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Spectral synthesis and Euclidean motions

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Abstract: This paper exhibits the basics of non-commutative spectral analysis and spectral synthesis. Classical spectral synthesis deals with translation invariant function spaces on commutative groups. The building bricks of such spaces are the exponentials and exponential monomials. The main result is due to L. Schwartz about spectral synthesis on the reals. Counterexamples show, however, that this theorem cannot be generalized directly for functions in several variables. In this work the background of a possible extension is studied and presented. The idea is based on Gelfand pairs and spherical functions. "Translation" invariance is replaced by invariance with respect to the action of affine groups.

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1. Introduction

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. We consider the space $\mathcal{C}(G)$ of all complex valued

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continuous functions on a locally compact Abelian group G, which is a locally convex topological linear space with respect to the pointwise linear operations (addition, multiplication with scalars) and to the topology of uniform convergence on compact sets. Closed linear translation invariant subspaces of this space are called *varieties.* Continuous homomorphisms of G into the additive topological group of complex numbers, and into the multiplicative topological group of nonzero complex numbers, respectively, are called *additive*, and *exponential functions*, respectively. A function is a *polynomial* if it belongs to the algebra generated by all continuous additive functions. An *exponential monomial* is the product of a polynomial and an exponential. It turns out that finite dimensional varieties are exactly those spanned by exponential monomials. Hence exponential monomials can be considered as basic building blocks of varieties. A given variety may or may not contain any nonzero exponential monomial. If every nonzero subvariety of it contains an exponential monomial, then we say that *spectral analysis* holds for the variety. There is, however, an even stronger property of some varieties, namely, if all exponential monomials in the variety span a dense subspace of the variety. In this case we say that the variety is *synthesizable.* If every subvariety of a variety is synthesizable, then we say that *spectral synthesis* holds for the variety. It follows, that for spectral synthesis for a variety implies spectral analysis. If spectral analysis, respectively, spectral synthesis holds for every variety on an Abelian group, then we say that *spectral analysis*, respectively, *spectral synthesis* holds *on the Abelian group.* A famous and pioneer result of L. Schwartz [1] exhibits the situation by showing that if the group is the reals with the Euclidean topology, then spectral synthesis holds: there are sufficiently many exponential monomials in every variety in the sense that their linear hull is dense in the variety. Besides partial generalizations of Schwartz's theorem (see e.g. [2, 5]), attempts to extend Schwartz's result to higher dimensions failed. Finally, in [6] counterexamples are given proving the impossibility of a direct extension.

In a former paper (see [13]) we presented a possible way how to extend Schwartz's result to functions in several variables. Our idea is based on the observation that the basic tools of commutative spectral analysis and synthesis can be adopted in non-commutative situations using the theory of Gelfand pairs and spherical functions (see [3, 7, 8, 4, 9]). Our point is to show that the original version of Schwartz's result can be considered as a "spherical spectral synthesis" result on the affine group $\mathrm{Aff}\, SO(n) = SO(n) \ltimes \mathbb{R}^n$ of *proper Euclidean motions* with $n = 1$, as that group is identical with $\mathbb{R}$. Hence, a proper generalization is to prove the corresponding "spherical" result for $n > 1$ – and this is possible. Of course, it is reasonable to ask what is the situation on other affine groups. Here we consider the case of the group of *all Euclidean motions*, that is the affine group $\mathrm{Aff}\, O(n) = O(n) \ltimes \mathbb{R}^n$ of all orthogonal transformations on $\mathbb{R}^n$. We note that this group reduces to $\mathbb{Z}_2 \ltimes \mathbb{R}$ in the case $n = 1$, so the situation is different from the case of the special orthogonal group. Nevertheless, in this paper we show that spherical spectral synthesis holds on the affine group $\mathrm{Aff}\, O(n)$ for each $n \geq 1$. We shall use the notation and terminology of [13], but in the following section we give a short summary about the necessary preliminaries.

2. Preliminaries on spherical spectral synthesis

Let G be a locally compact topological group and K a compact subgroup with normalized Haar measure ω. The set $\mathcal{C}(G)$ of all continuous complex valued functions on G is a locally convex topological vector space when equipped with the pointwise linear operations (addition and multiplication by complex scalars) and with the topology of uniform convergence on compact sets. The topological dual of this space is identified with the space $\mathcal{M}_c(G)$ of all compactly supported complex Borel measures on G which is a unital topological algebra when equipped with the pointwise linear operations, with the convolution of measures and with the weak*-topology. The Dirac measure of the form δ_y is supported at y in G, and all Dirac measures span a dense subalgebra in $\mathcal{M}_c(G)$.

Functions f in $\mathcal{C}(G)$ satisfying $f(kxl) = f(x)$ for each x in G and k, l in K are called *K-invariant*. They form a closed subspace $\mathcal{C}(G//K)$ in $\mathcal{C}(G)$. The notation is related to the fact that these functions can be considered as continuous functions on the *double coset space* $G//K$ with respect to K. In fact, $\mathcal{C}(G//K)$ can be identified with the space of all continuous complex valued functions on the locally compact factor space $G//K$. The dual $\mathcal{M}_c(G//K)$ of $\mathcal{C}(G//K)$ can be identified with a closed subalgebra of $\mathcal{M}_c(G)$, its elements are called *K-invariant measures* (see [13]). Clearly, $\mathcal{C}(G//K)$ is a module over the algebra $\mathcal{M}_c(G//K)$

(see [13]). We say that (G, K) is a *Gelfand pair*, if $\mathcal{M}_c(G//K)$ is commutative.

The *projection* of $\mathcal{C}(G)$ onto $\mathcal{C}(G//K)$ is defined in the following way: for each f in $\mathcal{C}(G)$ we let

$$f^{\#}(x) = \int_K \int_K f(kxl)\, d\omega(k)\, d\omega(l) \ \ (x \in G);$$

then $f \mapsto f^{\#}$ is a continuous linear mapping. The function $f^{\#}$ is called the *projection of* f, and f is K-invariant if and only if $f = f^{\#}$.

The adjoint mapping from $\mathcal{M}_c(G//K)$ onto $\mathcal{M}_c(G)$ is defined as

$$\langle \mu^{\#}, f \rangle = \langle \mu, f^{\#} \rangle;$$

$\mu^{\#}$ is called the *projection of* μ, and μ is K-invariant if and only if $\mu = \mu^{\#}$.

Using the projection we define *K-translation* as follows: for each f in $\mathcal{C}(G//K)$ and y in G we let

$$\tau_y f(x) = \delta_{y^{-1}}^{\#} * f(x) = \int_K f(ykx)\, d\omega(k) \ \ (x \in G).$$

We call $\tau_y f$ the *K-translate* of f with *increment y*. It turns out that all K-translations form a commuting family if and only if (G, K) is a Gelfand pair (see [13]).

The closed linear subspace V in $\mathcal{C}(G//K)$ is called a *K-variety* if it is K-translation invariant: for each f in V the function $\tau_y f$ is in V whenever y is in G. K-varieties are exactly the closed submodules of the module $\mathcal{C}(G//K)$ over the algebra $\mathcal{M}_c(G//K)$. The smallest K-variety including f is called the *K-variety of f* and it is denoted by $\tau(f)$.

Now we are about to formulate the basic problems of spectral analysis and spectral synthesis for K-varieties. Observe that in the case of commutative G the space $G//K$ can be identified with the locally compact factor group G/K and $\mathcal{C}(G//K)$ is the space of continuous complex functions on this group. Moreover, K-translation coincides with ordinary translation hence a reasonable concept of spectral analysis and spectral synthesis for K-varieties will provide a proper generalization of the theory. For this we need the basic functions: the substitutes of exponential monomials. From now on we always assume that (G, K) is a Gelfand pair.

The normalized common K-invariant eigenfunctions of all K-translations are called *K-spherical functions*; we note that we do not assume boundedness of K-spherical functions. In other words, the nonzero K-invariant continuous function $f : G \to \mathbb{C}$ is a K-spherical function if and only if it satisfies

$$\int_K f(xky)\, d\omega(k) = f(x)f(y)$$

for each x, y in G. It follows $f(e) = 1$, where e is the identity in G. It is easy to see that K-spherical functions are exactly the normalized common eigenfunctions of all K-translations (see [13]).

Given the K-spherical function s and the element y in G we define the *modified difference* corresponding to s with increment y as

$$\Delta_{s;y} = \delta^{\#}_{y^{-1}} - s(y)\delta_e.$$

Obviously, the normalized K-invariant function s is a K-spherical function if and only if

$$\Delta_{s;y} * s(x) = 0$$

for each x, y in G. The iterates of modified differences are used to define K-monomials as follows: the continuous K-invariant function f on G is called a *K-monomial* if there is a K-spherical function s and a natural number n such that

$$\Delta_{s;y_1,y_2,\dots,y_{n+1}} * f(x) = [\Pi_{k=1}^{n+1} \Delta_{s;y_k}] * f(x) = 0$$

holds for each $x, y_1, y_2, \dots, y_{n+1}$ in G. Here Π denotes convolution product. If f is nonzero, then s is uniquely determined and the we say that f is an *s-monomial*, and its *degree* is defined as the smallest n satisfying the above requirement.

Now we are in the position to define spectral analysis and spectral synthesis for K-varieties. Given a K-variety V we say that *K-spectral analysis holds for V* if every nonzero K-variety in V contains a K-spherical function. This equivalent to the requirement that every nonzero K-variety in V contains a nonzero K-monomial. If all K-monomials in K span a dense subspace in K, then we say that *V is K-synthesizable.* We say that *K-spectral synthesis holds for V* if every K-variety in V is K-synthesizable. If K-spectral analysis, resp. K-spectral synthesis holds for every K-variety on G, then we say that *K-spectral analysis*, resp. *K-spectral synthesis holds on G*. It is obvious that for commutative G these concepts coincide with the corresponding concepts related to ordinary spectral analysis, resp. spectral synthesis on the locally compact Abelian group G/K. Nevertheless, we may have Gelfand pairs (G,K) with different choices of non-commutative G, where the compact subgroup K is not even normal subgroup. In [13] we considered the case where G is the affine group $\mathrm{Aff}\,SO(n)$ of the special orthogonal group $SO(n)$ over $\mathbb{R}^n$ and we proved K-spectral synthesis for this group when K is the compact subgroup of $\mathrm{Aff}\,SO(n)$ topologically isomorphic to $SO(n)$. As $SO(1)$ is the trivial group, hence $\mathrm{Aff}\,SO(1)$ is topologically isomorphic to $\mathbb{R}$, this result provides a proper generalization of L. Schwartz's spectral synthesis result for functions in several variables. In the subsequent paragraphs we shall present a similar result with $O(n)$ instead of $SO(n)$. In the real case $n=1$ this gives a "spherical" spectral synthesis result for even continuous functions.

3. Affine groups

In this section we specialize the above setting to affine groups. Let n be a positive integer and V an n-dimensional vector space over the field $\mathbb{F}$. We denote by $GL_n(V,\mathbb{F})$ the *general linear group* of V, i.e. the group of all linear automorphisms of V. Sometimes we can use any of the notations $GL(V,\mathbb{F})$, $GL(n,\mathbb{F})$ or $GL(V)$. Given a subgroup H in $GL(V,\mathbb{F})$ the *affine group* $\mathrm{Aff}\,(H,V)$ of H over V is an extension of H by the group of translations in V: it is the semidirect product

$$\mathrm{Aff}\,(H,V) = H \ltimes V.$$

We can also write simply $\mathrm{Aff}\,H$ for $\mathrm{Aff}\,(H,V)$. We recall that $\mathrm{Aff}\,H$ is a group with basic set $H\times V$ and the group operation is defined by

$$(h,u)\cdot(k,v) = \big(h\circ k, u + h\cdot v\big)$$

for each h,k in H and u,v in V. Here $\circ$ is the composition of automorphisms and $h\cdot v$ is the image of v under the automorphism h. The identity of this operation is $(id,0)$, and the inverse of (h,u) is $(h^{-1},-h^{-1}\cdot u)$ as it is easy to check. Hence the operation in $\mathrm{Aff}\,H$ imitates the composition of the affine mappings of the form

$$(h,u)\cdot v = h\cdot v + u$$

with h in H and u, v in V. It is easy to verify that the elements of $\mathrm{Aff}\, H$ of the form $(h, 0)$ form a subgroup in $\mathrm{Aff}\, H$ isomorphic to H – we shall identify this subgroup with H. Similarly, the elements of $\mathrm{Aff}\, H$ of the type (id, u) form a normal subgroup in $\mathrm{Aff}\, H$ which is isomorphic to V – we shall identify this normal subgroup with V.

In this work we shall deal with the case $V = \mathbb{R}^n$ only, and we assume that K is a compact subgroup of $GL(\mathbb{R}^n, \mathbb{R})$ which will be written simply as $GL(\mathbb{R}^n)$. We let $G = \mathrm{Aff}\,(K, \mathbb{R}^n) = \mathrm{Aff}\, K$, the *affine group* of K over $\mathbb{R}^n$. Clearly, G is a locally compact group when equipped with the product topology. Let $f : \mathrm{Aff}\, K \to \mathbb{C}$ be a continuous K-invariant function. This means

$$f(h, x) = f\big((k, 0) \cdot (h, x) \cdot (l, 0)\big)$$

holds for each h, k, l in K and x in $\mathbb{R}^n$. This is equivalent to

$$f(h, x) = f(k \circ h \circ l, k \cdot x)$$

for each h, k, l in K and x in $\mathbb{R}^n$. Choosing an arbitrary k in K and taking $l = h^{-1} \circ k^{-1}$ we have that f is K-invariant if and only if

$$f(h, x) = f(id, k \cdot x)$$

holds for each h, k in K and x in $\mathbb{R}^n$. In other words, f is independent of the first variable, hence it can be identified with a continuous function on $\mathbb{R}^n$, and this function is invariant with respect to the action of K on $\mathbb{R}^n$. The set of all continuous K-invariant functions on $\mathrm{Aff}\, K$ is identified with the closed linear subspace of all continuous complex valued functions φ in $\mathcal{C}(\mathbb{R}^n)$ satisfying

$$\varphi(x) = \varphi(k \cdot x)$$

for each k in K and x in $\mathbb{R}^n$. Obviously, we may consider $\mathcal{C}(\mathbb{R}^n)$ as embedded in $\mathcal{C}(\mathrm{Aff}\, K)$, since $\mathbb{R}^n$ is embedded in $\mathrm{Aff}\, K$. We shall denote the space of all continuous K-invariant functions – as a subspace of $\mathcal{C}(\mathbb{R}^n)$ – by $\mathcal{C}_K(\mathbb{R}^n)$. Similarly, the topological dual of this space is identified with a closed subspace of the space $\mathcal{M}_c(\mathbb{R}^n)$ of all compactly supported complex Borel measures on $\mathbb{R}^n$ when the latter is equipped with the weak*-topology. The dual of $\mathcal{C}_K(\mathbb{R}^n)$ will be denoted by $\mathcal{M}_K(\mathbb{R}^n)$, its elements are the K-invariant measures when considered as measures on

Aff K – their support is a compact subset of $\mathbb{R}^n$. Hence the measure μ in $\mathcal{M}_c(\mathbb{R}^n)$ is K-invariant if and only if

$$\int_{\mathbb{R}^n} f\,d\mu = \int_{\mathbb{R}^n} f(kv)\,d\mu(v)$$

holds for each f in $\mathcal{C}(\mathbb{R}^n)$ and k in K. Finally, in this situation convolution of K-invariant measures, and also convolution of K-invariant measures with K-invariant functions reduces to ordinary convolution of measures, and to ordinary convolution of measures with functions on $\mathbb{R}^n$, as it is easy to check.

4. Euclidean motions

In this paper we focus on the affine group $\operatorname{Aff} O(n)$ of $O(n) = O(n,\mathbb{R})$, the *orthogonal group*. The elements of this group are called *Euclidean motions*. In other words

$$\operatorname{Aff} O(n) = O(n,\mathbb{R}) \ltimes \mathbb{R}^n$$

with the operation

$$(O,x)\cdot(P,y) = (O\circ P, x + O\cdot y)$$

for each orthogonal operators O, P and vectors x, y in $\mathbb{R}^n$. This operation corresponds to the composition of the affine mappings $\varphi : u \mapsto x + O\cdot u$ and $\psi : u \mapsto y + P\cdot u$. Indeed, we have

$$\varphi\circ\psi(u) = x + O\cdot\psi(u) = x + O\cdot(y + P\cdot u) = x + O\cdot y + (O\circ P)\cdot u.$$

The orthogonal group $O(n)$ is compact and its normalized Haar measure will be denoted by ω.

For each f in $\mathcal{C}(\operatorname{Aff} O(n))$ the $O(n)$-projection of f is given by

$$f^{\#}(h,x) = \int_{O(n)}\int_{O(n)} f(l,kx)\,d\omega(l)\,d\omega(k),$$

for each h in $O(n)$ and x in $\mathbb{R}^n$. Clearly, this is independent of h and we simply write $f^{\#}(x)$ for $f^{\#}(h,x)$. Accordingly, the projection of the measure $\delta_{(h,x)}$ is

$$\langle\delta^{\#}_{(h,x)}, f\rangle = f^{\#}(x) = \int_{O(n)} f(l,kx)\,d\omega(k)\,d\omega(l)$$

for each h in $O(n)$ and x in $\mathbb{R}^n$. We can simply write $\delta_x^\#$ for $\delta_{(h,x)}^\#$, as this measure is in $\mathcal{M}_c(\mathbb{R}^n)$, it is independent of h. It follows that the $O(n)$-translate with y in $\mathbb{R}^n$ of each f in $\mathcal{C}_r(\mathbb{R}^n)$ is given by the equation

$$\tau_y f(x) = \delta_{y^{-1}}^\# * f(x) = \int_{O(n)} f(x + ky)\, d\omega(k)$$

for each x in $\mathbb{R}^n$. Clearly, the K-translation operators τ_y for each y in $\mathbb{R}^n$ form a commuting family, hence $(\mathrm{Aff}\, O(n), O(n))$ is a Gelfand pair. The continuous radial function $f : \mathbb{R}^n \to \mathbb{C}$ is an $O(n)$-spherical function if and only if it satisfies

$$\int_{O(n)} f(x + ky)\, d\omega(k) = f(x)f(y), \;\; f(0) = 1.$$

The following theorem is of basic importance (see [13]).

Theorem 4.1. The continuous $O(n)$-invariant function $f : \mathbb{R}^n \to \mathbb{C}$ is an $O(n)$-spherical function if and only if it is C^∞ and it is an eigenfunction of the Laplacian on $\mathbb{R}^n$ with $f(0) = 1$.

The proof can be found in [9]. Although the proof in [9] is given for $SO(n)$, but it is easy to see that it works for $O(n)$ as well. In fact, $O(n)$-invariant functions and measures coincide with $SO(n)$-invariant functions and measures unless $n = 1$ as it is shown by the following theorem.

Theorem 4.2. For $n \geq 2$ the space of $O(n)$-invariant functions on $\mathbb{R}^n$ coincides with the space of $SO(n)$-invariant functions on $\mathbb{R}^n$.

Proof. We note that for $n \geq 1$ $SO(n)$ acts transitively on S^{n-1}: if $\|x\| = \|y\| = 1$, then there exists k in $SO(n)$ such that $k \cdot x = y$. Now suppose that f is $SO(n)$-invariant on $\mathbb{R}^n$: $f(k \cdot x) = f(x)$ holds for each x in $\mathbb{R}^n$ and k in $SO(n)$. Let $\|x\| = \|y\|$ for $x, y \neq 0$ in $\mathbb{R}^n$. Then we have

$$\frac{x}{\|x\|} = \frac{y}{\|y\|} = 1,$$

hence there exists k in $SO(n)$ such that

$$k \cdot \frac{x}{\|x\|} = \frac{y}{\|y\|},$$

i.e. $k \cdot x = y$, hence $f(x) = f(k \cdot x) = f(y)$. In other words, we have proved that if x, y are in $\mathbb{R}^n$ with $\|x\| = \|y\|$, then $f(x) = f(y)$.

Now let x be in $\mathbb{R}^n$ and k in $O(n)$. Then $\|x\| = \|k \cdot x\|$, hence $f(x) = f(k \cdot x)$, and our statement is proved. $\Diamond$

Corollary 4.3. For $n \geq 2$ the space of $O(n)$-invariant measures on $\mathbb{R}^n$ coincides with the space of $SO(n)$-invariant measures on $\mathbb{R}^n$.

Hence for $n \geq 2$ the $O(n)$-invariant functions on the group $\mathrm{Aff}\, O(n)$ are those functions on $\mathbb{R}^n$ which are rotation invariant, i.e. they depend on the norm only. These functions are called *radial*, and the space of all continuous radial functions will be denoted by $\mathcal{C}_r(\mathbb{R}^n)$. Similarly, the space of all compactly supported $O(n)$-invariant measures will be denoted by $\mathcal{M}_r(\mathbb{R}^n)$ – they are called *radial measures*.

In the case $n = 1$ every continuous function on $\mathbb{R}$ and every measure in $\mathcal{M}_c(\mathbb{R})$ is $SO(1)$-invariant, as $SO(1) = \{id\}$, but $O(1) = \{id, -id\}$, hence $O(1)$-invariant functions are exactly the even functions on $\mathbb{R}$.

5. Spherical spectral synthesis on the Euclidean motion group

In order to prove $O(n)$-spectral analysis and $O(n)$-spectral synthesis for the Euclidean motion group we need some general results from [13]. In the subsequent paragraphs $\mathrm{Ann}\, V$ denotes the annihilator in $\mathcal{M}_c(G//K)$ of the submodule V in $\mathcal{C}(G//K)$. We recall (see [13]) that a maximal ideal in a commutative complex algebra is called an *exponential maximal ideal*, if its residue algebra is the complex field.

Theorem 5.1. Suppose that (G, K) is a Gelfand pair. Let V be a K-variety on G. Then K-spectral analysis holds for V if and only if every maximal ideal in $\mathcal{M}_c(G//K)$ containing $\mathrm{Ann}\, V$ is exponential. In other words, K-spectral analysis holds for V if and only if every maximal ideal in the residue algebra $\mathcal{M}_c(G//K)/\mathrm{Ann}\, V$ is exponential.

Corollary 5.2. K-spectral analysis holds on G if and only if every maximal ideal inthe algebra $\mathcal{M}_c(G//K)$ is exponential.

For synthesizability of varieties we have the following result (see [10, 11, 12, 13]).

Theorem 5.3. The nonzero K-variety V is K-synthesizable if and only if

$$\operatorname{Ann} V = \bigcap_M \bigcap_{n\in\mathbb{N}} (\operatorname{Ann} V + M^{n+1}),$$

where the first intersection is taken for all exponential maximal K-ideals M containing $\operatorname{Ann} V$ and $\mathcal{M}_c(G//K)/M^{n+1}$ is finite dimensional.

In [13] we introduced the following definition: let R be a commutative complex topological algebra with unit. The proper closed ideal I in R is called *synthesizable* if

$$I = \bigcap_M \bigcap_{n\in\mathbb{N}} (I + M^{n+1}), \tag{5.1}$$

where the first intersection is taken for all exponential maximal ideals M containing I and R/M^{n+1} is finite dimensional. Accordingly, we say that spectral analysis holds on R, if every maximal ideal is exponential, and spectral synthesis holds on R, if every closed ideal I in R satisfies the above equation. In particular, K-spectral analysis holds on the Gelfand pair (G, K) if spectral analysis holds on $\mathcal{M}_c(G//K)$, and K-spectral synthesis holds on G if and only if this spectral synthesis holds on $\mathcal{M}_c(G//K)$. The following theorem is a simple consequence of the above definitions.

Theorem 5.4. Let R, Q be commutative complex topological algebras with unit. If spectral analysis, resp. spectral synthesis holds on R, and there exists a continuous surjective homomorphism $\Phi : R \to Q$, then spectral analysis, resp. spectral synthesis holds on Q.

Proof. Let M be a maximal ideal with Q, then $M = \Phi(N)$ with some ideal N in R such that $N = \Phi^{-1}(M)$. Let $\psi : Q \to Q/M$ denote the natural mapping, then ψ is continuous and open. We define

$$F(r) = \psi\big(\Phi(r)\big)$$

for each r in R, then $F : R \to Q/M$ is a continuous homomorphism. Clearly, F is surjective. If $F(r) = 0$, then $\Phi(r)$ is in $\operatorname{Ker}\psi = M$, that is, r is in N. It follows that $R/N \cong Q/M$, a field, hence N is a maximal ideal. If N is exponential, then M is exponential, too, which proves the statement about spectral analysis.

Let J be a proper closed ideal in Q and let $I = \Phi^{-1}(J)$. Then I is a proper closed ideal in R, hence it is synthesizable, by assumption. It follows that (5.1) holds. Then we have

$$J = \bigcap_{\Phi(M)} \bigcap_{n\in\mathbb{N}} (J + \Phi(M)^{n+1}), \tag{5.2}$$

and here the first intersection extends for all maximal ideals $\Phi(M)$ containing J. Indeed, the left hand side is clearly a subset of the right hand side. Suppose now that $q = \Phi(r)$ is not in J, then r is not in I. By equation (5.1), there exists a maximal ideal M with $I \subseteq M$, and a natural number n_0 such that r is not in $I + M^{n_0+1}$, hence $q = \Phi(r)$ is not in $J + \Phi(M)^{n_0+1}$. It follows that (5.2) holds.

What is left is to show that $Q/\Phi(M)^{n+1}$ is finite dimensional for every maximal ideal M with $I \subseteq M$ and for each natural number n. We define $F : R/M^{n+1} \to Q/\Phi(M)^{n+1}$ by

$$F(r + M^{n+1}) = \Phi(r) + \Phi(M)^{n+1}$$

for each r in R. We have to show that the value of F is independent of the choice of r in the coset $r+M^{n+1}$. Suppose that $r-r_1$ is in M^{n+1}, that is $r - r_1 = \sum x_1x_2\cdots x_{n+1}$, where the sum is finite, and $x_1, x_2, \dots, x_{n+1}$ is in M. Then

$$\Phi(r) = \Phi(r_1) + \sum \Phi(x_1)\Phi(x_2)\cdots\Phi(x_{n+1}),$$

hence $\Phi(r)$ and $\Phi(r_1)$ are in the same coset of $\Phi(M)^{n+1}$. As F is clearly a surjective homomorphism, we infer that $Q/\Phi(M)^{n+1}$ is finite dimensional and the proof is complete. $\Diamond$

Using the above results we can extend Schwartz's spectral synthesis result for the Euclidean motion group in the following form.

Theorem 5.5. For each positive integer n $O(n)$-spectral synthesis holds for the Gelfand pair $(\mathrm{Aff}\, O(n), \mathbb{R}^n)$.

Proof. For $n \geq 2$ the statement follows from Corollary 4.3 and form the results in [13]. The case $n = 1$ can be treated separately, using the ideas in [13]. We recall again that the $O(1)$-invariant continuous functions on $\mathbb{R}$ are exactly the continuous even functions. It is enough to show that there exists a surjective algebra homomorphism of the measure algebra $\mathcal{M}_c(\mathbb{R})$ onto $\mathcal{M}_r(\mathbb{R})$. Indeed, for each μ in $\mathcal{M}_c(\mathbb{R})$ and f continuous even function we define μ_r as the restriction of μ to the space of continuous even functions. Clearly, the mapping $\mu \mapsto \mu_r$ is an algebra homomorphism of $\mathcal{M}_c(\mathbb{R})$ onto $\mathcal{M}_r(\mathbb{R}^n)$. $\Diamond$

6. $O(n)$-spherical functions and monomials

The radial eigenfunctions of the Laplacian in $\mathbb{R}^n$ for $n \geq 2$ can be obtained from the radial form of the Laplacian: suppose that $s_\lambda : \mathbb{R}^n \to \mathbb{C}$ is C^∞, and

$$\Delta s_\lambda = \lambda s_\lambda, \;\; s_\lambda(0) = 1$$

holds with some complex λ. If s_λ is radial, then $s_\lambda(x) = \varphi(\|x\|)$ holds with some C^∞ even function $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying the Bessel differential equation

$$\frac{d^2\varphi}{dr^2} + \frac{n-1}{r}\frac{d\varphi}{dr} = \lambda\varphi$$

and $\varphi(0) = 1$. Using this the explicit form of s_λ can be given as

$$s_\lambda(x) = \Gamma\Big(\frac{n}{2}\Big) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\Gamma\big(k+\frac{n}{2}\big)} \Big(\frac{\|x\|}{2}\Big)^{2k}$$

for each x in $\mathbb{R}^n$.

In the case of $n = 1$ $O(1)$-spherical functions are the functions of the form

$$s_\lambda(x) = \cosh \lambda x$$

for each x in $\mathbb{R}$, where λ is any complex number (see [13]).

The $O(n)$-monomials can be described in the same way as in [14]: given the $O(n)$-spherical function s_λ with some complex λ all s_λ-monomials of degree at most d are linear combinations of the functions $\frac{d^j s_\lambda}{d\lambda^j}$ with $j = 0, 1, \ldots, d$. The explicit form can be given using the formula

$$\frac{d^j s_\lambda}{d\lambda^j}(x) = \Gamma\Big(\frac{n}{2}\Big) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!\Gamma\big(k+j+\frac{n}{2}\big)} \Big(\frac{\|x\|}{2}\Big)^{2(k+j)}$$

for each x in $\mathbb{R}^n$.

For instance, in the special case $n = 1$ we can formulate our result as follows.

Theorem 6.1. Let V be a linear space of even continuous complex valued functions on the real line which is closed under uniform convergence on compact sets. Suppose that for each f in V the function $x \mapsto f(x+y) + f(x-y)$ is in V whenever y is in $\mathbb{R}$. Then there exists a complex λ such that the function $x \mapsto \cosh \lambda x$ is in V. In addition, every function in V is a uniform limit on compact sets of a sequence of linear combinations of functions of the form $x \mapsto x^k \sinh \lambda x$ and $x \mapsto x^k \cosh \lambda x$ where λ is a complex number such that these functions are in V.

References

[1] SCHWARTZ, L.: *Théorie générale des fonctions moyenne-périodiques*, Ann. of Math. **48(4)**, (1947), 857–929.

[2] EHRENPREIS, L.: *Mean periodic functions. I. Varieties whose annihilator ideals are principal*, Amer. J. Math., **77**, (1955), 293–328.

[3] HELGASON, S.: Differential geometry and symmetric spaces, Academic Press, New York-London, 1962.

[4] HELGASON, S.: *Groups and geometric analysis*, volume 83 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2000.

[5] ELLIOTT, R. J. : Some results in spectral synthesis. *Proc. Cambridge Philos. Soc.*, 61:395–424, 1965.

[6] GUREVIČ, D. I.: Counterexamples to a problem of L. Schwartz. *Funkcional. Anal. i Priložen.*, 9(2):29–35, 1975.

[7] DIEUDONNÉ, J.: Treatise on analysis. Vol. VI, Academic Press, Inc., New York-London, 1978.

[8] TAKEUCHI, M.: Modern spherical functions, American Mathematical Society, Providence, RI, 1994.

[9] VAN DIJK, G.: Introduction to harmonic analysis and generalized Gelfand pairs, Walter de Gruyter & Co., Berlin, 2009.

[10] SZÉKELYHIDI, L.: Harmonic and spectral analysis, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

[11] SZÉKELYHIDI, L.: *Annihilator methods in discrete spectral synthesis*, Acta Math. Hung., **143(2)**, (2014), 141–366,

[12] SZÉKELYHIDI, L.: *Annihilator methods for spectral synthesis on locally compact Abelian groups*, Monatsh. Math., **180(2)**, (2016), 157–371,

[13] SZÉKELYHIDI, L.: *Spherical spectral synthesis*, arXiv:1607.07079v1 [math.FA] (to appear in Acta Math. Hung.)

[14] SZÉKELYHIDI, L.: *On spectral synthesis in several variables*, arXiv:1607.07079v1 [math.FA] (to appear in Advances in Operator Theory)