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Distinguishing between sharp and non sharp normal numbers

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Abstract: In 2015, De Koninck, Kátai and Phong introduced the concept of sharp normal numbers and proved that almost all real numbers are sharp normal numbers in the sense of the Lebesgue measure. They also proved that although the Champernowne number is normal in base 2, it is not sharp in that base. Here, we prove that various real numbers are sharp normal numbers, while others are not.

1. Introduction and notation

Given an integer $q \geq 2$, in a recent paper, De Koninck, Kátai and Phong [4] introduced the concept of *base q strong normal number*, shortly after called *base-q sharp normal number* as follows. First recall that the

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discrepancy of a set of N real numbers $x_1, \ldots, x_N$ is the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N} \sum_{\substack{\nu=1 \\ \{x_\nu\}\in[a,b)}}^{N} 1 - (b-a) \right|. \tag{1.1}$$

Here and in what follows, $\{y\}$ stands for the fractional part of the real number y. Recall also that a sequence $(x_n)_{n\geq 1}$ of real numbers is said to be *uniformly distributed modulo 1* if for each subinterval $[a,b)$ of $[0,1)$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : \{x_n\} \in [a,b)\} = b - a.$$

We then say that a real number α is *normal in base* q (or q-*normal*) if the sequence $(\alpha q^n)_{n\geq 1}$ is uniformly distributed modulo 1.

This paves the way for the introduction of the notions of "sharp distribution modulo 1" and of a "sharp normal number".

For each positive integer N, let

$$M = M_N = \lfloor \delta_N \sqrt{N} \rfloor, \qquad \text{where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty. \tag{1.2}$$

We shall say that a sequence of real numbers $(x_n)_{n\geq 1}$ is *sharply uniformly distributed* modulo 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \qquad \text{as} \qquad N \to \infty$$

for every choice of δ_N satisfying (1.2). Given a fixed integer $q \geq 2$, we then say that an irrational number α is a *sharp normal number* in base q (or a sharp q-normal number) if the sequence $(\alpha q^n)_{n\geq 1}$ is sharply uniformly distributed modulo 1. In [4], it is shown that, given a fixed base $q \geq 2$, the Lebesgue measure of the set of all those real numbers $\alpha \in [0,1]$ which are not sharp q-normal is equal to 0.

In a more recent paper [5], we proved that, given a fixed integer $q \geq 2$ and letting $\tau_q(n)$ stand for the number of ways of writing n as a product of q positive integers, then, if α is a sharp normal number in base q, the sequence $(\alpha\tau_q(n))_{n\geq 1}$ is uniformly distributed modulo 1. In that same paper, other properties of sharp normal numbers were established.

Given an integer $q \geq 2$ and a real number $\gamma \in (0,1)$, we will say that a real number α is a γ-sharp normal number in base q if, by setting $x_n = \{\alpha q^n\}$ for $n = 1, 2, \ldots$ and

$$M = M_N = \lfloor \delta_N N^\gamma \rfloor, \qquad \text{where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty, \tag{1.3}$$

we have that

$$D(x_{N+1},\ldots,x_{N+M}) \to 0 \qquad \text{as} \qquad N \to \infty$$

for every choice of δ_N satisfying (1.3).

Observe that in [4], it was shown that the binary Champernowne number

$$\theta := 0.1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,1101\,1110\,1111\,\ldots$$

is not a sharp normal number. Similarly, one can prove that θ is not a γ-sharp normal number for any $\gamma \in (0,1)$.

Here, we further explore the topic of γ-sharp normal numbers.

2. Main results

From here on, we let q stand for a fixed integer ≥ 2. Let $\wp = \{p_1, p_2, \ldots\}$ stand for the set of all primes. Given a positive integer n, we let $\overline{n}$ stand for the concatenation of the base q digits of the number n.

In 1946, Copeland and Erdős [2] showed that the now called *Copeland-Erdős number*

$$\theta := 0.\overline{p_1}\,\overline{p_2}\,\overline{p_3}\ldots$$

is q-normal. Here, we will prove the following.

Theorem 2.1. Given any $\gamma \in (0,1)$, the number θ is not a binary γ-sharp normal number.

In the same 1946 paper, Copeland and Erdős conjectured that if $f \in \mathbb{Z}[x]$ is a polynomial of positive degree such that $f(x) > 0$ for $x > 0$, then the number $\beta = 0.\overline{f(1)}\,\overline{f(2)}\,\overline{f(3)}\ldots$ is a normal number in base 10. This was proved to be true in 1952 by Davenport and Erdős [3]. Here we prove the following.

Theorem 2.2. Given a positive integer r, the real number

$$\beta = 0.\overline{1^r}\,\overline{2^r}\,\overline{3^r}\,\ldots$$

is not a binary sharp normal number.

Fix an integer $q \geq 2$. Given an integer $n \geq 2$, let $p(n)$ stand for its smallest prime factor and write $\overline{p(n)}$ for the concatenation of the digits of $p(n)$ in base q. In 2014, we showed [6] that the number $\eta = 0.\overline{p(2)}\,\overline{p(3)}\,\overline{p(4)}\ldots$ is a q-normal number. Here, we prove the following.

Theorem 2.3. Given an arbitrary real number $\gamma \in (0,1)$, the real number

$$\eta = 0.\overline{p(2)}\,\overline{p(3)}\,\overline{p(4)}\ldots$$

is a γ-sharp normal number in base q.

Fix an integer $q \geq 2$. Let $\wp_0, \wp_1, \ldots, \wp_{q-1}, \mathcal{R}$ be disjoint sets of primes such that

$$\wp = \wp_0 \cup \wp_1 \cup \cdots \cup \wp_{q-1} \cup \mathcal{R}$$

and such that $\#\mathcal{R} < \infty$. Assume also that

$$\max_{0 \leq i < j \leq q-1} \max_{\frac{x}{\log^5 x} \leq y \leq x} \left| \frac{\pi([x, x+y] \cap \wp_i)}{\pi([x, x+y] \cap \wp_j)} - 1 \right| \to 0 \quad \text{as } x \to \infty.$$

More over let Λ stand for the empty word and for each $p \in \wp$, let

$$H(p) := \begin{cases} \ell & \text{if} \quad p \in \wp_\ell, \\ \Lambda & \text{if} \quad p \in \mathcal{R}. \end{cases}$$

Given an integer $n \geq 2$ written as $n = q_1^{a_1} \cdots q_r^{a_r}$, where $q_1 < \cdots < q_r$ are primes and each $a_i \in \mathbb{N}$, let

$$S(n) := H(q_1) \ldots H(q_r).$$

Further set $S(1) = 1$. In 2011, we showed [7] that the number $0.\text{Concat}(\overline{S(n)} : n \in \mathbb{N})$ is a q-normal number. Here, we prove the following.

Theorem 2.4. Given an arbitrary real number $\gamma \in (0,1)$, the real number

$$0.\overline{S(1)}\,\overline{S(2)}\,\overline{S(3)}\,\ldots$$

is a γ-sharp normal number in base q.

We also have the following.

Theorem 2.5. Fix an integer $q \geq 2$. Given any pair of prime numbers $u < v$, let $\epsilon(u,v)$ stand for the unique integer $\ell \in \{0,1,\ldots,q-1\}$ such that

$$\frac{\ell}{q} \leq \frac{\log u}{\log v} < \frac{\ell+1}{q}.$$

For each positive integer $n = q_1^{a_1} \cdots q_r^{a_r}$, let

$$\xi(n) = \begin{cases} \epsilon(q_1,q_2)\,\epsilon(q_2,q_3)\ldots\epsilon(q_{r-1},q_r) & \text{if} \quad \omega(n) \geq 2, \\ \Lambda & \text{if} \quad \omega(n) \leq 1. \end{cases}$$

Then, given any real number $\gamma \in (0,1)$, the number

$$0.\mathrm{Concat}(\xi(n) : n \in \mathbb{N})$$

is a γ-sharp normal number in base q.

Let $\mathcal{P}$ be a set of primes and set $\pi_{\mathcal{P}}(x) := \#\{p \leq x : p \in \mathcal{P}\}$. Moreover, let $\mathcal{N} = \{n_1, n_2, \ldots\}$ be the semi-group generated by $\mathcal{P}$. Let $F(x) \in \mathbb{Z}[x]$ be a monic polynomial of positive degree t. Assume that there exists a positive constant τ such that

$$\lim_{x\to\infty} \frac{\pi_{\mathcal{P}}(x)}{\mathrm{li}(x)} = \tau,$$

where $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$. Fix an integer $q \geq 2$. Given a positive integer n, let $\overline{n}$ stand for the concatenation of the digits of n in base q and consider the real number

$$\eta_0 = 0.\overline{F(n_1)}\,\overline{F(n_2)}\,\overline{F(n_3)}\,\ldots$$

It was proved by German and Kátai [8] that η_0 is a q-normal number. Their proof uses essentially the same method as the one used in the paper of Bassily and Kátai [1], along with other ideas of E. Wirsing, H. Davenport and L.K. Hua. Using these ideas, one could prove the following.

Theorem 2.6. The q-normal number η_0 is not sharp.

3. Proof of Theorem 2.1

First observe that it has been proved by Montgomery [9] that, given any small $\varepsilon > 0$,

$$\pi(x+y)-\pi(x) = (1+o(1))\frac{y}{\log x} \quad \text{uniformly for} \quad x^{\frac{7}{12}+\varepsilon} \le y \le x. \tag{3.1}$$

Let $t \ge 2$ be an integer sufficiently large so that $\gamma \le 1 - \dfrac{1}{2^t}$. Moreover, for each integer $k \ge 1$, let $x_k = 2^{2^k}$ and $y_k = x_k^{1-1/2^t} = 2^{2^k - 2^{k-t}}$. Then, let $q_1 < q_2 < \cdots < q_R$ be all the primes located in the interval $(x_k, x_k + y_k]$, where clearly $R = R(k)$. For each $j \in \{1, \ldots, R\}$, let a_j be defined implicitly by $q_j = x_k + a_j$. Then, $a_j \le y_k$ and in light of (3.1), we have

$$R = \pi(x_k + y_k) - \pi(x_k) = (1+o(1))y_k/\log x_k \qquad (k \to \infty).$$

Given an integer $n \ge 1$, let $\alpha(n)$ stand for the sum of its binary digits. Adopting the argument of Erdős and Copeland used in [2], we can say that for every arbitrarily small $\delta > 0$, there exists a constant $\kappa = \kappa(\delta) > 0$ such that

$$\#\left\{m \le y_k : \alpha(m) > (1+\delta)2^{k-1}\left(1 - \frac{t}{2}\right)\right\} < y_k^{1-\kappa},$$

provided k is sufficiently large. It follows from this observation that

$$\begin{aligned} T &:= \sum_{j=1}^{R} \alpha(q_j) = R + \sum_{j=1}^{R} \alpha(a_j) \\ &\le R + (1+\delta)2^{k-1}\left(1 - \frac{t}{2}\right)R + 2^k y_k^{1-\kappa} \\ &\le (1+2\delta)2^{k-1}\left(1 - \frac{t}{2}\right)R, \end{aligned} \tag{3.2}$$

provided k is large enough.

Letting $\lambda(n)$ stand for the number of binary digits of n and observing that $\lambda(\overline{q_j}) = 2^k + 1$ for $j = 1, \ldots, R$, it follows from (3.2) that

$$T < \left(\frac{1}{2} - \varepsilon\right)\sum_{j=1}^{R} \lambda(\overline{q_j}). \tag{3.3}$$

However, if θ were to be a binary γ-sharp normal number, we would need to have

$$\frac{T}{\sum_{j=1}^{R} \lambda(\overline{q_j})} \to \frac{1}{2} \qquad (k \to \infty),$$

which clearly contradicts (3.3). We may therefore conclude that θ is not a binary γ-sharp normal number.

4. Proof of Theorem 2.2

Given an integer $n \in [2^k, 2^{k+1})$, write its binary expansion as $n = \sum_{\nu=0}^{k} \epsilon_\nu(n) 2^\nu$. In [1], the following result was proved.

Lemma 4.1. Let $N = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$ and let $F(x) \in \mathbb{Z}[x]$ be a polynomial of positive degree r such that $F(n) > 0$ for $n \geq 1$. If

$$N^{1/3} \leq \ell \leq rN - N^{1/3},$$

then,

$$\frac{1}{x} \#\{n \leq x : \epsilon_\ell(F(n)) = 1\} = \frac{1}{2} + O\left(\frac{1}{\log^A x}\right),$$

where A is some positive constant which may depend on the particular polynomial F.

In order to prove Theorem 2.2, we use Lemma 4.1 with $F(n) = n^r$. Let $M = M_k := 2^k$ and let

$$f(m) = (4M^2 + m)^r = (2M)^{2r} + g(m), \tag{4.1}$$

where

$$g(m) = \sum_{j=0}^{r-1} \binom{r}{j} (2M)^{2j} m^{r-j}.$$

Recalling that $\alpha(n)$ stands for the sum of the binary digits of n, whereas $\lambda(n)$ stands for the number of binary digits of n, our goal will be to estimate $A_M := \sum_{m=1}^{M} \alpha(f(m))$ and to compare it with $L_M := \sum_{m=1}^{M} \lambda(f(m))$.

Now, let

$$I_0 = [0, 2k], \quad I_1 = [2k+1, 4k], \quad \ldots \quad , I_{r-1} = [2(r-1)k + 1, 2rk].$$

Given any $I \subseteq \mathbb{N}\cup\{0\}$, we shall be using the function $\alpha_I(n) := \sum_{\nu\in I} \epsilon_\nu(n)$.

It follows from (4.1) that

$$\alpha(f(m)) = 1 + \alpha(g(m)) = 1 + \sum_{j=0}^{r-1} \alpha_{I_j}(g(m)).$$

With M fixed, consider the expression

$$K_j := \sum_{m=1}^{M} \alpha_{I_j}(g(m)) \qquad (j = 0, 1, \ldots, r-1).$$

Observing that $\alpha_{I_0}(g(m)) = \alpha_{I_0}(m^r)$ and choosing $A = 2/3$ in Lemma 4.1, we get that

$$K_0 = kM + O(k^{1/3}M).$$

Similarly, we obtain that

$$K_1 = \sum_{m=1}^{M} \alpha_{I_1}\left(m^r + \binom{r}{1}(2M)^2 m^{r-1}\right) = kM + O(k^{1/3}M) \tag{4.2}$$

and more generally that

$$K_j = KM + O(k^{1/3}M) \qquad (j = 2, \ldots, r-2). \tag{4.3}$$

We also get that

$$\alpha_{I_{r-1}}(g(m)) = \alpha_{I_{r-1}}\left(\binom{r}{r-1} 2^{(k+1)2(r-1)} m\right) = \alpha_{[0,k]}(m), \tag{4.4}$$

implying that

$$K_{r-1} = \frac{k}{2}M + O(k^{1/3}M). \tag{4.5}$$

Therefore, gathering (4.2), (4.3), (4.4) and (4.5), we obtain that

$$A_M = M + (r-1)kM + \frac{k}{2}M + O(k^{1/3}M) = \left(r - \frac{1}{2}\right)kM + O(k^{1/3}M).$$

Since $\lambda(f(m)) = 2(k+1)r + 1$ for $m = 1, \ldots, M$, it follows that

$$L_M = \sum_{m=1}^{M} \lambda(f(m)) = (2(k+1)r + 1)M.$$

Combining these last two relations, we find that

$$\limsup_{M\to\infty} \frac{A_M}{L_M} = \frac{1}{2} - \frac{1}{2r}. \tag{4.6}$$

However, if β were to be a binary sharp normal number, we would need to have

$$\limsup_{M\to\infty} \frac{A_M}{L_M} = \frac{1}{2},$$

which is clearly in contradiction with (4.6). We may therefore conclude that β is not a binary sharp binary normal number.

5. Proof of Theorem 2.3

Given large numbers x and $y = y(x)$, we set

$$\begin{aligned} \eta_x &:= \overline{p(2)}\,\overline{p(3)}\,\overline{p(4)}\,\ldots\,\overline{p(\lfloor x\rfloor)}, \\ \mu = \mu_{x,y} &:= \overline{p(\lfloor x\rfloor+1)}\,\overline{p(\lfloor x\rfloor+2)}\,\ldots\,\overline{p(\lfloor x\rfloor+\lfloor y\rfloor)}. \end{aligned}$$

In [6], we proved that there exists an absolute constant $c > 0$ such that

$$\lambda(\eta_x) = (1+o(1))cx\,\log\log x \qquad (x\to\infty). \tag{5.1}$$

Pick an arbitrary positive number $\delta < 1$, let $y = y(x) = x^\delta$ and consider the interval $J_x = [x, x+y]$. Using standard sieve methods, given a fixed small number $\varepsilon > 0$, one can prove that, for any prime $Q \le x^\varepsilon$, for some absolute constants $C_1 > 0$ and $C_2 > 0$,

$$\sum_{\substack{n\in J_x\\ p(n)=Q}} 1 \le C_1 \frac{y}{Q} \prod_{\pi<Q}\left(1-\frac{1}{\pi}\right) \le C_2 \frac{y}{Q\log Q} \tag{5.2}$$

and that, for some absolute constant $C_3 > 0$,

$$\#\{n\in J_x : p(n) > x^\varepsilon\} \le C_3 \frac{y}{\log x}. \tag{5.3}$$

In light of (5.1), it is easily seen that, for some absolute constant $c_1 > 0$,

$$\lambda(\mu_{x,y}) = (1+o(1))c_1 y\,\log\log x \qquad (x\to\infty). \tag{5.4}$$

Let $\mathcal{A}_q := \{0, 1, \dots, q-1\}$. Moreover, let K be an arbitrary positive integer and let Υ_K be the set of the q-ary words of length K. Here, by a q-ary word of length K, we mean a block of K base q digits. Choose an arbitrary $\beta \in \Upsilon_K$. Given a word ξ whose digits belong to $\mathcal{A}_q$, let $\sigma(\xi, \beta)$ be the number of times that β appears as a subword of the word ξ. It is clear that

$$\sigma(\mu, \beta) = \sum_{n=\lfloor x \rfloor+1}^{\lfloor x \rfloor+\lfloor y \rfloor} \sigma(\overline{p(n)}, \beta) + O(y\,K)$$

and therefore that, if $\beta_1, \beta_2 \in \Upsilon_K$ with $\beta_1 \neq \beta_2$, then

$$|\sigma(\mu, \beta_1) - \sigma(\mu, \beta_2)| \leq \sum_{n=\lfloor x \rfloor+1}^{\lfloor x \rfloor+\lfloor y \rfloor} \left| \sigma(\overline{p(n)}, \beta_1) - \sigma(\overline{p(n)}, \beta_2) \right| + O(y\,K). \tag{5.5}$$

Clearly, the theorem will be proved if we can show that

$$\max_{\substack{\beta_1, \beta_2 \in \Upsilon_K \\ \beta_1 \neq \beta_2}} \frac{|\sigma(\mu, \beta_1) - \sigma(\mu, \beta_2)|}{\lambda(\mu)} \to 0 \quad (x \to \infty). \tag{5.6}$$

Indeed, if (5.6) holds, then, given any $\beta \in \Upsilon_K$,

$$\max_{\beta \in \Upsilon_K} \frac{1}{\lambda(\mu)} \left| \sigma(\mu, \beta) - \frac{\lambda(\mu)}{q^K} \right| \to 0 \quad (x \to \infty),$$

thereby implying that μ is a q-normal sequence, as requested.

Arguing as Copeland and Erdős did in their paper [2], we have that, given a fixed $\varepsilon_1 > 0$,

(5.7)

$$\#\left\{ Q \in \wp \cap [U, 2U] : \max_{\substack{\beta_1, \beta_2 \in \Upsilon_K \\ \beta_1 \neq \beta_2}} \frac{|\sigma(\overline{Q}, \beta_1) - \sigma(\overline{Q}, \beta_2)|}{\lambda(\overline{Q})} > \varepsilon_1 \right\} \leq c_2 U^{1-\kappa},$$

where κ and c_2 are positive constants depending on ε_1 and K.

Let us now say that Q is a *bad prime* if

$$\max_{\substack{\beta_1, \beta_2 \in \Upsilon_K \\ \beta_1 \neq \beta_2}} \frac{|\sigma(\overline{Q}, \beta_1) - \sigma(\overline{Q}, \beta_2)|}{\lambda(\overline{Q})} > \varepsilon_1.$$

Now, observe that, for each $\beta \in \Upsilon_K$,

$$\sum_{n=\lfloor x\rfloor+1}^{\lfloor x\rfloor+\lfloor y\rfloor} \sigma(\overline{p(n)},\beta) = \sum_{Q<x^{\varepsilon}} \sigma(\overline{Q},\beta)\cdot\#\{n\in J_x : p(n)=Q\} + O\left(\#\{n\in J_x : p(n)>x^{\varepsilon}\}\cdot\log x\right),$$

which in light of (5.3) can be written as

$$\sum_{n=\lfloor x\rfloor+1}^{\lfloor x\rfloor+\lfloor y\rfloor} \sigma(\overline{p(n)},\beta) = \sum_{Q<x^{\varepsilon}} \sigma(\overline{Q},\beta)\cdot\#\{n\in J_x : p(n)=Q\} + O(y).$$

It follows from this last estimate that

$$\begin{aligned} S &:= \sum_{n=\lfloor x\rfloor+1}^{\lfloor x\rfloor+\lfloor y\rfloor} \left|\sigma(\overline{p(n)},\beta_1)-\sigma(\overline{p(n)},\beta_2)\right| \\ &\le O(y)+\varepsilon_1\sum_{Q<x^{\varepsilon}}\lambda(\overline{Q})\cdot\#\{n\in J_x : p(n)=Q\}+B(x), \qquad (5.8) \end{aligned}$$

where $B(x)$ stands for the contribution of the bad primes.

Now, since, in light of (5.7), the number of bad primes $Q\in[2^u,2^{u+1}]$ is no larger than $c_2\cdot(2^u)^{1-\kappa}$, it follows, using (5.2), that there exists a positive constant c_3 such that

$$\begin{aligned} B(x) &\le c_3 \sum_{\substack{Q<x^{\varepsilon}\\ Q \text{ bad primes}}} \lambda(\overline{Q})\frac{y}{Q\log Q} \le c_3 y \sum_{\substack{Q<x^{\varepsilon}\\ Q \text{ bad primes}}} \frac{1}{Q} \\ &\le c_3 y \sum_{2^u\le x^{\varepsilon}} \frac{1}{2^u}\#\{Q\in[2^u,2^{u+1}] : Q \text{ is a bad prime}\} \\ &\le c_3c_2 y \sum_{2^u\le x^{\varepsilon}} \frac{1}{2^u}2^{u(1-\kappa)} = c_3c_2y\sum_{2^u\le x^{\varepsilon}}\frac{1}{2^{u\kappa}} \le c_3c_2y\sum_{u=1}^{\infty}\frac{1}{2^{u\kappa}} < c_4 y \qquad (5.9) \end{aligned}$$

for some positive constant c_4.

Substituting (5.9) in (5.8) and recalling (5.4), it follows from (5.5) that

$$\begin{aligned} \max_{\substack{\beta_1,\beta_2\in\Upsilon_K\\ \beta_1\ne\beta_2}} \frac{|\sigma(\mu,\beta_1)-\sigma(\mu,\beta_2)|}{\lambda(\mu)} &\le \frac{O(y)+\varepsilon_1\lambda(\mu)+O(y)}{\lambda(\mu)} \\ &\le \varepsilon_1+o(1) \quad (x\to\infty), \end{aligned}$$

which implies (5.6), thereby completing the proof of the theorem.

6. Proofs of Theorems 2.4, 2.5 and 2.6

The proofs of Theorems 2.4, 2.5 and 2.6 are similar to that of Theorem 2.3 and we will therefore omit them.

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