), (2.3a) and (2.4) we obtain the scalar curvature of the relative metric

$$S=-\frac{\kappa\,g^2v^2+2\kappa\,f\,g\,v+\delta^2 g^2(\kappa-\lambda)+2\kappa\,f^2-\delta' f\,g+2\delta\,(f'g-f\,g')}{2\delta^2\,(f+g\,v)}.$$

The scalar curvature of the relative metric G vanishes identically iff

$$\kappa=\delta^2 g^2(\kappa-\lambda)+2\kappa\,f^2-\delta' f\,g+2\delta\,(f'g-f\,g')=0,$$

that is, iff $\kappa=0$ and $f=\frac{1}{2}|\delta|^{1/2}\,g\left(\int|\delta|^{1/2}\,\lambda\,\mathrm{d}u+c\right)$, $c\in\mathbb{R}$. So, we have:

Proposition 2.2. The scalar curvature S of a right normalized skew ruled surface $\varPhi \subset \mathbb{E}^3$ vanishes identically iff $\varPhi$ is conoidal and the function f is given by

$$f = \frac{1}{2}|\delta|^{1/2}\, g \left(\int |\delta|^{1/2}\, \lambda \, \mathrm{d}u + c \right),\ c \in \mathbb{R}.$$

We distinguish the right normalizations in two types.

2.1. Right normalizations of the first type

We say that a right relative normalization $\overline{y}$ is of the first type if the relative image $\varPhi^*$ of $(\varPhi, \overline{y})$ degenerates into a curve. Obviously this occurs iff

$$\delta\, g' - \kappa\, f = 0$$

(cf. (2.2)). Thus, on account of (2.2) and (2.3a) we conclude that

Proposition 2.3. Let $(\varPhi, \overline{y})$ be a right normalized ruled surface. Then the following properties are equivalent:
(a) $\overline{y}$ is a right normalization of the first type.
(b) $(\varPhi, \overline{y})$ is relatively minimal.
(c) The function g is given by

$$g = \int \frac{\kappa\, f}{\delta}\, \mathrm{d}u + c,\ c \in \mathbb{R}.$$

The right normalized ruled surfaces with vanishing Pick invariant belong obviously to this subclass.

The relative image $\varPhi^*$ is the curve parametrized by

$$\overline{y} = \frac{\delta' f - \delta f' - \delta^2\, \kappa\, g}{\delta^2}\, \overline{e} + \frac{f}{\delta}\, \overline{n} - g\, \overline{z}.$$

2.2. Right normalizations of the second type

A right relative normalization $\overline{y}$ is said to be of the second type if the relative image $\varPhi^*$ of $(\varPhi, \overline{y})$ does not degenerate into a curve of $\mathbb{E}^3$. Then $\varPhi^*$ is a ruled surface whose generators are parallel to those of $\varPhi$. From (2.2) we find the following parametrization of the striction curve of $\varPhi^*$:

$$\varGamma^* \colon \overline{s}^* = \frac{\delta' f - \delta f' - \delta^2 \kappa\, g}{\delta^2}\, \overline{e} + \frac{f}{\delta}\, \overline{n} - g\, \overline{z}.$$

Consequently $\varPhi^*$ can be parametrized like (1.1) and (1.2):

$$\varPhi^*\colon \overline{y} = \overline{s}^* + v^*\,\overline{e}, \quad \text{where} \quad v^* := \frac{(\kappa f - \delta\, g')v}{\delta^2}.$$

Considering $\mathcal{D}$ as moving frame of $\varPhi^*$ we compute its fundamental invariants:

$$\kappa^* = \kappa, \quad \delta^* = \frac{\kappa f - \delta g'}{\delta},$$
$$\lambda^* = -\frac{\delta^3(\kappa g' + \kappa' g) + \delta^2(f + f'') - \delta(\delta'' f + 2\delta' f') + 2\delta'^2 f}{\delta^2(\kappa f - \delta g')}.$$

By using (1.7) we infer that

$$w^* = \sqrt{\det(g^*_{ij})} = |H|\, w$$

and, thus, by means of (1.8a), the Gaussian curvature $\widetilde{K}^*$ of $\varPhi^*$ is

$$\widetilde{K}^* = -\frac{\delta^6}{w^4(\kappa f - \delta\, g')^2}.$$

The focal surfaces, which are the loci of the edges of regression of the developable surfaces consisting of the relative normals along the relative lines of curvature, coincide. The parametrization of the unique relative focal surface of $\varPhi$, which initially reads

$$\overline{x}^* = \overline{s} + v\,\overline{e} + \frac{1}{H}\,\overline{y},$$

in view of (2.2) and (2.3a) becomes

$$\overline{x}^* = \overline{s} + \frac{(\delta' f - \delta f' - \delta^2\,\kappa\, g)\,\overline{e} + \delta\, f\,\overline{n} - \delta^2 g\,\overline{z}}{\delta\, g' - \kappa\, f},$$

i.e., *the focal surface degenerates into a curve $\varLambda^*$ and all relative normals along each generator form a pencil of straight lines.*

3. The Tchebychev vector field of a right normalization

In [6] it was shown that the coordinate functions of the *Tchebychev vector* $\overline{T}(u,v)$ of $(\varPhi, \overline{y})$, which is defined by

$$\overline{T} := T^m\,\overline{x}_{/m}, \quad \text{where} \quad T^m := \frac{1}{2}A_i^{im},$$

are given by
(3.1)

$$T^1 = \frac{w^2 q_{/2} + v\,q}{\delta\,w},\ T^2 = \frac{2\delta\,w^2 q_{/1} + \delta' q\,(\delta^2 - v^2)}{2\delta^2\,w} + \frac{T^1(\kappa w^2 + \delta' v - \delta^2\lambda)}{\delta}.$$

By means of (1.5) and (2.1) the Tchebychev vector of a right normalization can be expressed in terms of the moving frame $\mathcal{D}$ as follows:

$$\overline{T} = \frac{2\kappa\,g\,v^2 + (\delta' g + 2\delta\,g')v + 2\delta^2\kappa\,g - \delta' f + 2\delta f'}{2\delta^2}\,\overline{e} + \frac{g}{\delta}(v\,\overline{n} + \delta\,\overline{z}). \tag{3.2}$$

The vectors $\overline{T}$ are orthogonal to the generators iff $\langle \overline{e}, \overline{T} \rangle = 0$. Taking (3.2) into consideration we find

$$2\kappa\,g\,v^2 + (\delta' g + 2\delta\,g')v + 2\delta^2\kappa\,g - \delta' f + 2\delta f' = 0,$$

or, after successive differentiations of this last equation relative to v, iff

$$2\kappa\,g = \delta' g + 2\delta\,g' = 2\delta^2\kappa\,g - \delta' f + 2\delta f' = 0.$$

After standard treatment of this system we deduce that $\kappa = 0$, $g = c_1|\delta|^{-1/2}$, $c_1 \in \mathbb{R}^*$, and $f = c_2|\delta|^{1/2}$, $c_2 \in \mathbb{R}^*$. So, we have the following

Proposition 3.1. The Tchebychev vector field $\overline{T}$ of a right normalized skew ruled surface $\varPhi \subset \mathbb{E}^3$ is orthogonal to the generators of $\varPhi$ iff $\varPhi$ is conoidal and the functions g and f are given by

$$g = c_1|\delta|^{-1/2}, c_1 \in \mathbb{R}^* \textit{ and } f = c_2|\delta|^{1/2}, c_2 \in \mathbb{R}^*.$$

A curve $\varLambda$ on $\varPhi$ is defined by means of v as a function of u, i.e., $\varLambda\colon v = v(u)$. Then for its tangent vector we have

$$\overline{x}' = (\delta\,\lambda + v')\,\overline{e} + v\,\overline{n} + \delta\,\overline{z}. \tag{3.3}$$

From (3.2) and (3.3) it follows: $\overline{x}'$ and $\overline{T}$ are parallel or orthogonal iff

$$2\kappa\,g\,v^2 + (\delta' g + 2\delta\,g')\,v + 2\delta^2\kappa\,g - \delta' f + 2\delta f' - 2\delta g\,(\delta\,\lambda + v') = 0 \tag{3.4}$$

or

$$\left[2\kappa g v^2 + (\delta' g + 2\delta g')\,v + 2\delta^2\kappa\,g - \delta' f + 2\delta f'\right](\delta\,\lambda + v') + 2\delta\,g\,w^2 = 0, \tag{3.5}$$

respectively. Among the curves of $\varPhi$ we consider the following families:

- The u-curves, i.e., the curves of constant striction distance, whose differential equation is

$$v' = 0. \tag{3.6}$$

- The curved asymptotic lines, which are different from the rulings. The differential equation of the curved asymptotic lines, which initially reads $II = 0$, becomes on account of (1.6) and (1.7)

$$\kappa\, v^2 + \delta' v + \delta^2(\kappa - \lambda) - 2\delta\, v' = 0. \tag{3.7}$$

- The $\widetilde{K}$-curves, i.e., the curves along which the Gaussian curvature is constant (cf. [4]). The differential equation of the $\widetilde{K}$-curves is $\mathrm{d}\widetilde{K} = 0$, that is,

$$2\delta\, v\, v' + \delta' \left(\delta^2 - v^2\right) = 0. \tag{3.8}$$

From (3.6) and (3.4), resp. (3.5), we have: $\overline{T}$ is tangential or orthogonal to the u-curves iff

$$2\kappa\, g\, v^2 + (\delta' g + 2\delta g')\, v + 2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = 0 \tag{3.9}$$

or

$$2g(\kappa\,\lambda + 1)v^2 + \lambda(\delta' g + 2\delta g')v + 2\delta^2 g(\kappa\,\lambda + 1) + \lambda(2\delta f' - \delta' f) = 0, \tag{3.10}$$

respectively. From (3.9) we find that $\overline{T}$ is tangential to the u-curves iff

$$\kappa = \delta' g + 2\delta g' = 2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = 0,$$

that is, iff $\kappa = 0$, $g = c_1|\delta|^{-1/2}$, $c_1 \in \mathbb{R}^*$, and $f = |\delta|^{1/2}\left(c_1 \int \lambda\, \mathrm{d}u + c_2\right)$, $c_2 \in \mathbb{R}$.

From (3.10) we derive that $\overline{T}$ is orthogonal to the u-curves iff

$$\kappa\,\lambda + 1 = \lambda(\delta' g + 2\delta\, g') = 2\delta^2 g(\kappa\,\lambda + 1) + \lambda(2\delta\, f' - \delta' f) = 0.$$

By direct computation we deduce that $\kappa\,\lambda + 1 = 0$, i.e., the striction curve of Φ is an Euclidean line of curvature, $g = c_1|\delta|^{-1/2}$, $c_1 \in \mathbb{R}^*$ and $f = c_2|\delta|^{1/2}$, $c_2 \in \mathbb{R}^*$. Therefore, we obtain

Proposition 3.2. The Tchebychev vector field $\overline{T}$ of a right normalized skew ruled surface $\varPhi \subset \mathbb{E}^3$ is
(a) tangential to the u-curves of $\varPhi$ iff $\varPhi$ is conoidal and the functions g and f are given by

$$g = c_1|\delta|^{-1/2},\ c_1 \in \mathbb{R}^* \ \textit{and}\ f = |\delta|^{1/2}\left(c_1 \int \lambda \,\mathrm{d}u + c_2\right),\ c_2 \in \mathbb{R}.$$

(b) orthogonal to the u-curves of $\varPhi$ iff the striction curve of $\varPhi$ is an Euclidean line of curvature and the functions g and f are given by

$$g = c_1|\delta|^{-1/2},\ c_1 \in \mathbb{R}^* \ \textit{and}\ f = c_2|\delta|^{1/2},\ c_2 \in \mathbb{R}^*.$$

From (3.6) and (3.4) we infer that $\overline{T}$ is tangential or orthogonal to the curved asymptotic lines iff

$$\kappa g v^2 + 2\delta g' v + \delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = 0, \tag{3.11}$$

or

(3.12)
$$\begin{aligned}
&2\kappa^2 g v^4 + \kappa\left(3\delta' g + 2\delta g'\right) v^3 \\
&+ \left[4\delta^2 g\left(\kappa^2+1\right) + 2\delta^2\kappa\lambda g - \delta'\kappa f + \delta'^2 g + 2\delta\kappa f' + 2\delta\delta' g'\right] v^2 \\
&+ \left(3\delta^2\delta'\kappa g + \delta^2\delta'\lambda g - \delta'^2 f + 2\delta\delta' f' + 2\delta^3\kappa g' + 2\delta^3\lambda g'\right) v \\
&+ \delta^2\left(4\delta^2 g + 2\delta^2\kappa^2 g + 2\delta^2\kappa\lambda g - \delta'\kappa f - \delta'\lambda f + 2\delta\kappa f' + 2\delta\lambda f'\right) = 0,
\end{aligned}$$

respectively. From (3.11) we have that $\overline{T}$ is tangential to the curved asymptotic lines iff

$$\kappa = g' = \delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = 0,$$

from which we take $\kappa = 0$, $g = c_1 \in \mathbb{R}^*$ and

$$f = |\delta|^{1/2}\left(\frac{c_1}{2}\int |\delta|^{1/2}\lambda \,\mathrm{d}u + c_2\right),\ c_2 \in \mathbb{R}.$$

From (3.12) we deduce that $\overline{T}$ is orthogonal to the curved asymptotic lines iff

$$\begin{aligned}
\kappa &= \kappa\left(3\delta' g + 2\delta g'\right) = \\
&= 4\delta^2 g\left(\kappa^2+1\right) + 2\delta^2\kappa\lambda g - \delta'\kappa f + \delta'^2 g + 2\delta\kappa f' + 2\delta\delta' g' = \\
&= 3\delta^2\delta'\kappa g + \delta^2\delta'\lambda g - \delta'^2 f + 2\delta\delta' f' + 2\delta^3\kappa g' + 2\delta^3\lambda g' = \\
&= 4\delta^2 g + 2\delta^2\kappa^2 g + 2\delta^2\kappa\lambda g - \delta'\kappa f - \delta'\lambda f + 2\delta\kappa f' + 2\delta\lambda f' = 0,
\end{aligned}$$

from which we obtain, initially, $\kappa = 0$. Solving the arising system of differential equations we arrive at a contradiction. So, we have

Proposition 3.3. The Tchebychev vector field $\overline{T}$ of a right normalized skew ruled surface $\varPhi \subset \mathbb{E}^3$ is
(a) tangential to the curved asymptotic lines of $\varPhi$ iff $\varPhi$ is conoidal, the function g is a nonvanishing constant c_1 and the function f is given by

$$f = |\delta|^{1/2} \left(\frac{c_1}{2} \int |\delta|^{1/2} \lambda \, \mathrm{d}u + c_2 \right), \; c_2 \in \mathbb{R}.$$

but
(b) it cannot be orthogonal to the curved asymptotic lines of $\varPhi$.

From (3.8) and (3.4), resp. (3.5), we infer: $\overline{T}$ is tangential or orthogonal to the $\widetilde{K}$-curves iff

$$2\kappa g v^3 + 2\delta g' v^2 + \left[2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f'\right] v + \delta^2 \delta' g = 0 \tag{3.13}$$

or

$$\begin{aligned} &2\kappa\delta' g v^4 + \left[4\delta^2 g(\kappa\lambda + 1) + \delta'(\delta' g + 2\delta g')\right] v^3 \\ &+ \left(2\delta^2\delta'\lambda g - \delta'^2 f + 2\delta\delta' f' + 4\delta^3\lambda g'\right) v^2 \\ &+ \delta^2 \left[4\delta^2 g(\kappa\lambda + 1) - 2\delta'\lambda f - \delta'^2 g + 4\delta\lambda f' - 2\delta\delta' g'\right] v \\ &- \delta^2\delta' \left(2\delta^2\kappa g - \delta' f + 2\delta f'\right) = 0, \end{aligned} \tag{3.14}$$

respectively. From (3.13) we find that $\overline{T}$ is tangential to the $\widetilde{K}$-curves iff

$$\kappa = g' = 2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = \delta' = 0,$$

i.e., iff $\kappa = 0$, $\delta = c_1 \in \mathbb{R}^*$, $g = c_2 \in \mathbb{R}^*$ and $f = c_1 c_2 \int \lambda \, \mathrm{d}u + c_3$, $c_3 \in \mathbb{R}$. From (3.14) we deduce that $\overline{T}$ is orthogonal to the $\widetilde{K}$-curves iff

$$\begin{aligned} &\kappa\delta' = 4\delta^2 g(\kappa\lambda + 1) + \delta'(\delta' g + 2\delta g') = \\ &\quad = 2\delta^2\delta'\lambda g - \delta'^2 f + 2\delta\delta' f' + 4\delta^3\lambda g' = 0, \\ &4\delta^2 g(\kappa\lambda + 1) - 2\delta'\lambda f - \delta'^2 g + 4\delta\lambda f' - 2\delta\delta' g' = \\ &\quad = \delta' \left(2\delta^2\kappa g - \delta' f + 2\delta f'\right) = 0, \end{aligned}$$

that is, iff $\delta = c \in \mathbb{R}^*$ or $\kappa = 0$. If $\delta = c \in \mathbb{R}^*$, we deduce that $\kappa\lambda + 1 = 0$, i.e., $\varPhi$ is an Edlinger surface[1], $g = c_1 \in \mathbb{R}^*$ and $f = c_2 \in \mathbb{R}^*$. If $\kappa = 0$ and

[1] i.e., a ruled surface whose osculating quadrics are rotational hyperboloids. The Edlinger surfaces are characterized by the conditions $\delta' = \kappa\,\lambda + 1 = 0$ (see [1, p. 36], [4]).

$\delta \neq c \in \mathbb{R}^*$ we arrive at a contradiction. Thus, the following has been shown

Proposition 3.4. The Tchebychev vector field $\overline{T}$ of a right normalized skew ruled surface $\varPhi \subseteq \mathbb{E}^3$ is
(a) tangential to the $\widetilde{K}$-curves of $\varPhi$ iff $\varPhi$ is conoidal of constant distribution parameter c_1, the function g is a nonvanishing constant c_2 and the function f is given by

$$f = c_1 c_2 \int \lambda \, \mathrm{d}u + c_3, \; c_3 \in \mathbb{R}.$$

(b) orthogonal to the $\widetilde{K}$-curves of $\varPhi$ iff $\varPhi$ is an Edlinger surface and the functions g and f are nonvanishing constants c_1 and c_2, respectively.

The following table summarizes the results:

$\overline{T}$ is ...	Type of the ruled surface $\varPhi$	g	f
orthogonal to the generators	conoidal	$g = c_1 \|\delta\|^{-1/2}$, $c_1 \in \mathbb{R}^*$	$f = c_2 \|\delta\|^{1/2}$, $c_2 \in \mathbb{R}^*$
tangential to the u-curves	conoidal	$g = c_1 \|\delta\|^{-1/2}$, $c_1 \in \mathbb{R}^*$	$f = \|\delta\|^{1/2} \left(c_1 \int \lambda \, \mathrm{d}u + c_2\right)$, $c_2 \in \mathbb{R}$
orthogonal to the u-curves	the striction curve is an Euclidean line of curvature	$g = c_1 \|\delta\|^{-1/2}$, $c_1 \in \mathbb{R}^*$	$f = c_2 \|\delta\|^{1/2}$, $c_2 \in \mathbb{R}^*$
tangential to the curved asympt. lines	conoidal	$g = c_1 \in \mathbb{R}^*$	$f = \|\delta\|^{1/2} \left(\frac{c_1}{2} \int \|\delta\|^{1/2} \lambda \, \mathrm{d}u + c_2\right)$, $c_2 \in \mathbb{R}$
orthogonal to the curved asympt. lines	-	-	-
tangential to the $\widetilde{K}$-curves	conoidal, $\delta = c_1 \in \mathbb{R}^*$	$g = c_2 \in \mathbb{R}^*$	$f = c_1 c_2 \int \lambda \, \mathrm{d}u + c_3$, $c_3 \in \mathbb{R}$
orthogonal to the $\widetilde{K}$-curves	Edlinger surface	$g = c_1 \in \mathbb{R}^*$	$f = c_2 \in \mathbb{R}^*$

The divergence $\operatorname{div}^I \overline{T}$ of $\overline{T}$ with respect to the first fundamental form I of $\varPhi$, which initially reads (see [9])

$$\operatorname{div}^I \overline{T} = \frac{(w T^i)_{/i}}{w}$$

becomes, on account of (3.1) and (2.1),

$$\operatorname{div}^I \overline{T} = \\ = \frac{6\kappa g v^3 + 6\delta g' v^2 + \left(6\delta^2 \kappa g - 2\delta^2 \lambda g - \delta' f + 2\delta\, f'\right) v + \delta^2 \left(\delta' g + 4\delta\, g'\right)}{2\delta^2 w^2},$$

from which we have that the Tchebychev vector field $\overline{T}$ is incompressible with respect to the first fundamental form of $\varPhi$ ($\operatorname{div}^I \overline{T} = 0$) iff

$$\kappa = g' = 6\delta^2 \kappa g - 2\delta^2 \lambda g - \delta' f + 2\delta f' = \delta' g + 4\delta g' = 0,$$

or iff $\kappa = 0$, $g = c_1 \in \mathbb{R}^*$, $\delta = c_2 \in \mathbb{R}^*$ and $f = c_1 c_2 \int \lambda \,\mathrm{d}u + c_3$, $c_3 \in \mathbb{R}$. Therefore, we arrive at

Proposition 3.5. The Tchebychev vector field $\overline{T}$ of a right normalized skew ruled surface $\varPhi \subset \mathbb{E}^3$ is incompressible with respect to the first fundamental form of $\varPhi$ iff $\varPhi$ is conoidal of constant distribution parameter c_2, the function g is a nonvanishing constant c_1 and the function f is given by

$$f = c_1 c_2 \int \lambda \,\mathrm{d}u + c_3,\ c_3 \in \mathbb{R}.$$

Let, now, $\operatorname{div}^G \overline{T}$ be the divergence of $\overline{T}$ with respect to the relative metric of $(\varPhi, \overline{y})$. Analogously to the above computation, by using (1.6), we get

$$\operatorname{div}^G \overline{T} = \frac{\kappa g^2 v^2 + 2\kappa f g v - \delta^2 g^2 (\kappa - \lambda) + \delta' f g - 2\delta g f' + 2\delta f g'}{\delta^2 (g v + f)}.$$

The Tchebychev vector field $\overline{T}$ is incompressible with respect to the relative metric, that is, $\operatorname{div}^G \overline{T} = 0$ iff

$$\kappa = -\delta^2 g^2 (\kappa - \lambda) + \delta' f g - 2\delta g f' + 2\delta f g' = 0,$$

i.e., iff $\kappa = 0$ and $f = \frac{1}{2}|\delta|^{1/2} g \left(\int |\delta|^{1/2} \lambda \,\mathrm{d}u + c\right)$, $c \in \mathbb{R}$. So, by taking into consideration Proposition 2.2, we deduce:

Proposition 3.6. Let $\varPhi \subset \mathbb{E}^3$ be a right normalized skew ruled surface. The following properties are equivalent:
(a) The Tchebychev vector field $\overline{T}$ is incompressible with respect to the relative metric.
(b) The scalar curvature S of the relative metric vanishes identically.
(c) $\varPhi$ is conoidal and the function f is given by

$$f = \frac{1}{2}|\delta|^{1/2} g \left(\int |\delta|^{1/2} \lambda \,\mathrm{d}u + c\right),\ c \in \mathbb{R}.$$

4. The support vector field of a right normalization

Let
$$\overline{Q} := \frac{1}{4}\,\triangledown^{G}\left(\frac{1}{q},\overline{x}\right)$$
be the support vector $\overline{Q}(u,v)$ of $(\varPhi,\overline{y})$, which is introduced in [6]. On account of (1.5), (1.6) and (2.1) we express the support vector in terms of the moving frame $\mathcal{D}$ as follows:

(4.1)
$$\overline{Q} = -w\frac{(\delta\, g' - \kappa\, f)\, v + \delta^2\kappa\, g - \delta' f + \delta f'}{4\delta^2(g\, v + f)}\overline{e} + \frac{f\, v - \delta^2 g}{4\delta\, w(g\, v + f)}\,(v\,\overline{n} + \delta\,\overline{z})\,.$$

The vectors $\overline{Q}$ are orthogonal to the generators iff $\langle\overline{e},\overline{Q}\rangle = 0$. Taking (4.1) into consideration we have
$$(\delta g' - \kappa f)\, v + \delta^2\kappa g - \delta' f + \delta f' = 0,$$
that is, iff
$$\delta g' - \kappa f = \delta^2\kappa g - \delta' f + \delta f' = 0,$$
from which we find that $\varPhi$ is relative minimal and $f = \pm\delta\,|c - g^2|^{1/2}$, $c \in \mathbb{R}$, $g^2 \neq c$. Thus, we arrive at:

Proposition 4.1. The support vector field $\overline{Q}$ of a right normalized skew ruled surface $\varPhi \subset \mathbb{E}^3$ is orthogonal to the generators of $\varPhi$ iff $\varPhi$ is relative minimal and the function f is given by
$$f = \pm\delta\left|c - g^2\right|^{1/2},\ c \in \mathbb{R},\ g^2 \neq c.$$

We will investigate, now, the right normalized ruled surfaces $\varPhi$, whose support vectors are tangent or orthogonal to the above mentioned geometrically distinguished families of curves of $\varPhi$. From (3.3) and (4.1) we have: $\overline{x}'$ and $\overline{Q}$ are parallel or orthogonal iff

(4.2) $$w^2\left[(\delta g' - \kappa f)\, v + \delta^2\kappa g - \delta' f + \delta f'\right] + \delta\left(f v - \delta^2 g\right)(\delta\lambda + v') = 0$$

or

(4.3) $$-(\delta\lambda + v')\left[(\delta g' - \kappa f)\, v + \delta^2\kappa g - \delta' f + \delta f'\right] + \delta\left(f v - \delta^2 g\right) = 0.$$

From (3.7) and (4.2), resp. (4.3), we find: $\overline{Q}$ is tangential or orthogonal to the u-curves iff

$$(4.4)\quad \begin{aligned}&(\kappa f-\delta g')\,v^3+\left(-\delta^2\kappa g+\delta' f-\delta f'\right)v^2+\delta^2\left[f\,(\kappa-\lambda)-\delta g'\right]v\\&-\delta^2\left[\delta^2 g\,(\kappa-\lambda)-\delta' f+\delta f'\right]=0\end{aligned}$$

or

$$(4.5)\quad \left[f\,(\kappa\lambda+1)-\delta\lambda g'\right]v-\delta^2 g\,(\kappa\lambda+1)+\lambda\,(\delta' f-\delta f')=0,$$

respectively. From (4.4) we infer that $\overline{Q}$ is tangential to the u-curves iff

$$\kappa f-\delta g'=-\delta^2\kappa g+\delta' f-\delta f'=f\,(\kappa-\lambda)-\delta g'=\delta^2 g\,(\kappa-\lambda)-\delta' f+\delta f'=0,$$

that is, iff $\varPhi$ is relative minimal, $\lambda=0$, i.e., $\varPhi$ is orthoid[2] and $f=\pm\delta\left|c-g^2\right|^{1/2}$, $c\in\mathbb{R}$, $g^2\neq c$. From (4.5) we take that $\overline{Q}$ is orthogonal to the u-curves iff

$$f\,(\kappa\lambda+1)-\delta\lambda g'=-\delta^2 g\,(\kappa\lambda+1)+\lambda\,(\delta' f-\delta f')=0,$$

i.e., iff $\kappa\lambda+1=\frac{\delta\lambda g'}{f}$ and $f=\pm\delta\left|c-g^2\right|^{1/2}$, $c\in\mathbb{R}$, $g^2\neq c$, hence $\kappa=\pm g'\left|c-g^2\right|^{-1/2}-\lambda^{-1}$, $\lambda\neq 0$. Therefore, we obtain

Proposition 4.2. The support vector field $\overline{Q}$ of a right normalized skew ruled surface $\varPhi\subset\mathbb{E}^3$ is
(a) tangential to the u-curves of $\varPhi$ iff $\varPhi$ is an orthoid, relative minimal surface and the function f is given by

$$f=\pm\delta\left|c-g^2\right|^{1/2},\ c\in\mathbb{R},\ g^2\neq c.$$

(b) orthogonal to the u-curves of $\varPhi$ iff the conical curvature and the function f are given by

$$\kappa=\pm g'\left|c-g^2\right|^{-1/2}-\lambda^{-1},\ c\in\mathbb{R},\ \lambda\neq 0,\ g^2\neq c \textit{ and } f=\pm\delta\left|c-g^2\right|^{1/2}.$$

From (3.6) and (4.2) we have, that $\overline{Q}$ is tangential or orthogonal to the curved asymptotic lines iff

$$(4.6)\quad \begin{aligned}&(\kappa f-2\delta g')\,v^3+\left(-\delta^2\kappa g+\delta' f-2\delta f'\right)v^2+\delta^2\left[f\,(\kappa-\lambda)+\delta' g-2\delta g'\right]v\\&-\delta^2\left[\delta^2 g\,(\kappa-\lambda)-2\delta' f+2\delta f'\right]=0,\end{aligned}$$

[2] that is, a ruled surface whose striction curve is an orthogonal trajectory of the generators. The ortoid ruled surfaces are characterized by the condition $\lambda=0$.

or

(4.7)
$$\begin{aligned}
&\kappa\left(\kappa f-\delta g'\right)v^3+\left(-\delta^2\kappa^2 g+2\delta'\kappa f-\delta\kappa f'-\delta\delta' g'\right)v^2\\
&+\left[\delta^2 f\left(\kappa^2+2\right)+\delta^2\kappa\left(\lambda f-\delta' g\right)+\delta'\left(\delta' f-\delta f'\right)-\delta^3 g'\left(\kappa+\lambda\right)\right]v\\
&-\delta^2\left[\delta^2 g\left(\kappa^2+2\right)+\delta^2\kappa\lambda g+\left(\kappa+\lambda\right)\left(\delta f'-\delta' f\right)\right]=0,
\end{aligned}$$

respectively. From (4.6) we infer that $\overline{Q}$ is tangential to the curved asymptotic lines iff

$$\begin{aligned}
\kappa f-2\delta g' &= -\delta^2\kappa g+\delta' f-2\delta f'=f\left(\kappa-\lambda\right)+\delta' g-2\delta g'=\\
&=\delta^2 g\left(\kappa-\lambda\right)-2\delta' f+2\delta f'=0.
\end{aligned}$$

Treating the above system in the standard way we find that $\lambda=\delta'=0$. If $\kappa=0$, $\varPhi$ is right helicoid[3], $f=c_1\in\mathbb{R}^*$ and $g=c_2\in\mathbb{R}^*$. If $\kappa\neq 0$, $\varPhi$ is orthoid of constant distribution parameter c_3, $\kappa=\pm 2g'\left|c_4-g^2\right|^{-1/2}$, $c_4\in\mathbb{R}^*$, $g'\neq 0$, $g^2\neq c_4$ and $f=\pm c_3\left|c_4-g^2\right|^{1/2}$. From (4.7) we deduce that $\overline{Q}$ is orthogonal to the curved asymptotic lines iff

$$\begin{aligned}
&\kappa\left(\kappa f-\delta g'\right)=\left(-\delta^2\kappa^2 g+2\delta'\kappa f-\delta\kappa f'-\delta\delta' g'\right)=0\\
&\left[\delta^2 f\left(\kappa^2+2\right)+\delta^2\kappa\left(\lambda f-\delta' g\right)+\delta'\left(\delta' f-\delta f'\right)-\delta^3 g'\left(\kappa+\lambda\right)\right]=0\\
&\delta^2 g\left(\kappa^2+2\right)+\delta^2\kappa\lambda g+\left(\kappa+\lambda\right)\left(\delta f'-\delta' f\right)=0.
\end{aligned}$$

From the system we have, initially, that $\kappa=0$ or $\varPhi$ is a relative minimal surface. If $\kappa=0$ we have $\delta'=0$ or $g'=0$. In both cases the arising systems of differential equations lead to a contradiction. If $\varPhi$ is a relative minimal surface and $\kappa\neq 0$ we arrive again to a contradiction.

So, we can state

Proposition 4.3. The support vector field $\overline{Q}$ of a right normalized skew ruled surface $\varPhi\subset\mathbb{E}^3$ is
(a) tangential to the curved asymptotic lines of $\varPhi$ iff

- (i) $\varPhi$ is right helicoid, the function f is a nonvanishing constant c_1 and the function g is a nonvanishing constant c_2, or
- (ii) $\varPhi$ is orthoid of constant distribution parameter c_3 and the conical curvature and the function f are given by
$$\kappa=\pm 2g'\left|c_4-g^2\right|^{-1/2},\ c_4\in\mathbb{R}^*,\ g'\neq 0,\ g^2\neq c_4$$
$$\text{and } f=\pm c_3\left|c_4-g^2\right|^{1/2}$$

[3] The right helicoids are characterized by the conditions $\delta=c\in\mathbb{R}^*$ and $\kappa=\lambda=0$.

but
(b) it cannot be orthogonal to the curved asymptotic lines of $\varPhi$.

From (3.8) and (4.2), resp. (4.3), we deduce: $\overline{Q}$ is tangential or orthogonal to the $\widetilde{K}$-curves iff

(4.8)
$$2\left(\kappa f-\delta g'\right)v^{4}-\left(2\delta^{2}\kappa g-\delta' f+2\delta f'\right)v^{3}+\delta^{2}\left[2f\left(\kappa-\lambda\right)+\delta' g-2\delta g'\right]v^{2}$$
$$-\delta^{2}\left[2\delta^{2}g\left(\kappa-\lambda\right)-3\delta' f+2\delta f'\right]v-\delta^{4}\delta' g=0$$

or

(4.9)
$$\delta'\left(\kappa f-\delta g'\right)v^{3}+\left[2\delta^{2}f\left(\kappa\lambda+1\right)-\delta^{2}\delta'\kappa g+\delta'^{2}f-\delta\delta' f'-2\delta^{3}\lambda g'\right]v^{2}$$
$$-\delta^{2}\left[2\delta^{2}g\left(\kappa\lambda+1\right)+\delta'\kappa f+2\lambda\left(\delta f'-\delta' f\right)-\delta\delta' g'\right]v$$
$$+\delta^{2}\delta'\left(\delta^{2}\kappa g-\delta' f+\delta f'\right)=0,$$

respectively. From (4.8) we have that $\overline{Q}$ is tangential to the $\widetilde{K}$-curves iff

$$\kappa f-\delta g'=2\delta^{2}\kappa g-\delta' f+2\delta f'=2f\left(\kappa-\lambda\right)+\delta' g-2\delta g'=0,$$
$$2\delta^{2}g\left(\kappa-\lambda\right)-3\delta' f+2\delta f'=\delta'=0,$$

from which we take that $\varPhi$ is relative minimal, $\delta=c_1\in\mathbb{R}^*$, $\lambda=0$ and $f=\pm\left|c_2-c_1^2g^2\right|^{1/2}$, $c_2\in\mathbb{R}$, $c_1^2g^2\neq c_2$. From (4.9) we infer that $\overline{Q}$ is orthogonal to the $\widetilde{K}$-curves iff

$$\delta'\left(\kappa f-\delta g'\right)=2\delta^{2}f\left(\kappa\lambda+1\right)-\delta^{2}\delta'\kappa g+\delta'^{2}f-\delta\delta' f'-2\delta^{3}\lambda g'=0,$$
$$2\delta^{2}g\left(\kappa\lambda+1\right)+\delta'\kappa f+2\lambda\left(\delta f'-\delta' f\right)-\delta\delta' g'=\delta'\left(\delta^{2}\kappa g-\delta' f+\delta f'\right)=0,$$

that is, iff $\varPhi$ is relative minimal or $\delta=c_1\in\mathbb{R}^*$. If $\varPhi$ is relative minimal we arrive at a contradiction.

If $\delta=c_1\in\mathbb{R}^*$, we obtain $\kappa\lambda+1=\frac{c_1\lambda g'}{f}$ and $f=\pm\left|c_2-c_1^2g^2\right|^{1/2}$, $c_2\in\mathbb{R}$, $c_1^2g^2\neq c_2$, hence $\kappa=\pm c_1g'\left|c_2-c_1^2g^2\right|^{-1/2}-\lambda^{-1}$, $\lambda\neq0$. Thus, we deduce

Proposition 4.4. The support vector field $\overline{Q}$ of a right normalized skew ruled surface $\varPhi\subset\mathbb{E}^3$ is
(a) tangential to the $\widetilde{K}$-curves of $\varPhi$ iff $\varPhi$ is an orthoid, relative minimal surface of constant distribution parameter c_1 and the function f is given by
$$f=\pm\left|c_2-c_1^2g^2\right|^{1/2},\ c_2\in\mathbb{R},\ c_1^2g^2\neq c_2.$$

(b) orthogonal to the $\widetilde{K}$-curves of Φ iff Φ has constant distribution parameter c_1 and the conical curvature and the function f are given by

$$\kappa = \pm c_1 g' \left|c_2 - c_1^2 g^2\right|^{-1/2} - \lambda^{-1},\ c_2 \in \mathbb{R},\ \lambda \neq 0,\ c_1^2 g^2 \neq c_2$$
$$\text{and } f = \pm \left|c_2 - c_1^2 g^2\right|^{1/2}.$$

The following table summarizes the results:

$\overline{Q}$ is ...	Type of the ruled surface Φ	f, g
orthogonal to the generators	relative minimal	$f=\pm\delta\left\|c-g^2\right\|^{1/2}$, $c\in\mathbb{R}$, $g^2\neq c$
tangential to the u-curves	orthoid, relative minimal	$f=\pm\delta\left\|c-g^2\right\|^{1/2}$, $c\in\mathbb{R}$, $g^2\neq c$
orthogonal to the u-curves	$\kappa=\pm g'\left\|c-g^2\right\|^{-1/2}-\lambda^{-1}$, $c\in\mathbb{R}$, $\lambda\neq 0$, $g^2\neq c$	$f=\pm\delta\left\|c-g^2\right\|^{1/2}$
tangential to the curved asympt. lines	right helicoid	$f=c_1\in\mathbb{R}^*$, $g=c_2\in\mathbb{R}^*$
	orthoid, $\delta=c_3\in\mathbb{R}^*$, $\kappa=\pm 2g'\left\|c_4-g^2\right\|^{-1/2}$, $c_4\in\mathbb{R}^*$, $g'\neq 0$, $g^2\neq c_4$	$f=\pm c_3\left\|c_4-g^2\right\|^{1/2}$
orthogonal to the curved asympt. lines	-	-
tangential to the $\widetilde{K}$-curves	orthoid, relative minimal, $\delta=c_1\in\mathbb{R}^*$	$f=\pm\left\|c_2-c_1^2g^2\right\|^{1/2}$, $c_2\in\mathbb{R}$, $c_1^2g^2\neq c_2$
orthogonal to the $\widetilde{K}$-curves	$\delta=c_1\in\mathbb{R}^*$, $\kappa=\pm c_1g'\left\|c_2-c_1^2g^2\right\|^{-1/2}-\lambda^{-1}$, $c_2\in\mathbb{R}$, $\lambda\neq 0$, $c_1^2g^2\neq c_2$	$f=\pm\left\|c_2-c_1^2g^2\right\|^{1/2}$

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