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Relationships between inclusions for relations and inequalities for corelations

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Abstract: Let X and Y be quite arbitrary sets. Then, a function U on the power set $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ will be called a corelation on X to Y. Thus, complementation and closure (interior) operations on X are corelations on X. Moreover, for any two corelations U and V on X to Y, we shall write $U \leq V$ if $U(A) \subseteq V(A)$ for all $A \subseteq X$. Thus, the family of all corelations on X to Y also forms a complete poset (partially ordered set).

Formerly, we have established a partial Galois connection $(\triangleright, \triangleleft)$ between relations and corelations. Now, by using this, we shall establish some further relationships between inclusions for relations and inequalities for corelations.

For instance, for some very particular corelations U and V on X to Y, with $U^{\triangleleft} \leq V^{\triangleleft}$, we shall prove the existence of an union-preserving corelation Φ on X to Y which separates U and V in the sense that $U \leq \Phi \leq V$.

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1. Introduction

Let X and Y be quite arbitrary sets without having any particular algebraic or topological structure. Moreover, denote the power sets (families of all subsets) of X and Y by $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, respectively.

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In our former paper [17], a function U on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ has been briefly called a corelation on X to Y. Thus, complementation and closure (interior) operations on X are corelations on X.

If R is a relation on X to Y, i.e., $R \subseteq X \times Y$, then the function $R^{\triangleright}$, defined by $R^{\triangleright}(A) = R[A] = \bigcup_{x \in A} R(x)$ for all $A \subseteq X$, can be easily seen to be a union-preserving corelation on X to Y which may be identified with R.

Conversely, if U is a corelation on X to Y, then we may naturally define a relation $U^{\triangleleft}$ on X to Y such that $U^{\triangleleft}(x) = U(\{x\})$ for all $x \in X$. Moreover, for the corelation U, we may also naturally write $U^{\circ} = \big(U^{\triangleleft}\big)^{\triangleright}$.

Namely, for any two corelations U and V on X to Y, we may also naturally write $U \leq V$ if $U(A) \subseteq V(A)$ for all $A \subseteq X$. Thus, the family of all corelations on X to Y also forms a complete poset (partially ordered set).

Moreover, we can show that the functions $\triangleright$ and $\triangleleft$ establish a partial Galois connection in the sense that, for an arbitrary relation R and a quasi-increasing corelation U on X to Y, we have $R^{\triangleright} \leq U$ if and only if $R \subseteq U^{\triangleright}$.

Now, a corelation U on X to Y may be briefly called open (quasi-increasing) if $U \leq U^{\circ}$ $\big(U^{\circ} \leq U\big)$. Moreover, we can easily see that U is union-preserving if and only if $U = U^{\circ}$. That is, U is both open and quasi-increasing.

In our present paper, by using the functions $\triangleright$, $\triangleleft$ and $\circ$, we shall establish some further connections between inclusions for relations and inequalities for corelations. For instance, we shall show that $U \leq V \implies U^{\circ} \leq V^{\circ} \iff U^{\triangleleft} \subseteq V^{\triangleleft}$.

Moreover, we shall show that for an open corelation U and a quasi-increasing corelation V on X to Y, with $U^{\triangleleft} \leq V^{\triangleleft}$, there exists a union-preserving corelation Φ on X to Y which separates U and V in the sense that $U \leq \Phi \leq V$.

2. A few basic facts on corelations

The following definition was first introduced in our former paper [17]. It differs from that of Pöschel and Rössinger [11].

Definition 2.1. A function U on one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$ is called a *corelation* on X to Y.

Remark 2.2. Note that if a subset A of X is not in the domain of U, then by the corresponding definition for relations we have $U(A) = \emptyset$. Therefore, every corelation on X to Y is actually a corelation of X to Y.

Definition 2.3. A corelation U on X to Y is called

(1) *increasing* if $U(A) \subseteq U(B)$ for all $A \subseteq B \subseteq X$;

(2) *quasi-increasing* if $U(\{x\}) \subseteq U(A)$ for all $x \in A \subseteq X$;

(3) *union-preserving* if $U\big(\bigcup \mathcal{A}\big) = \bigcup_{A \in \mathcal{A}} U(A)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$.

Remark 2.4. In particular, a corelation U on X to itself will be simply called a corelation on X.

Thus, a corelation U on X may be called *extensive, intensive, involutive* and *idempotent* if $A \subseteq U(A)$, $U(A) \subseteq A$, $U\big(U(A)\big) = A$ and $U\big(U(A)\big) = U(A)$ for all $A \subseteq X$, respectively.

Moreover, an increasing involutive (idempotent) corelation is called a *involution (projection) operation.* While, an extensive (intensive) projection operation is called a *closure (interior) operation.*

Simple reformulations of properties (1) and (2) in Definition 2.3 give the following two theorems.

Theorem 2.5. For a corelation U on X to Y, the following assertions are equivalent:

(1) U is quasi-increasing;

(2) $\bigcup_{x \in A} U\big(\{x\}\big) \subseteq U(A)$ for all $A \subseteq X$.

Theorem 2.6. For a corelation U on X to Y, the following assertions are equivalent:

(1) U is increasing;

(2) $\bigcup_{A \in \mathcal{A}} U(A) \subseteq U\big(\bigcup \mathcal{A}\big)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$;

(3) $U\big(A_1\big) \cup U(A_2) \subseteq U\big(A_1 \cup A_2\big)$ for all $A_1, A_2 \subseteq X$.

Hence, it is clear that in particular we also have the following

Corollary 2.7. For a corelation U on X to Y, the following assertions are equivalent:

(1) U is union-preserving;

(2) U is increasing and $U\big(\bigcup \mathcal{A}\big) \subseteq \bigcup_{A\in\mathcal{A}} U(A)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$.

However, it is now more important to note that we also have the following theorem which has also been proved, in a different way, by Pataki [10].

Theorem 2.8. For a corelation U on X to Y, the following assertions are equivalent:

(1) U is union-preserving;

(2) $U(A) = \bigcup_{x\in A} U\big(\{x\}\big)$ for all $A \subseteq X$.

Proof. To prove the implication (2) $\Longrightarrow$ (1), note that if (2) holds, then U is increasing.

Therefore, by Theorem 2.6, we have $\bigcup_{A\in\mathcal{A}} U(A) \subseteq U\big(\bigcup \mathcal{A}\big)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$. Thus, to obtain (1), we need only prove the converse inclusion.

For this, note that if $\mathcal{A} \subseteq \mathcal{P}(X)$, then by (2) we have

$$U\big(\bigcup \mathcal{A}\big) = \bigcup_{x\in\bigcup\mathcal{A}} U\big(\{x\}\big).$$

Therefore, if $y \in U\big(\bigcup \mathcal{A}\big)$, then there exists $x \in \bigcup \mathcal{A}$ such that $y \in U\big(\{x\}\big)$. Thus, in particular there exists $A_0 \in \mathcal{A}$ such that $x \in A_0$, and so $\{x\} \subseteq A_0$. Hence, by using the increasingness of U, we can already see that

$$y \in U\big(\{x\}\big) \subseteq U(A_0) \subseteq \bigcup_{A\in\mathcal{A}} U(A).$$

Therefore, $U\big(\bigcup \mathcal{A}\big) \subseteq \bigcup_{A\in\mathcal{A}} U(A)$ also holds. $\Diamond$

From this theorem, by Theorem 2.5, it is clear that in particular we also have

Corollary 2.9. For a corelation U on X to Y, the following assertions are equivalent:

(1) U is union-preserving;

(2) U is quasi-increasing and $U(A) \subseteq \bigcup_{x \in A} U\big(\{x\}\big)$ for all $A \subseteq X$.

Now, for a preliminary illustration of the usefulness of Theorem 2.8, we can easily establish the following

Example 2.10. If R is a relation on X to Y, and $U(A) = R[A]$ for all $A \subseteq X$, then U is the unique union-preserving corelation on X to Y such that $R(x) = U\big(\{x\}\big)$ for all $x \in X$.

To check the union-preservingness of U, recall that $R[A] = \bigcup_{x \in A} R(x)$ with $R(x) = \big\{ y \in Y : \ (x, y) \in R \big\}$.
Therefore,

$$U(A) = R[A] = \bigcup_{x \in A} R(x) = \bigcup_{x \in A} R\big[\{x\}\big] = \bigcup_{x \in A} U\big(\{x\}\big)$$

for all $A \subseteq X$. Thus, Theorem 2.8 can be applied to get the required assertion.

Remark 2.11. Conversely, we can also easily see that if U is a union-preserving corelation U on X to Y, and R is a relation on X to Y such that $R(x) = U\big(\{x\}\big)$ for all $x \in X$, then $U(A) = R[A]$ for all $A \subseteq X$.

3. Some pointwise operations and an inequality for corelations

Here, to distinguish the pointwise complements and differences for corelations from the global ones, we shall use bold notations.

Definition 3.1. If U and V are corelations on X to Y, then for any $A \subseteq X$ we define

$$U^{\mathbf{c}}(A) = U(A)^c = Y \setminus U(A) \quad \text{and} \quad \big(U \boldsymbol{\setminus} V\big)(A) = U(A) \setminus V(A).$$

Remark 3.2. Thus, if in particular $U(A) = Y$ for all $A \subseteq X$, then

$$(U \setminus V)(A) = U(A) \setminus V(A) = Y \setminus V(A) = V(A)^c = V^{\boldsymbol{c}}(A)$$

for all $A \subseteq X$. Therefore, in this particular case, we have $U \setminus V = V^{\boldsymbol{c}}$.

Moreover, to distinguish the pointwise intersections and unions for corelations from the global ones, we shall use lattice theoretic notations.

Definition 3.3. If $\mathcal{U}$ is a family of corelations on X to Y, then for any $A \subseteq X$ we define

$$\big(\bigwedge \mathcal{U}\big)(A) = \bigcap_{U \in \mathcal{U}} U(A) \qquad \text{and} \qquad \big(\bigvee \mathcal{U}\big)(A) = \bigcup_{U \in \mathcal{U}} U(A)\,.$$

Remark 3.4. Thus, for any two corelations U and V on X to Y, we also write

$$U \wedge V = \bigwedge \{U, V\} \qquad \text{and} \qquad U \vee V = \bigvee \{U, V\}\,.$$

Now, by using the corresponding definitions and Theorem 2.8, we can easily prove the following two theorems.

Theorem 3.5. If $\mathcal{U}$ is a family of increasing (quasi-increasing) corelations on X to Y, then $\bigwedge \mathcal{U}$ and $\bigvee \mathcal{U}$ are also increasing (quasi-increasing) corelations on X to Y.

Theorem 3.6. If $\mathcal{U}$ is a family of union-preserving corelations on X to Y, then $\bigvee \mathcal{U}$ is also a union-preserving corelation on X to Y.

Proof. Under the notation $V = \bigvee \mathcal{U}$, for any $A \subseteq X$ we have

$$V(A) = \big(\bigvee \mathcal{U}\big)(A) = \bigcup_{U \in \mathcal{U}} U(A)\,.$$

Hence, by using that each member of $\mathcal{U}$ is unioin-preserving, we can see that

$$V(A) = \bigcup_{U \in \mathcal{U}} U(A) = \bigcup_{U \in \mathcal{U}} \bigcup_{x \in A} U\big(\{x\}\big) = \bigcup_{x \in A} \bigcup_{U \in \mathcal{U}} U\big(\{x\}\big) = \bigcup_{x \in A} V\big(\{x\}\big)\,.$$

Therefore, by Theorem 2.8, the corelation V is also union-preserving. $\Diamond$

The following example shows that the corresponding assertion fails to hold for the corelation $\bigwedge \mathcal{U}$.

Example 3.7. Let X be a set such that $\operatorname{card}(X) > 1$, and for any $A \subseteq X$ define

$$U(A) = \Delta_X[A] \qquad \text{and} \qquad V(A) = \Delta_X^c[A],$$

where Δ_X is the identity function of X and $\Delta_X^c = X^2 \setminus \Delta_X$.

Then, by Example 2.10, it is clear that U and V are union-preserving corelations on X. Moreover, by taking $A \subseteq X$, with $\operatorname{card}(A) = 2$, we can easily see that

$$\big(U \wedge V\big)(A) = A, \qquad \text{but} \qquad \bigcup_{x \in A} \big(U \wedge V\big)\big(\{x\}\big) = \emptyset.$$

Therefore, the increasing corelation $U \wedge V$ is very far from being union-preserving.

In the sequel, since set inclusion is not, in general, a convenient partial order for functions, we shall use the following

Definition 3.8. For any two sets X and Y, denote by $\mathcal{Q}(X, Y)$ the family of all corelations on X to Y.

Moreover, for any two U, $V \in \mathcal{Q}(X, Y)$, define $U \leq V$ if $U(A) \subseteq V(A)$ for all $A \subseteq X$.

Thus, we can easily prove the following

Theorem 3.9. With the above inequality relation $\leq$, the family $\mathcal{Q}(X, Y)$ forms a complete poset.

Proof. It is clear that the relation $\leq$ considered in Definition 3.8 is a partial order (reflexive, transitive and antisymmetric) relation on $\mathcal{Q}(X, Y)$.

Moreover, if $\mathcal{U} \subseteq \mathcal{Q}(X, Y)$ and $V = \bigvee \mathcal{U}$, i. e.,

$$V(A) = \bigcup_{U \in \mathcal{U}} U(A)$$

for all $A \subseteq X$, then it can be easily seen that $V = \sup\big(\mathcal{U}\big)$. Thus, the poset $\mathcal{Q}(X, Y)$ is sup-complete.

The fact that $\mathcal{Q}(X, Y)$ is inf-complete can be proved quite similarly by showing that $\bigwedge \mathcal{U} = \inf\big(\mathcal{U}\big)$. $\Diamond$

Remark 3.10. Note that, by a basic theorem of Birkhoff [1, p. 112], a poset is inf-complete if and only if it is sup-complete.

Moreover, by our former papers [3, 22], this theorem can be extended to an arbitrary goset (generalized ordered set) even with a simpler proof.

Definition 3.11. In the sequel, the families of the quasi-increasing, increasing and union-preserving members of $\mathcal{Q}(X, Y)$ will be denoted by $\mathcal{Q}_1(X, Y)$, $\mathcal{Q}_2(X, Y)$ and $\mathcal{Q}_3(X, Y)$, respectively.

Remark 3.12. Thus, we evidently have

$$\mathcal{Q}_3(X, Y) \subseteq \mathcal{Q}_2(X, Y) \subseteq \mathcal{Q}_1(X, Y) \subseteq \mathcal{Q}(X, Y).$$

Moreover, by using Theorems 3.9 and 3.6, we can also prove the following

Theorem 3.13. With the corresponding restriction of the inequality relation $\leq$ considered in Definition 3.8, the family $\mathcal{Q}_i(X, Y)$, with $i = 1, 2, 3$, is also a complete poset.

Remark 3.14. Now, by Remark 3.10, $\inf(\mathcal{U})$ also exists in $\mathcal{Q}_3(X, Y)$. However, because of Example 3.7, it can be strictly smaller than $\bigwedge \mathcal{U}$. Therefore, the latter notation may cause some confusions.

4. Two natural maps between relations and corelations

In [24], by using the corresponding definitions of Höhle and Kubiak [9] and the notations of Davey and Priestley [6, p. 55], we have introduced the following

Definition 4.1. For any relation R and corelation U on X to Y, we define a corelation $R^{\triangleright}$ and a relation $U^{\triangleleft}$ on X to Y such that

$$R^{\triangleright}(A) = R[A] \qquad \text{and} \qquad U^{\triangleleft}(x) = U\big(\{x\}\big)$$

for all $A \subseteq X$ and $x \in X$.

Moreover, for the corelation U, we also define

$$U^{\circ} = U^{\triangleleft\triangleright} = \big(U^{\triangleleft}\big)^{\triangleright}.$$

From Example 2.10, by the above definition, it is clear that we have

Theorem 4.2. If R is a relation on X to Y, then $R^{\triangleright}$ is a union-preserving corelation on X to Y.

Moreover, by using the above definition and the latter theorem, we can also easily prove the following

Theorem 4.3. If U is a corelation on X to Y, then U° is a union-preserving corelation on X to Y such that, for any $A \subseteq X$, we have

$$U^{\circ}\big(A\big) = U^{\triangleleft}[\,A\,] = \bigcup_{x \in A} U\big(\{x\}\big)\,.$$

Proof. To check this equality, note that, by the corresponding definitions, we have

$$U^{\circ}(A) = \big(U^{\triangleleft}\big)^{\triangleright}(A) = U^{\triangleleft}[\,A\,] = \bigcup_{x \in A} U^{\triangleleft}(x) = \bigcup_{x \in A} U\big(\,\{x\}\big)\,.$$

$\Diamond$

Thus, in particular we can also state

Corollary 4.4. If U is a corelation on X to Y, then for any $x \in X$ we have

$$U^{\circ}\big(\{x\}\big) = U^{\triangleleft}(x) = U\big(\{x\}\big)\,.$$

Now, for a preliminary illustration of Theorem 4.3, we can easily establish the following two examples

Example 4.5. If U is a corelation on X such that $U\big(\{x\}\big) = \{x\}$ for all $x \in X$, then $U^{\circ}(A) = A$ for all $A \subseteq X$. That is, U° is the identity corelation on X.

Example 4.6. If U is the complement operation on X, i. e., $U\,(A) = A^{c}$ for all $A \subseteq X$, then U° is a union-preserving corelation on X such that, for any $A \subseteq X$, we have

$$U^{\circ}(A) = \begin{cases} \emptyset & \text{if} \quad \operatorname{card}(A) = 0\,, \\ A^{c} & \text{if} \quad \operatorname{card}(A) = 1\,, \\ X & \text{if} \quad \operatorname{card}(A) > 1\,. \end{cases}$$

To check this, note that, by Theorem 4.3 and De Morgan's law, we have

$$U^{\circ}\big(A\big) = \bigcup_{x \in A} U\big(\{x\}\big) = \bigcup_{x \in A} \{x\}^{c} = \Big(\bigcap_{x \in A} \{x\}\Big)^{c}.$$

Now, by using Definition 4.1 and Theorem 4.3, we can also easily prove

Theorem 4.7. For any relation R and corelation U on X to Y, we have

(1) $R^{\triangleright\triangleleft} = R$; (2) $R^{\triangleright\circ} = R^{\triangleright}$;

(3) $U^{\circ\triangleleft} = U^{\triangleleft}$; (4) $U^{\circ\circ} = U^{\circ}$.

Proof. By the corresponding definitions, it is clear that

$$R^{\triangleright\triangleleft}(x) = \big(R^{\triangleright}\big)^{\triangleleft}(x) = R^{\triangleright}\big(\{x\}\big) = R\,\big[\,\{x\}\,\big] = R(x)$$

for all $x \in X$. Therefore, (1) is true.

Now, by using assertion (1) and Definition 4.1, we can easily see that

$$R^{\triangleright\circ} = R^{\triangleright\triangleleft\triangleright} = R^{\triangleright} \qquad \text{and} \qquad U^{\circ\triangleleft} = U^{\triangleleft\triangleright\triangleleft} = U^{\triangleleft}.$$

Therefore, (2) and (3) are also true.

Furthermore, by using Theorem 4.3 and its corollary, we can also easily see that

$$U^{\circ\circ}(A) = \bigcup_{x\in A} U^{\circ}\big(\{x\}\big) = \bigcup_{x\in A} U\big(\{x\}\big) = U^{\circ}(A)$$

for all $A \subseteq X$. Therefore, (4) is also true. $\Diamond$

By Theorems 2.5, 2.8 4.2, and 4.3, it is clear that we also have the following two theorems.

Theorem 4.8. For a corelation U on X to Y, the following assertions are equivalent:

(1) $U^{\circ} \leq U$; (2) U is quasi-increasing;

Remark 4.9. In the sequel, a corelation U on X to Y will be called *open* if the converse inequality $U \leq U^{\circ}$ holds true.

Theorem 4.10. For a corelation U on X to Y, the following assertions are equivalent:

(1) $U^{\circ} = U$;

(2) U is union-preserving;

(3) $U = R^{\triangleright}$ for some relation R on X to Y.

Finally, we note that from Remark 4.9 and Theorems 4.8 and 4.10, by Theorem 4.3, it is clear that the following theorem is also true.

Theorem 4.11. If U is a corelation on X to Y, then

(1) U is open if and only if $U(A) \subseteq U^{\triangleleft}[A]$ for all $A \subseteq X$;

(2) U is quasi-increasing if and only if $U^{\triangleleft}[A] \subseteq U(A)$ for all $A \subseteq X$;

(3) U is union-preserving if and only if $U(A) = U^{\triangleleft}[A]$ for all $A \subseteq X$.

Thus, in particular, we can also state

Corollary 4.12. If U is a corelation on X to Y, then the following assertions are equivalent:

(1) U is union-preserving; (2) U is open and quasi-increasing.

5. Some further properties of the maps $\triangleright$ and $\triangleleft$

Theorem 5.1. For any two relations R and S on X to Y, the following assertions are equivalent:

(1) $R \subseteq S$; (2) $R^{\triangleright} \leq S^{\triangleright}$.

Proof. If (1) holds, then by the corresponding definitions, it is clear that (2) also holds.

Conversely, if (2) holds, then by the forthcoming Theorem 5.3 we can see that $R^{\triangleright\triangleleft} \subseteq S^{\triangleright\triangleleft}$. Thus, by Theorem 4.7, assertion (1) also holds. $\Diamond$

Remark 5.2. From this theorem, we can see that $\triangleright$ is an increasing, injective function of $\mathcal{P}(X \times Y)$ to $\mathcal{Q}(X, Y)$. Moreover, from Theorem 4.10, we can see that it is actually onto $\mathcal{Q}_3(X, Y)$.

Concerning corelations, we can only prove the following less convenient

Theorem 5.3. If U and V are corelations on X to Y, then among the assertions

(1) $U \leq V$; (2) $U^{\circ} \leq V^{\circ}$; (3) $U^{\triangleleft} \subseteq V^{\triangleleft}$;

the implications $(1) \Longrightarrow (2) \Longleftrightarrow (3)$ hold.

Proof. If (1) holds, then by Theorem 4.3 and the corresponding definitions, it is clear that (2) and (3) also hold.

While, if (3) holds, then by Theorem 5.1 we also have $U^{\lhd\rhd} \leq V^{\lhd\rhd}$. Hence, since $\lhd\rhd = \circ$, it is clear that (2) also holds.

Conversely if (2) holds, then because of (1) $\Longrightarrow$ (3) we also have $U^{\circ\lhd} \subseteq V^{\circ\lhd}$. Hence, since by Theorem 4.7 we have $\circ\lhd = \lhd$, it is clear that (3) also holds. $\lozenge$

Remark 5.4. The fact that (3) does not implies (1) even if either U or V is union-preserving is quite obvious from Example 4.5.

However, from Theorem 5.3, by using Theorem 4.10, we can still infer

Corollary 5.5. For any two union-preserving corelations U and V on X to Y, the following assertions are equivalent:

(1) $U \leq V$; (2) $U^{\lhd} \subseteq V^{\lhd}$.

Now, in addition to Theorem 5.3, we can also easily prove the following

Theorem 5.6. For any two corelations U and V on X to Y,

(1) $U^{\circ} \leq V$ implies $U^{\circ} \leq V^{\circ}$;

(2) $U^{\circ} \leq V^{\circ}$ implies $U^{\circ} \leq V$ if V is quasi-increasing.

Proof. If $U^{\circ} \leq V$, then by Theorem 5.3 we also have $U^{\circ\circ} \leq V^{\circ}$. Hence, since by Theorem 4.7 we have $\circ\circ = \circ$, it is clear that $U^{\circ} \leq V^{\circ}$, and thus (1) also holds.

Moreover, if V is quasi-increasing then by Theorem 4.8 we $V^{\circ} \leq V$. Hence, by the transitivity of $\leq$, it is clear that $U^{\circ} \leq V^{\circ}$ implies $U^{\circ} \leq V$, and thus (2) also holds. $\lozenge$

Now, as an immediate consequence of Theorems 5.3 and 5.6, we can also state

Corollary 5.7. For any two corelations U and V on X to Y,

(1) $U^{\circ} \leq V$ implies $U^{\lhd} \subseteq V^{\lhd}$;

(2) $U^{\lhd} \subseteq V^{\lhd}$ implies $U^{\circ} \leq V$ if V is quasi-increasing.

Remark 5.8. From Theorem 5.3, we we can see that $\circ$ and $\lhd$ are increasing functions of $\mathcal{Q}(X, Y)$ to $\mathcal{P}(X \times Y)$.

Moreover, from Theorems 4.7, 4.8, 4.10 and 5.6 and Corollary 5.4, we can see that the maps $\circ$ and $\lhd$ have some further useful properties.

Now, by using our former theorems, we can also easily prove the following two theorems.

Theorem 5.9. If U is a corelation on X to Y, then

$$R^{\rhd} \leq U \quad \text{implies} \quad R \subseteq U^{\lhd}$$

for any relation R on X to Y.

Proof. If $R^{\rhd} \leq U$ holds, then by Theorem 5.3 we also have $R^{\rhd\lhd} \subseteq U^{\lhd}$. Hence, since by Theorem 4.7 we have $R^{\rhd\lhd} = R$, it is clear that $R \subseteq U^{\lhd}$ also holds. $\Diamond$

Theorem 5.10. For a corelation U on X to Y, the following assertions are equivalent:

(1) U is quasi-increasing;

(2) $R \subseteq U^{\lhd}$ implies $R^{\rhd} \leq U$ for any relation R on X to Y.

Proof. If (1) holds, then by Theorem 4.8, we have $U^{\circ} \leq U$. Moreover, if $R \subseteq U^{\lhd}$, then by Theorem 5.1 we also have $R^{\rhd} \leq U^{\lhd\rhd}$, and thus $R^{\rhd} \leq U^{\circ}$. Hence, by the transitivity of $\leq$, it is clear $R^{\rhd} \subseteq U$, and thus (2) also holds.

Conversely if (2) holds, then from the trivial inequality $U^{\lhd} \leq U^{\lhd}$, we can infer that $U^{\lhd\rhd} \leq U$, and thus $U^{\circ} \leq U$. Therefore, by Theorem 4.8, assertion (1) also holds. $\Diamond$

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 5.11. For an arbitrary relation R and a quasi-increasing corelation U on X to Y, the following assertions are equivalent:

(1) $R^{\rhd} \leq U$; (2) $R \subseteq U^{\lhd}$.

Remark 5.12. This corollary shows that the function $\rhd$ and the restriction of $\lhd$ to $\mathcal{Q}_1(X, Y)$ establish a Galois connection between the complete posets $\mathcal{P}(X \times Y)$ and $\mathcal{Q}_1(X, Y)$.

Therefore, the extensive theory of Galois connections [2, 8, 6] could have been applied here. However, because of the simplicity of Definition 4.1, it was more convenient to use some, more elementary, direct proofs.

Interesting examples, applications and generalizations for Galois connections can also be found in the books Ganter and Wille [7] and Denecke, Erné and Wismath [5], and our former papers listed in the References.

6. Some applications of Theorems 5.3 and 4.10

Now, as an immediate consequence of Theorems 5.3, we can state the following

Theorem 6.1. If U, V and Φ are corelations on X to Y, then among the assertions

$$(1)\quad U \leq \Phi \leq V; \qquad (2)\quad U^{\circ} \leq \Phi^{\circ} \leq V^{\circ}; \qquad (3)\quad U^{\triangleleft} \subseteq \Phi^{\triangleleft} \subseteq V^{\triangleleft};$$

the implications $(1) \Longrightarrow (2) \Longleftrightarrow (3)$ hold.

From this theorem, by using Theorem 4.10 and the corresponding definition, we can easily derive the following

Theorem 6.2. If U and V are arbitrary and Φ is a union-preserving corelation on X to Y, then among the assertions

(1) $U \leq \Phi \leq V$, (2) $U^{\circ} \leq \Phi \leq V^{\circ}$;

(3) $U^{\circ}(A) \subseteq \Phi(A) \subseteq V^{\circ}(A)$ for all $A \subseteq X$;

(4) $U\big(\{x\}\big) \subseteq \Phi\big(\{x\}\big) \subseteq V\big(\{x\}\big)$ for all $x \in X$;

the implications $(1) \Longrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4)$ hold.

Proof. By Theorem 4.10, we have $\Phi = \Phi^{\circ}$. Hence, by Theorem 6.1, we can see that (1) implies (2), and (2) is equivalent to the inclusion $U^{\triangleleft} \subseteq \Phi^{\triangleleft} \subseteq V^{\triangleleft}$.

However, by the corresponding definitions, this inclusion is equivalent to (4), and (2) is equivalent to (3). Therefore, the required implications are true. $\Diamond$

Now, by using this theorem, we can also easily prove the following

Theorem 6.3. If U and V are corelations and R is a relation on X to Y such that

$$U^{\triangleleft} \subseteq R \subseteq V^{\triangleleft},$$

then $\Phi = R^{\triangleright}$ is a union-preserving corelation on X to Y such that

$$U^{\circ} \leq \Phi \leq V^{\circ}.$$

Proof. By Theorem 4.2, it is clear that Φ is a union-preserving corelation on X to Y. Moreover, from the assumption of the theorem we can see that

$$U^{\triangleleft}(x) \subseteq R\,(x) \subseteq V^{\triangleleft}(x)\,, \qquad \text{and thus} \qquad U\big(\{x\}\big) \subseteq R\,(x) \subseteq V\big(\{x\}\big)$$

for all $x \in X$. Hence, since

$$R\,(x) = R\,[\,\{x\}\,] = R^{\triangleright}\big(\{x\}\big) = \Phi\big(\{x\}\big)\,,$$

we can infer that

$$U\big(\{x\}\big) \subseteq \Phi\,\big(\{x\}\big) \subseteq V\big(\{x\}\big)$$

for all $x \in X$. Thus, by Theorem 6.2, the required inequalities are also true. $\Diamond$

From this theorem, it is clear that we can also state

Corollary 6.4. If U and V are corelations on X to Y such that

$$U^{\triangleleft} \subseteq V^{\triangleleft},$$

then there exists a union-preserving corelation Φ on X to Y such that

$$U^{\circ} \leq \Phi \leq V^{\circ}.$$

Proof. To check this, note that, for the relation $R = U^{\triangleleft}$ or $V^{\triangleleft}$, Theorem 6.3 can be applied. $\Diamond$

Now, by using this corollary, we also easily establish the following sandwich theorem with union-preservingness.

Theorem 6.5. If U is an open and V is a quasi-increasing corelation on X to Y such that
$$U^{\triangleleft} \subseteq V^{\triangleleft},$$
then there exists a union-preserving corelation Φ on Y to Y such that
$$U \leq \Phi \leq V.$$

Proof. By Corollary 6.4, there exists a union-preserving corelation Φ on X to Y such that
$$U^{\circ} \leq \Phi \leq V^{\circ}.$$
Moreover, by Remark 4.9 and Theorem 4.8, we now have
$$U \leq U^{\circ} \qquad \text{and} \qquad V^{\circ} \leq V.$$
Hence, by the transitivity of the relation $\leq$, it is clear that the required inequalities are also true. $\Diamond$

Remark 6.6. Note that, by Theorems 6.2 and 6.1, the inclusion $U^{\triangleleft} \subseteq V^{\triangleleft}$ is a natural necessary condition for the existence of such corelation Φ.

However, the openness of U and the quasi-increasingness of V are far from being necessary. Therefore, Theorem 6.5 should be substantially improved.

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