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FUNCTIONS AND RELATIONS WHICH HAVE A KEY ROLE IN STUDY OF BICENTRIC POLYGONS WHERE CONICS ARE CIRCLES

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Abstract: The bicentric n-gons with incircle and those with excircle are in the focus of our interest. Remodeled and extended results are presented concerning Fuss' relations and geometrical configurations with incircle and those with excircle. These results are companion to ones exposed in [10]. Several new conjectures are also posed and discussed.

1. Introduction

A polygon which is both chordal and tangential is shortly called bicentric polygon. The relation (condition) that an n-sided polygon be a bicentric one is called *Fuss' relation for bicentric n-gons* and denoted by $F_n(R, r, d) = 0$ in honor to Swiss mathematician Nicolaus Fuss who first found the relation for bicentric quadrilateral. This relation is given by

$$(R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0,$$

where R and r are radii of circumcircle and incircle, respectively, and d is distance between centers of circumcircle and incircle, see [3]. Fuss also

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found relations for bicentric n-gons where $5 \leq n \leq 8$, [4].

The main and key result in theory of bicentric polygons is Poncelet's celebrated closure theorem which can be stated as follows [5]:

> *Let C and D be two nested conics such that there is an n-sided polygon inscribed in D and circumscribed around C. Then for every point X on D there is an n-sided polygon inscribed in D and circumscribed around C such that the point X is one of its vertices. Hence, for every starting point X there is a polygon with the same n-periodicity.*

Berger, Cayley, Dörrie, Jacobi and others have worked on number of problems related to this inspiring result. The problem of establishing Fuss' relation for bicentric quadrilateral is listed in [2, pp. 188–192] as one of the 100 great problems of elementary mathematics; however, we point out that the case $n \geq 3$ has been solved only recently (see [6] for odd n and [11] for n being even).

In the following we shall restrict ourselves to the case when conics are circles and deal with properties of the functions given in [10]. Some frequently used notation and used in sequel are three positive real numbers R_0, r_0, d_0 which satisfy $R_0 > r_0 + d_0$ and $F_n(R_0, r_0, d_0) = 0$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be circles in the same plane so that

$$\begin{aligned} R_0 &= \text{ radius of } \mathcal{C}_1,\ r_0 = \text{ radius of } \mathcal{C}_2, \\ d_0 &= \text{ distance between the centers of } \mathcal{C}_1 \text{ and } \mathcal{C}_2. \end{aligned}$$

By the Poncelet closure theorem, since $F_n(R_0, r_0, d_0) = 0$, there is for any point X of $\mathcal{C}_1$ a bicentric n-gon inscribed in $\mathcal{C}_1$ and circumscribed about $\mathcal{C}_2$ with X one of its vertices. This fact will be shortly said that (R_0, r_0, d_0) *has n-closure.*

The class of all bicentric n-gons inscribed in $\mathcal{C}_1$ and circumscribed about $\mathcal{C}_2$ will be denoted by $C(R_0, r_0, d_0)$. According to [10, Definition 1]:

> *Let (R_0, r_0, d_0) be positive real numbers and $R_0 > r_0 + d_0$. Then $(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0)$ is a triple obtained from the triple (R_0, r_0, d_0) such that R_0 and d_0 are mutually interchanged, that is*
>
> $$(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0) = (d_0, r_0, R_0). \tag{1.1}$$
>
> *This kind of triples will be called* dual triples. *Thus, (d_0, r_0, R_0) is dual to the triple (R_0, r_0, d_0) and* vice versa.

Concerning to this definition we give the following extension:

Let (R_0, r_0, d_0) *be as in the previous definition. Then the triples* $(1, r_0/R_0, d_0/R_0)$ *and* $(1, r_0/d_0, R_0/d_0)$ *will be called* relational dual triples.

The first part of these definitions refers to bicentric polygons with incircle and the second is connected to bicentric polygons with excircle.

Next, let $R, r, d > 0$ consist the triple (R, r, d). Then it will be frequently said that (R, r, d) is positive or to write $(R, r, d) \in \mathbb{R}_+^3$. Now we recall

Definition 1.1. [8, Definition 1] Let $\mathbb{S}$ be the set given by

$$\mathbb{S} = \left\{(R, r, d) \in \mathbb{R}_+^3 : R > r + d\right\}.$$

Let f_1, $f_2 : \mathbb{S} \to \mathbb{S}$ be functions on the set $\mathbb{S}$ defined as follows. Let $(R_0, r_0, d_0) \in \mathbb{S}$. Then

$$f_1(R_0, r_0, d_0) = (R_1, r_1, d_1), \tag{1.2}$$

where

$$\begin{cases} R_1^2 &= R_0\left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \\ r_1^2 &= (R_0 + r_0)^2 - d_0^2, \\ d_1^2 &= R_0\left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \end{cases} \tag{1.3}$$

and

$$f_2(R_0, r_0, d_0) = (R_2, r_2, d_2), \tag{1.4}$$

where

$$\begin{cases} R_2^2 &= R_0\left(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2}\right), \\ r_2^2 &= (R_0 - r_0)^2 - d_0^2, \\ d_2^2 &= R_0\left(R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - d_0^2}\right). \end{cases} \tag{1.5}$$

In [7, 8] are proved the following findings. Let $(R_i, r_i, d_i),\ i = 1, 2$ and $f_1,\ f_2$ be as in the last definition. Then

$$
\begin{cases}
R_1 > r_1 + d_1, \quad R_2 > r_2 + d_2, \\
R_1 d_1 = R_2 d_2 = R_0 d_0, \\
R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_0^2 + d_0^2 - r_0^2, \\
\dfrac{R_1^2 - d_1^2}{2r_1} = \dfrac{R_2^2 - d_2^2}{2r_2} = R_0, \\
\dfrac{2R_1 r_1 d_1}{R_1^2 - d_1^2} = \dfrac{2R_2 r_2 d_2}{R_2^2 - d_2^2} = d_0,
\end{cases}
\tag{1.6}
$$

and

$$
\begin{aligned}
&-\left(R_1^2 + d_1^2 - r_1^2\right) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 r_1 d_1}{R_1^2 - d_1^2}\right)^2 \\
&= -\left(R_2^2 + d_2^2 - r_2^2\right) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 r_2 d_2}{R_2^2 - d_2^2}\right)^2 = r_0^2.
\end{aligned}
\tag{1.7}
$$

Let $\mathbb{K}$ denote the set given by

$$\mathbb{K} = \left\{(R, r, d) \in \mathbb{S} \colon (R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0\right\},$$

that is $\mathbb{K}$ denotes the set of all (positive) solutions of Fuss' relations for bicentric quadrilaterals.

Moreover, in [8, Theorem 2] we have established the following result:

Let (R, r, d) be a triple of the set $\mathbb{S} \setminus \mathbb{K}$ and let g be function on the set $\mathbb{S} \setminus \mathbb{K}$ given by $g(R, r, d) = (\hat{R}, \hat{r}, \hat{d})$, where

$$
\begin{cases}
\hat{R} = \dfrac{R^2 - d^2}{2r}, \qquad \hat{d} = \dfrac{2Rdr}{R^2 - d^2} \\
\hat{r} = \sqrt{-\left(R^2 + d^2 - r^2\right) + \left(\dfrac{R^2 - d^2}{2r}\right)^2 + \left(\dfrac{2Rdr}{R^2 - d^2}\right)^2}.
\end{cases}
\tag{1.8}
$$

Then $\mathbb{S} \setminus \mathbb{K}$ is maximal subset of $\mathbb{S}$ such that

$$(\hat{R}, \hat{r}, \hat{d}) \in \mathbb{S} \setminus \mathbb{K} \Longrightarrow (R, r, d) \in \mathbb{S} \setminus \mathbb{K}. \tag{1.9}$$

Notice 1.2. In [8, Theorem 2] it is shown that in the case when $(R_0, r_0, d_0) \in \mathbb{K}$ then $\hat{r}_0 = 0$.

Definition 1.3. Let $(R_0, r_0, d_0) \in \mathbb{R}^3_+$ be a solution of Fuss' relation $F_n(R, r, d) = 0$. Let $\mathcal{C}_1, \mathcal{C}_2$ be circles such that $\mathcal{C}_2$ is completely inside of $\mathcal{C}_1$. Let $A_1 \cdots A_n$ be a bicentric n-gon from the class $C(R_0, r_0, d_0)$ and let $T_1, \cdots, T_n$ be touching points of its sides (segments) $A_1A_2, \cdots, A_nA_1$ and circle $\mathcal{C}_2$, respectively. Then $|A_iT_i|$, $i = 1, \cdots, n$, are so-called *tangent lengths* of the n-gon $A_1 \cdots A_n$. If

$$\sum_{i=1}^{n} \arctan \frac{|A_iT_i|}{r_0} = k\pi,$$

where $k \in \mathbb{N}$ then n-gon $A_1 \cdots A_n$ is *k-circumscribed* and k is *rotation number for n*.

If n-gons from the class $C(R_0, r_0, d_0)$ are k-circumscribed, then Fuss' relation for this class of the n-gons is denoted by $F_n^{(k)}(R, r, d) = 0$.

The term cycle will be also used in the following. Let $(R_{k_1}, r_{k_1}, d_{k_1}) \in \mathbb{R}^3_+$ be a solution of Fuss' relation $F_n(R, r, d) = 0$, where $n \geq 3$ is an odd integer. Then there is an integer $m \geq 1$ such that

$$g^m (R_{k_1}, r_{k_1}, d_{k_1}) = (R_{k_1}, r_{k_1}, d_{k_1}),$$

where $k_1, \cdots, k_m$ are rotation numbers for n. Then $(k_1, \ldots, k_m)$ is called a *cycle for n*. For example, the cycles for $n = 3, 5, 7, 9$ are (1), $(1, 2)$, (1,2,3), (1,2,4), respectively.

The following conjecture [8, Conjecture 2] is also of importance. *Let (R_k, r_k, d_k) be a solution of Fuss' relation $F_n(R, r, d) = 0$, where $n \geq 3$ is an odd integer. Let*

$$g(R_k, r_k, d_k) = (R_l, r_l, d_l), \tag{1.10a}$$

where k and l are rotation numbers for n. Then

$$f_1(R_l, r_l, d_l) = (R_k, r_k, d_k), \quad \text{if } l \text{ is even}, \tag{1.10b}$$

$$f_2(R_l, r_l, d_l) = (R_k, r_k, d_k), \quad \text{if } l \text{ is odd}. \tag{1.10c}$$

From (1.9) follow the following two assertions:

Assertion 1.4. Let $(R_0, r_0, d_0) \in \mathbb{S} \setminus \mathbb{K}$. Then the solutions of the system $(\hat{R}, \hat{r}, \hat{d}) = (R_0, r_0, d_0)$ are (R_i, r_i, d_i), $i = 1, 2$, described by (1.2) and (1.4).

In turn, g is a left inverse of f_1, f_2, that is $gf_j(R_0, r_0, d_0) = (R_0, r_0, d_0)$, $j = 1, 2$.

Assertion 1.5. Let $(R_0, r_0, d_0) \in \mathbb{S}$. Then there are two triples in $\mathbb{S}$ which g maps onto (R_0, r_0, d_0); these are $f_1(R_0, r_0, d_0)$ and $f_2(R_0, r_0, d_0)$.

Now let $(R_0, r_0, d_0) \in \mathbb{S}$ and $i_1, \ldots, i_n \in \{1, 2\}$ and $n \in \mathbb{N}$. Then by definition

$$(R_{i_1\ldots i_n}, r_{i_1\ldots i_n}, d_{i_1\ldots i_n}) = f_{i_n} \ldots f_{i_1}(R_0, r_0, d_0),$$

compare to Figure 1. It can be shown that

(1.11a)
$$\frac{R^2_{i_1\ldots i_n} + d^2_{i_1\ldots i_n} - r^2_{i_1\ldots i_n}}{2R_{i_1\ldots i_n} d_{i_1\ldots i_n}} = \frac{R_0^2 + d_0^2 - r_0^2}{2R_0 d_0} = I, \quad i_1, \ldots, i_n \in \{1, 2\},$$

where I is the invariant of the above described corresponding pencil. Indeed, it is sufficient to show that

$$\frac{R_1^2 + d_1^2 - r_1^2}{2R_1 d_1} = \frac{R_2^2 + d_2^2 - r_2^2}{2R_2 d_2} = \frac{R_0^2 + d_0^2 - r_0^2}{2R_0 d_0},$$

since the analogy is complete.

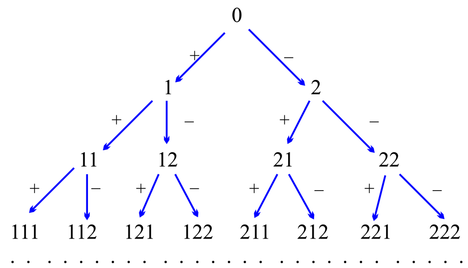

Figure 1: The arrow + refers to $f_1(R_i, r_i, d_i)$, the arrow - refers to $f_2(R_i, r_i, d_i)$.

It is often convenient the use of the triple $(1, \rho, \delta)$, normalized with respect to R, instead of (R, r, d), writing $\rho = \frac{r}{R}$, $\delta = \frac{d}{R}$, see e.g. [1]. So

$$\frac{R_0^2 + d_0^2 - r_0^2}{2R_0 d_0} = I \quad \Rightarrow \quad \frac{1 + \delta_0^2 - \rho_0^2}{2\delta_0} = I,$$

where

(1.11b) $$\rho_0^2 = 1 - 2I\delta_0 + \delta_0^2.$$

Supposing that (R_0, r_0, d_0) is not from $\mathbb{K}$ and using the relations given by (1.8), it is easy to show that

$$\frac{(\hat{R}_0)^2 + (\hat{d_0}^2) - (\hat{r}_0)^2}{2\hat{R}_0\hat{d}_0} = I \quad \text{or} \quad \frac{1 + (\overset{\triangle}{\delta_0})^2 - (\overset{\triangle}{\rho_0})^2}{2\overset{\triangle}{\delta_0}} = I$$

from which follows

$$(1.11c) \qquad (\overset{\triangle}{\rho_0})^2 = 1 - 2I\overset{\triangle}{\delta_0} + (\overset{\triangle}{\delta_0})^2.$$

where $\overset{\triangle}{\delta_0} = \hat{d}_0/\hat{R}_0$, $\overset{\triangle}{\rho_0} = \hat{r}_0/\hat{R}_0$.

Let (R_0, r_0, d_0) be any given triple from $\mathbb{S} \setminus \mathbb{K}$. Then the triples

$$(R_0, r_0, d_0), \quad (d_0, r_0, R_0), \quad (\hat{R}_0, \hat{r}_0, \hat{d}_0), \quad (\hat{d}_0, \hat{r}_0, \hat{R}_0)$$

belong to the same pencil since

$$\frac{R_0^2 + d_0^2 - r_0^2}{2R_0d_0} = \frac{d_0^2 + R_0^2 - r_0^2}{2d_0R_0} = \frac{\hat{R}_0^2 + \hat{d}_0^2 - \hat{r}_0^2}{2\hat{R}_0\hat{d}_0} = \frac{\hat{d}_0^2 + \hat{R}_0^2 - \hat{r}_0^2}{2\hat{d}_0\hat{R}_0} = I,$$

where I is invariant of the corresponding pencil. The above equalities can be written as

$$(1.11d) \qquad \begin{aligned} \frac{1 + \delta_0^2 - \rho_0^2}{2\delta_0} &= \frac{1 + (\overset{*}{\delta_0})^2 - (\overset{*}{\rho_0})^2}{2\overset{*}{\delta_0}} = \frac{1 + (\overset{\triangle}{\delta_0})^2 - (\overset{\triangle}{\rho_0})^2}{2\overset{\triangle}{\delta_0}} = \\ &= \frac{1 + (\overset{\square}{\delta_0})^2 - (\overset{\square}{\rho_0})^2}{2\overset{\square}{\delta_0}} = I, \end{aligned}$$

where

$$\delta_0 = \frac{d_0}{R_0}, \quad \overset{*}{\delta_0} = \frac{R_0}{d_0}, \quad \overset{\triangle}{\delta_0} = \frac{\hat{d}_0}{\hat{R}_0}, \quad \overset{\square}{\delta_0} = \frac{\hat{R}_0}{\hat{d}_0},$$
$$\rho_0 = \frac{r_0}{R_0}, \quad \overset{*}{\rho_0} = \frac{r_0}{d_0}, \quad \overset{\triangle}{\rho_0} = \frac{\hat{r}_0}{\hat{R}_0}, \quad \overset{\square}{\rho_0} = \frac{\hat{r}_0}{\hat{d}_0}$$

and from the equalities (1.11d) it follows

$$(1.11e) \qquad \begin{aligned} &\rho_0^2 = 1 - 2I\delta_0 + (\delta_0)^2, (\overset{*}{\rho_0})^2 = 1 - 2I\overset{*}{\delta_0} + (\overset{*}{\delta_0})^2, (\overset{\triangle}{\rho_0})^2 \\ &= 1 - 2I\overset{\triangle}{\delta_0} + (\overset{\triangle}{\delta_0})^2, (\overset{\square}{\rho_0})^2 = 1 - 2I\overset{\square}{\delta_0} + (\overset{\square}{\delta_0})^2. \end{aligned}$$

Now, using the above relations will be about some functions which play key role in research of bicentric polygons where conics are circles.

Let $(R_0, r_0, d_0) \in \mathbb{S} \setminus \mathbb{K}$. The relation given by

$$\frac{\hat{d}_0}{\hat{R}_0} = \frac{4\delta_0(1 - 2I\delta_0 + \delta_0^2)}{(1-\delta_0^2)^2} \tag{1.12a}$$

will be important in the sequel; one obtains from

$$\hat{d}_0 = \frac{2R_0 r_0 d_0}{R_0^2 - d_0^2}$$

such that the both sides of this relation be divided by $\hat{R}_0$, that is by $\dfrac{R_0^2 - d_0^2}{2r_0}$. Since $\hat{R}_0 = \dfrac{R_0^2 - d_0^2}{2r_0}$ we can write

$$\frac{\hat{d}_0}{\hat{R}_0} = \frac{2R_0 r_0 d_0}{R_0^2 - d_0^2} : \frac{R_0^2 - d_0^2}{2r_0} = \frac{4R_0 r_0^2 d_0 : R_0^4}{(R_0^2 - d_0^2)^2 : R_0^4} = \frac{4\rho_0^2 \delta_0}{(1-\delta_0^2)^2}.$$

Thus

$$\overset{\triangle}{\delta}_0 = \frac{4\delta_0 \rho_0^2}{(1-\delta_0^2)^2}, \tag{1.12b}$$

where $\overset{\triangle}{\delta}_0 = \hat{d}_0/\hat{R}_0$ and $\rho_0^2 = 1 - 2I\delta_0 + \delta_0^2$, compare (1.11b) and (1.11e).

Using computer algebra it can be found that Eq. (1.12a) has four solutions in δ_0:

$$\begin{aligned}
(\delta_0)_1 &= \frac{1 + \overset{\triangle}{\rho}_0 - \sqrt{2(1 - I\,\overset{\triangle}{\delta}_0 + \overset{\triangle}{\rho}_0)}}{\overset{\triangle}{\delta}_0}, \\
(\delta_0)_2 &= \frac{1 - \overset{\triangle}{\rho}_0 - \sqrt{2(1 - I\,\overset{\triangle}{\delta}_0 - \overset{\triangle}{\rho}_0)}}{\overset{\triangle}{\delta}_0}, \\
(\delta_0)_3 &= \frac{1 + \overset{\triangle}{\rho}_0 + \sqrt{2(1 - I\,\overset{\triangle}{\delta}_0 + \overset{\triangle}{\rho}_0)}}{\overset{\triangle}{\delta}_0}, \\
(\delta_0)_4 &= \frac{1 - \overset{\triangle}{\rho}_0 + \sqrt{2(1 - I\,\overset{\triangle}{\delta}_0 - \overset{\triangle}{\rho}_0)}}{\overset{\triangle}{\delta}_0},
\end{aligned} \tag{1.12c}$$

where $\overset{\triangle}{\rho_0}=\hat{r}_0/\hat{R}_0$ and I denotes the invariant of the corresponding pencil.

Conjecture 1.6. [10] There exist functions $\gamma_i, \varphi_i, i=1,2$ defined by

$$\gamma_1\left(\overset{\triangle}{\delta_0}\right)=(\delta_0)_1,\ \gamma_2\left(\overset{\triangle}{\delta_0}\right)=(\delta_0)_2,\ \varphi_1\left(\overset{\triangle}{\delta_0}\right)=(\delta_0)_3,\ \varphi_2\left(\overset{\triangle}{\delta_0}\right)=(\delta_0)_4\,,$$

and

$$\begin{aligned}
\gamma_1(\overset{\triangle}{\delta_0}) &= \frac{1+\overset{\triangle}{\rho_0}-\sqrt{2(1-I\,\overset{\triangle}{\delta_0}+\overset{\triangle}{\rho_0})}}{\overset{\triangle}{\delta_0}},\\
\gamma_2(\overset{\triangle}{\delta_0}) &= \frac{1-\overset{\triangle}{\rho_0}-\sqrt{2(1-I\,\overset{\triangle}{\delta_0}-\overset{\triangle}{\rho_0})}}{\overset{\triangle}{\delta_0}},\\
\varphi_1(\overset{\triangle}{\delta_0}) &= \frac{1+\overset{\triangle}{\rho_0}+\sqrt{2(1-I\,\overset{\triangle}{\delta_0}+\overset{\triangle}{\rho_0})}}{\overset{\triangle}{\delta_0}},\\
\varphi_2(\overset{\triangle}{\delta_0}) &= \frac{1-\overset{\triangle}{\rho_0}+\sqrt{2(1-I\,\overset{\triangle}{\delta_0}-\overset{\triangle}{\rho_0})}}{\overset{\triangle}{\delta_0}}.
\end{aligned} \tag{1.13}$$

Now we formulate notation which will be used in defining functions $\sigma_i, \tau_i, i=1,2$.

Let $(\hat{R}_0,\ \hat{r}_0,\ \hat{d}_0)$ be given by Eq. (1.8). Then the dual of the triple $(\hat{R}_0, \hat{r}_0, \hat{d}_0)$ is

$$(\hat{d}_0,\hat{r}_0,\hat{R}_0)= \\ =\left(\frac{2R_0r_0d_0}{R_0^2-d_0^2},\sqrt{-(R_0^2+d_0^2-r_0^2)+\left(\frac{R_0^2-d_0^2}{2r_0}\right)^2+\left(\frac{2R_0r_0d_0}{R_0^2-d_0^2}\right)^2},\frac{R_0^2-d_0^2}{2r_0}\right). \tag{1.14}$$

Thus, instead of the triple $(\hat{R}_0, \hat{r}_0, \hat{d}_0)$ used in obtaining (1.12b) we apply the dual triple $(\hat{d}_0, \hat{r}_0, \hat{R}_0)$. So, we conclude

$$\overset{\square}{\delta_0}=\frac{(1-\delta_0^2)^2}{4\delta_0\,(1-2\delta_0 I+\delta_0^2)} \tag{1.15}$$

or

$$\overset{\square}{\delta_0}= \frac{(1-\delta_0^2)^2}{4\delta_0\rho_0^2},$$

where $\overset{\square}{\delta_0}= \hat{R}_0/\hat{d}_0$ and the last relation follows from $\hat{R}_0 = R_0^2 - d_0^2/2r_0$ dividing both sides with $\hat{d}_0$, that is by $2R_0r_0d_0/(R_0^2-d_0^2)$, consult (1.14) and Cf. with (1.8) recalling that $\overset{\triangle}{\delta_0}$ in (1.12) equals $\hat{d}_0/\hat{R}_0 < 1$ and in (1.15) is $\overset{\square}{\delta_0}= \hat{R}_0/\hat{d}_0 > 1$. So we have the following equality

$$\overset{\triangle}{\delta_0} \cdot \overset{\square}{\delta_0}= \frac{\hat{d}_0}{\hat{R}_0} \cdot \frac{\hat{R}_0}{\hat{d}_0} = \frac{4\delta_0\rho_0^2}{(1-\delta_0^2)^2} \cdot \frac{(1-\delta_0^2)^2}{4\delta_0\rho_0^2} = 1.$$

The equation in δ_0 given by (1.15) has the following four solutions

$$(\delta_0)_1 = \overset{\square}{\delta_0} + \overset{\square}{\rho_0} - \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} + \overset{\square}{\rho_0} - I\right)},$$

$$(\delta_0)_2 = \overset{\square}{\delta_0} - \overset{\square}{\rho_0} - \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} - \overset{\square}{\rho_0} - I\right)},$$

$$(\delta_0)_3 = \overset{\square}{\delta_0} + \overset{\square}{\rho_0} + \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} + \overset{\square}{\rho_0} - I\right)},$$

$$(\delta_0)_4 = \overset{\square}{\delta_0} - \overset{\square}{\rho_0} + \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} - \overset{\square}{\rho_0} - I\right)},$$

where $\overset{\square}{\rho_0}= \hat{r}_0/\hat{d}_0$ and I remain the same as in (1.12).

Conjecture 1.7. [10] There exist functions $\sigma_i, \tau_i,\ i=1,2$ defined as

$$\sigma_1\Big(\frac{\hat{R}_0}{\hat{d}_0}\Big) = \overset{\square}{\delta_0} + \overset{\square}{\rho_0} - \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} + \overset{\square}{\rho_0} - I\right)},$$

$$\sigma_2\Big(\frac{\hat{R}_0}{\hat{d}_0}\Big) = \overset{\square}{\delta_0} - \overset{\square}{\rho_0} - \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} - \overset{\square}{\rho_0} - I\right)},$$

(1.16)

$$\tau_1\Big(\frac{\hat{R}_0}{\hat{d}_0}\Big) = \overset{\square}{\delta_0} + \overset{\square}{\rho_0} + \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} + \overset{\square}{\rho_0} - I\right)},$$

$$\tau_2\Big(\frac{\hat{R}_0}{\hat{d}_0}\Big) = \overset{\square}{\delta_0} - \overset{\square}{\rho_0} + \sqrt{2\,\overset{\square}{\delta_0}\left(\overset{\square}{\delta_0} - \overset{\square}{\rho_0} - I\right)}.$$

It is worth to mention that the functions given by (1.13) refer to bicentric polygons with incircle and the functions given by (1.16) are associated with the excircle. Also we already mentioned that for each triple from $\mathbb{S} \setminus \mathbb{K}$ we get relations like those established for $(R_0, r_0, d_0) \in \mathbb{S} \setminus \mathbb{K}$. Accordingly for any equality like (1.12b) we arrive at four solutions in the form of (1.12c). This shows that the definition of $\gamma_i, \varphi_i, i = 12$, is correct. The same holds true for $\sigma_i, \tau_i, i = 1, 2$.

In turn, with these argumentation the Conjectures 1.6, 1.7 have been proved.

2. Properties of the functions $\gamma_i, \varphi_i, \sigma_i, \tau_i, i = 1, 2$

The article primarily deals with connections between bicentric polygons with incircle and those with excircle. The connection is precisely established and cornerstone relations were inferred for bicentric polygons where conics are circles. Firstly, we express $\gamma_i, \varphi_i, \sigma_i, \tau_i, i = 1, 2$ in a more appropriate form. Namely, using (1.11d) for invariant I and writing $\hat{d}_0/\hat{R}_0, \hat{r}_0/\hat{R}_0, \hat{R}_0/\hat{d}_0, \hat{r}_0/\hat{d}_0$ instead of $\overset{\triangle}{\delta}_0, \overset{\triangle}{\rho}_0, \overset{\square}{\delta}_0, \overset{\square}{\rho}_0$ respectively, we get

$$\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\hat{R}_0 + \hat{r}_0 - \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0}, \tag{2.1a}$$

$$\gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\hat{R}_0 - \hat{r}_0 - \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0}, \tag{2.1b}$$

$$\varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\hat{R}_0 + \hat{r}_0 + \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0}, \tag{2.1c}$$

$$\varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\hat{R}_0 - \hat{r}_0 + \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0}, \tag{2.1d}$$

$$\sigma_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{\hat{R}_0 + \hat{r}_0 - \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0}, \tag{2.1e}$$

$$\text{(2.1f)} \qquad \sigma_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{\hat{R}_0 - \hat{r}_0 - \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0},$$

$$\text{(2.1g)} \qquad \tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{\hat{R}_0 + \hat{r}_0 + \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0},$$

$$\text{(2.1h)} \qquad \tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{\hat{R}_0 - \hat{r}_0 + \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0},$$

where

$$\gamma_i\left(\frac{\hat{d}_0}{\hat{R}_0}\right) \cdot \varphi_i\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = 1,\ i = 1,2, \tau_i\left(\frac{\hat{R}_0}{\hat{d}_0}\right) \cdot \sigma_i\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = 1,\ i = 1,2.$$

Also, let us remark that from (2.1) it follows:

$$\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \sigma_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right), \qquad \gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \sigma_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right),$$
$$\varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right), \qquad \varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right).$$

So the functions γ_i, φ_i refer to bicentric polygons with incircle and the functions σ_i, τ_i refer to bicentric polygons with excircle being throughout $i = 1, 2$.

Theorem 2.1. Let (R_0, r_0, d_0) be a triple from the set $\mathbb{S} \setminus \mathbb{K}$ and let $F_n(R_0, r_0, d_0) = 0$. Then

$$\text{(2.2)} \qquad \text{either} \quad \gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{d_0}{R_0} \quad \text{or} \quad \gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{d_0}{R_0},$$

where $\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right)$ and $\gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right)$ stand on the right hand sides of (2.1a) and (2.1b) respectively.

Proof. First we consider the following equation in s:

$$\text{(2.3)} \qquad \gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = s,$$

where $\hat{R}_0, \hat{r}_0, \hat{d}_0$ are given by

$$\hat{R}_0 = \frac{R_0^2 - d_0^2}{2r_0},\ \hat{r}_0 = \sqrt{-(R_0^2 + d_0^2 - r_0^2) + (\hat{R}_0)^2 + (\hat{d}_0)^2}, \quad \hat{d}_0 = \frac{2R_0 r_0 d_0}{R_0^2 - d_0^2}. \tag{2.4}$$

After rationalization and factorization (2.3) becomes

$$(d_0 s - R_0) \cdot (d_0 - R_0 s) \cdot (d_0^4 s - d_0^2 r_0^2 s - 2d_0^2 R_0^2 s - d_0 r_0^2 R_0 s^2 - d_0 r_0^2 R_0 - r_0^2 R_0^2 s + R_0^4 s) = 0,$$

which solutions are

$$s_1 = \frac{d_0}{R_0}, \qquad s_2 = \frac{R_0}{d_0}, \tag{2.5}$$

$$s_3 = -\frac{-d_0^4 + d_0^2 r_0^2 + 2d_0^2 R_0^2 + r_0^2 R_0^2 - R_0^4}{2d_0 r_0^2 R_0} + \frac{(d_0^2 - R_0^2)\sqrt{((d_0 - r_0)^2 - R_0^2)((d_0 + r_0)^2 - R_0^2)}}{2d_0 r_0^2 R_0},$$

$$s_4 = -\frac{-d_0^4 + d_0^2 r_0^2 + 2d_0^2 R_0^2 + r_0^2 R_0^2 - R_0^4}{2d_0 r_0^2 R_0} - \frac{(d_0^2 - R_0^2)\sqrt{((d_0 - r_0)^2 - R_0^2)((d_0 + r_0)^2 - R_0^2)}}{2d_0 r_0^2 R_0}.$$

It is worth to notice that $s_1 s_2 = s_3 s_4 = 1$. ◊

The same solutions, obtained in the same way, have the equation $\gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = s$. It turns out that

$$s_1 s_3 = \gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) \cdot \gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right), \tag{2.6a}$$

$$s_2 s_4 = \tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) \cdot \tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right), \tag{2.6b}$$

where $\tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right)$ and $\tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right)$ are described in (2.1g), (2.1h) respectively.

Thus s_1 and s_3 refer to bicentric n-gons with incircle and s_2 and s_4 refer to bicentric n-gons with excircle. So, establishing the equality related to (2.6a) we mutually obtain the equality related to (2.6b).

Corollary 2.2. Let (R_0, r_0, d_0) be as in Theorem 2.1. Then

either $\gamma_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0)$ or $\gamma_2(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0)$.

The proof of the next result contains the same lines as the proof of the previous Theorem 2.1.

Theorem 2.3. Let (R_0, r_0, d_0) be as in Theorem 2.1. Then

$$\text{either} \quad \tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{R_0}{d_0} \quad \text{or} \quad \tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{R_0}{d_0}. \tag{2.7}$$

Corollary 2.4. It holds

either $\tau_1(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (d_0, r_0, R_0)$ or$\tau_2(\hat{\mathrm{d}}_0, \hat{\mathrm{r}}_0, \hat{\mathrm{R}}_0) = (\mathrm{d}_0, \mathrm{r}_0, \mathrm{R}_0)$.

So the relation (2.6a) refers to bicentric n-gon with incircle, while (2.6b) is associated to bicentric n-gon with excircle.

Theorem 2.5. Let (R_0, r_0, d_0) be as in Theorem 2.1. Then

$$\text{either} \quad \varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{R_0}{d_0} \quad \text{or} \quad \varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{R_0}{d_0}. \tag{2.8}$$

Proof. Since the right hand sides of the relations

$$\varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\hat{R}_0 + \hat{r}_0 + \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0},$$

$$\varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\hat{R}_0 - \hat{r}_0 + \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}}{\hat{d}_0}$$

are the same as the right hand sides of the relations (2.1g) and (2.1h), the solutions of $\varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = s$ and $\varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = s$ coincide with the solutions of $\tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = s$ and $\tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = s$. ◊

Corollary 2.6. There holds

either $\varphi_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (d_0, r_0, R_0)$ or $\varphi_2(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (d_0, r_0, R_0)$.

Theorem 2.7. Let (R_0, r_0, d_0) be as in Theorem 2.1. Then

$$\text{either} \quad \sigma_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{d_0}{R_0} \quad \text{or} \quad \sigma_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = \frac{d_0}{R_0}. \tag{2.9}$$

Proof. Comparing the right hand sides of (2.1e) and (2.1f) and (2.1a) and (2.1b), we deduce that $\sigma_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = s$, $\sigma_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right) = s$ have the same solutions as $\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = s$, $\gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = s$. ♢

Corollary 2.8. It holds

$$\text{either} \quad \sigma_1(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (R_0, r_0, d_0) \quad \text{or} \quad \sigma_2(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (R_0, r_0, d_0).$$

Theorem 2.9. Let (R_0, r_0, d_0) be as in Theorem 2.1 and let $\overset{\blacktriangle}{R}_i, \overset{\blacktriangle}{r}_i, \overset{\blacktriangle}{d}_i, i = 1, 2$ be defined as

$$\left(\overset{\blacktriangle}{R}_1\right)^2 = \hat{R}_0\left(\hat{R}_0 + \hat{r}_0 + \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}\right), \tag{2.10a}$$

$$\left(\overset{\blacktriangle}{r}_1\right)^2 = (\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2, \tag{2.10b}$$

$$\left(\overset{\blacktriangle}{d}_1\right)^2 = \hat{R}_0\left(\hat{R}_0 + \hat{r}_0 - \sqrt{(\hat{R}_0 + \hat{r}_0)^2 - (\hat{d}_0)^2}\right), \tag{2.10c}$$

$$\left(\overset{\blacktriangle}{R}_2\right)^2 = \hat{R}_0\left(\hat{R}_0 - \hat{r}_0 + \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}\right), \tag{2.10d}$$

$$\left(\overset{\blacktriangle}{r}_2\right)^2 = (\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2, \tag{2.10e}$$

$$\left(\overset{\blacktriangle}{d}_2\right)^2 = \hat{R}_0\left(\hat{R}_0 - \hat{r}_0 - \sqrt{(\hat{R}_0 - \hat{r}_0)^2 - (\hat{d}_0)^2}\right). \tag{2.10f}$$

Then the subsequent relations follow

$$\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\overset{\blacktriangle}{d}_1}{\overset{\blacktriangle}{R}_1}, \quad \gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\overset{\blacktriangle}{d}_2}{\overset{\blacktriangle}{R}_2}, \tag{2.11a}$$

$$\varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\overset{\blacktriangle}{R}_1}{\overset{\blacktriangle}{d}_1}, \quad \varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \frac{\overset{\blacktriangle}{R}_2}{\overset{\blacktriangle}{d}_2}, \tag{2.11b}$$

$$\sigma_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right)=\frac{\overset{\blacktriangle}{d}_1}{\overset{\blacktriangle}{R}_1},\quad \sigma_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right)=\frac{\overset{\blacktriangle}{d}_2}{\overset{\blacktriangle}{R}_2},\tag{2.11c}$$

$$\tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right)=\frac{\overset{\blacktriangle}{R}_1}{\overset{\blacktriangle}{d}_1},\quad \tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right)=\frac{\overset{\blacktriangle}{R}_2}{\overset{\blacktriangle}{d}_2}.\tag{2.11d}$$

Proof. We will use the following straightforward equalities

$$\overset{\blacktriangle}{R}_1\overset{\blacktriangle}{d}_1=\overset{\blacktriangle}{R}_2\overset{\blacktriangle}{d}_2=\hat{R}_0\hat{d}_0.\tag{2.12}$$

Indeed,

$$\begin{aligned}\left(\overset{\blacktriangle}{R}_1\overset{\blacktriangle}{d}_1\right)^2=&\left[\hat{R}_0\left(\hat{R}_0+\hat{r}_0+\sqrt{(\hat{R}_0+\hat{r}_0)^2-(\hat{d}_0)^2}\right)\right]\cdot\\ &\cdot\left[\hat{R}_0\left(\hat{R}_0+\hat{r}_0-\sqrt{(\hat{R}_0+\hat{r}_0)^2-(\hat{d}_0)^2}\right)\right]=\\ =&(\hat{R}_0\hat{d}_0)^2.\end{aligned}$$

Using (2.12) and (2.1) we arrive at

$$\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right)=\frac{\hat{R}_0\left(\hat{R}_0+\hat{r}_0-\sqrt{(\hat{R}_0+\hat{r}_0)^2-(\hat{d}_0)^2}\right)}{\hat{R}_0\hat{d}_0}=\frac{\left(\overset{\blacktriangle}{d}_1\right)^2}{\overset{\blacktriangle}{R}_1\overset{\blacktriangle}{d}_1}=\frac{\overset{\blacktriangle}{d}_1}{\overset{\blacktriangle}{R}_1}$$

$$\gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right)=\frac{\hat{R}_0\left(\hat{R}_0-\hat{r}_0-\sqrt{(\hat{R}_0-\hat{r}_0)^2-(\hat{d}_0)^2}\right)}{\hat{R}_0\hat{d}_0}=\frac{\left(\overset{\blacktriangle}{d}_2\right)^2}{\overset{\blacktriangle}{R}_2\overset{\blacktriangle}{d}_2}=\frac{\overset{\blacktriangle}{d}_2}{\overset{\blacktriangle}{R}_2},$$

which finally confrim (2.11). ◊

Corollary 2.10. For $i=1,2$ we have

$$\gamma_i\left(\frac{\hat{d}_0}{\hat{R}_0}\right)\cdot\varphi_i\left(\frac{\hat{d}_0}{\hat{R}_0}\right)=1,$$

$$\gamma_i\left(\frac{\hat{d}_0}{\hat{R}_0}\right)\cdot\tau_i\left(\frac{\hat{R}_0}{\hat{d}_0}\right)=1,$$

$$\sigma_i\left(\frac{\hat{R}_0}{\hat{d}_0}\right)\cdot\tau_i\left(\frac{\hat{R}_0}{\hat{d}_0}\right)=1.$$

Corollary 2.11. It holds

(2.13a)
$$\text{either} \quad \gamma_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (\overset{\blacktriangle}{R}_1, \overset{\blacktriangle}{r}_1, \overset{\blacktriangle}{d}_1) \quad \text{or} \quad \gamma_2(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (\overset{\blacktriangle}{R}_2, \overset{\blacktriangle}{r}_2, \overset{\blacktriangle}{d}_2),$$
(2.13b)
$$\text{either} \quad \varphi_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (\overset{\blacktriangle}{d}_1, \overset{\blacktriangle}{r}_1, \overset{\blacktriangle}{R}_1) \quad \text{or} \quad \varphi_2(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (\overset{\blacktriangle}{d}_2, \overset{\blacktriangle}{r}_2, \overset{\blacktriangle}{R}_2),$$
(2.13c)
$$\text{either} \quad \sigma_1(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (\overset{\blacktriangle}{R}_1, \overset{\blacktriangle}{r}_1, \overset{\blacktriangle}{d}_1) \quad \text{or} \quad \sigma_2(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (\overset{\blacktriangle}{R}_2, \overset{\blacktriangle}{r}_2, \overset{\blacktriangle}{d}_2),$$
(2.13d)
$$\text{either} \quad \tau_1(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (\overset{\blacktriangle}{d}_1, \overset{\blacktriangle}{r}_1, \overset{\blacktriangle}{R}_1) \quad \text{or} \quad \tau_2(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (\overset{\blacktriangle}{d}_2, \overset{\blacktriangle}{r}_2, \overset{\blacktriangle}{R}_2).$$

The proof follows from (2.14a), so it is omitted.

Corollary 2.12. It holds

$$\gamma_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \sigma_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right), \quad \gamma_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \sigma_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right), \tag{2.14a}$$
$$\varphi_1\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \tau_1\left(\frac{\hat{R}_0}{\hat{d}_0}\right), \quad \varphi_2\left(\frac{\hat{d}_0}{\hat{R}_0}\right) = \tau_2\left(\frac{\hat{R}_0}{\hat{d}_0}\right). \tag{2.14b}$$

Remark that for all functions $\gamma_1, \varphi_1, \sigma_1, \tau_1$ we get different kind then in the case of $\gamma_2, \varphi_2, \sigma_2, \tau_2$.

Now, we extend the meaning of definition of $\gamma_i, \varphi_i, \sigma_i, \tau_i,\ i = 1, 2$ carried by (1.13) and (1.16) omitting the symbol $\hat{}$ throughout. This results in

$$\gamma_1\left(\frac{d_0}{R_0}\right) = \frac{R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - (d_0)^2}}{d_0}, \tag{2.15a}$$
$$\gamma_2\left(\frac{d_0}{R_0}\right) = \frac{R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - (d_0)^2}}{d_0}, \tag{2.15b}$$
$$\varphi_1\left(\frac{d_0}{R_0}\right) = \frac{R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - (d_0)^2}}{d_0}, \tag{2.15c}$$
$$\varphi_2\left(\frac{d_0}{R_0}\right) = \frac{R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - (d_0)^2}}{d_0}, \tag{2.15d}$$

(2.15e) $$\sigma_1\left(\frac{R_0}{d_0}\right)=\frac{R_0+r_0-\sqrt{(R_0+r_0)^2-(d_0)^2}}{d_0},$$

(2.15f) $$\sigma_2\left(\frac{R_0}{d_0}\right)=\frac{R_0-r_0-\sqrt{(R_0-r_0)^2-(d_0)^2}}{d_0},$$

(2.15g) $$\tau_1\left(\frac{R_0}{d_0}\right)=\frac{R_0+r_0+\sqrt{(R_0+r_0)^2-(d_0)^2}}{d_0},$$

(2.15h) $$\tau_2\left(\frac{R_0}{d_0}\right)=\frac{R_0-r_0+\sqrt{(R_0-r_0)^2-(d_0)^2}}{d_0}.$$

Theorem 2.13. The following relations hold

(2.16a) $$\gamma_1\left(\frac{d_0}{R_0}\right)=\frac{d_1}{R_1},\ \gamma_2\left(\frac{d_0}{R_0}\right)=\frac{d_2}{R_2},$$

(2.16b) $$\varphi_1\left(\frac{d_0}{R_0}\right)=\frac{R_1}{d_1},\ \varphi_2\left(\frac{d_0}{R_0}\right)=\frac{R_2}{d_2},$$

(2.16c) $$\sigma_1\left(\frac{R_0}{d_0}\right)=\frac{d_1}{R_1},\ \sigma_2\left(\frac{R_0}{d_0}\right)=\frac{d_2}{R_2},$$

(2.16d) $$\tau_1\left(\frac{R_0}{d_0}\right)=\frac{R_1}{d_1},\ \tau_2\left(\frac{R_0}{d_0}\right)=\frac{R_2}{d_2},$$

where $(R_i, r_i, d_i),\ i=1,2$ are given by (1.3) and (1.5).

Proof. The relation $R_1d_1=R_2d_2=R_0d_0$, which we can approve as

$$R_1^2d_1^2=\left[R_0(R_0+r_0+\sqrt{(R_0+r_0)^2-d_0^2})\right]\cdot$$
$$\cdot\left[R_0(R_0+r_0-\sqrt{(R_0+r_0)^2-d_0^2})\right]=R_0d_0.$$

the asserted equalities by (2.15) we rewrite as

$$\gamma_1\left(\frac{d_0}{R_0}\right)=\frac{R_0\left(R_0+r_0-\sqrt{(R_0+r_0)^2-(d_0)^2}\right)}{R_0d_0}=\frac{d_1^2}{R_1d_1}=\frac{d_1}{R_1},$$
$$\gamma_2\left(\frac{d_0}{R_0}\right)=\frac{R_0\left(R_0-r_0-\sqrt{(R_0-r_0)^2-(d_0)^2}\right)}{R_0d_0}=\frac{d_2^2}{R_2d_2}=\frac{d_2}{R_2}.$$

The rest is obvious. ◊

Corollary 2.14. We have

$$\gamma_1\left(\frac{d_0}{R_0}\right)=\sigma_1\left(\frac{R_0}{d_0}\right),\quad \gamma_2\left(\frac{d_0}{R_0}\right)=\sigma_2\left(\frac{R_0}{d_0}\right),$$
$$\varphi_1\left(\frac{d_0}{R_0}\right)=\tau_1\left(\frac{R_0}{d_0}\right),\quad \varphi_2\left(\frac{d_0}{R_0}\right)=\tau_2\left(\frac{R_0}{d_0}\right).$$

The shorthand form of (2.16) read

$$\begin{aligned}
\gamma_1(R_0,r_0,d_0)&=(R_1,r_1,d_1), & \gamma_2(R_0,r_0,d_0)&=(R_2,r_2,d_2),\\
\varphi_1(R_0,r_0,d_0)&=(d_1,r_1,R_1), & \varphi_2(R_0,r_0,d_0)&=(d_2,r_2,R_2),\\
\sigma_1(d_0,r_0,R_0)&=(R_1,r_1,d_1), & \sigma_2(d_0,r_0,R_0)&=(R_2,r_2,d_2),\\
\tau_1(d_0,r_0,R_0)&=(d_1,r_1,R_1), & \tau_2(d_0,r_0,R_0)&=(d_2,r_2,R_2),
\end{aligned}$$

where we use the input triples for (2.16). Thus, if a triple (R_0, r_0, d_0) refers to bicentric polygons with incircle then the triple (d_0, r_0, R_0) refers to bicentric polygons with excircle.

Definition 2.15. Let $\Sigma = \tilde{\mathbb{S}} \setminus \tilde{\mathbb{K}}$ and $(d, r, R) \in \tilde{\mathbb{S}} \setminus \tilde{\mathbb{K}}$ if and only if $(R, r, d) \in \mathbb{S} \setminus \mathbb{K}$. Let $\tilde{g} : \tilde{\mathbb{S}} \setminus \tilde{\mathbb{K}} \to \tilde{\mathbb{S}} \setminus \tilde{\mathbb{K}}$ be a function for which $\tilde{g}(d, r, R) = (\hat{d}, \hat{r}, \hat{R})$, where $\hat{d}, \hat{r}, \hat{R}$ are given by (1.14).

The functions g and $\tilde{g}$ we call rational dual functions.

In the sequel we note that relations (1.3), (1.5), (1.6) and (1.7) ensure the validity of the previous theorem.

Theorem 2.16. Let $(R_0, r_0, d_0) \in \mathbb{S} \setminus \mathbb{K}$. Then

$$\begin{aligned}
g\gamma_1(R_0,r_0,d_0)&=g(R_1,r_1,d_1)=(\hat{R}_1,\hat{r}_1,\hat{d}_1)=(R_0,r_0,d_0),\\
\tilde{g}\varphi_1(R_0,r_0,d_0)&=\tilde{g}(d_1,r_1,R_1)=(\hat{d}_1,\hat{r}_1,\hat{R}_1)=(d_0,r_0,R_0),\\
g\sigma_1(d_0,r_0,R_0)&=g(R_1,r_1,d_1)=(\hat{R}_1,\hat{r}_1,\hat{d}_1)=(R_0,r_0,d_0),\\
\tilde{g}\tau_1(d_0,r_0,R_0)&=\tilde{g}(d_1,r_1,R_1)=(\hat{d}_1,\hat{r}_1,\hat{R}_1)=(d_0,r_0,R_0),
\end{aligned}$$

Analogous properties have $\gamma_2, \varphi_2, \sigma_2, \tau_2$.

Corollary 2.17. The function g is left inverse for both of the functions γ_1 and γ_2 and the function $\tilde{g}$ is left inverse for both of τ_1 and τ_2. None of the functions g and $\tilde{g}$ is left the inverse for φ_1, φ_2, σ_1, σ_2.

Here let us remark that there is a conjecture for τ_1 and τ_2 analogous to the Conjecture 2 for γ_1, γ_2. Indeed, let $\tilde{g}(d_k, r_k, R_k) = (d_l, r_l, R_l)$, where k and l are rotation numbers for an odd $n \geq 3$. Then

$$\tau_1(d_l, r_l, R_l) = (d_k, r_k, R_k) \qquad l \text{ even},$$
$$\tau_2(d_l, r_l, R_l) = (d_k, r_k, R_k) \qquad l \text{ odd}.$$

Corollary 2.18. The following implications are valid:

$$\begin{aligned}
\gamma_1(R_0, r_0, d_0) = (R_1, r_1, d_1) &\Rightarrow (\hat{R}_1, \hat{r}_1, \hat{d}_1) = (R_0, r_0, d_0),\\
\gamma_2(R_0, r_0, d_0) = (R_2, r_2, d_2) &\Rightarrow (\hat{R}_2, \hat{r}_2, \hat{d}_2) = (R_0, r_0, d_0),\\
\varphi_1(R_0, r_0, d_0) = (d_1, r_1, R_1) &\Rightarrow (\hat{d}_1, \hat{r}_1, \hat{R}_1) = (d_0, r_0, R_0),\\
\varphi_2(R_0, r_0, d_0) = (d_2, r_2, R_2) &\Rightarrow (\hat{d}_2, \hat{r}_2, \hat{R}_2) = (d_0, r_0, R_0),\\
\sigma_1(d_0, r_0, R_0) = (R_1, r_1, d_1) &\Rightarrow (\hat{R}_1, \hat{r}_1, \hat{d}_1) = (R_0, r_0, d_0),\\
\sigma_2(d_0, r_0, R_0) = (R_2, r_2, d_2) &\Rightarrow (\hat{R}_2, \hat{r}_2, \hat{d}_2) = (R_0, r_0, d_0),\\
\tau_1(d_0, r_0, R_0) = (d_1, r_1, R_1) &\Rightarrow (\hat{d}_1, \hat{r}_1, \hat{R}_1) = (d_0, r_0, R_0),\\
\tau_2(d_0, r_0, R_0) = (d_2, r_2, R_2) &\Rightarrow (\hat{d}_2, \hat{r}_2, \hat{R}_2) = (d_0, r_0, R_0).
\end{aligned}$$

Corollary 2.19. The solution r_0 of (1.7) has the following rational form

$$r_0 = \frac{|2R_i^2 d_i^2 + 2d_i^2 r_i^2 + 2r_i^2 R_i^2 - R_i^4 - d_i^4|}{2r_i(R_i^2 - d_i^2)}, \qquad i = 1, 2.$$

It is interesting to quote that for $i = 1$

$$R_0 = \frac{R_1^2 - d_1^2}{2r_1},\ r_0 = \frac{2R_1^2 d_1^2 + 2d_1^2 r_1^2 + 2r_1^2 R_1^2 - R_1^4 - d_1^4}{2r_1(R_1^2 - d_1^2)},\ d_0 = \frac{2R_1 r_1 d_1}{R_1^2 - d_2^2},$$

but in the case $i = 2$

$$R_0 = \frac{R_2^2 - d_2^2}{2r_2},\ r_0 = \frac{-(2R_2^2 d_2^2 + 2d_2^2 r_2^2 + 2r_2^2 R_2^2) + R_2^4 + d_2^4}{2r_2(R_2^2 - d_2^2)},\ d_0 = \frac{2R_2 r_2 d_2}{R_2^2 - d_2^2},$$

that is

$$2R_1^2 d_1^2 + 2d_1^2 r_1^2 + 2r_1^2 R_1^2 - R_1^4 - d_1^4 > 0, \tag{2.17}$$
$$-2(R_2^2 d_2^2 + d_2^2 r_2^2 + r_2^2 R_2^2) + R_2^4 + d_2^4 > 0. \tag{2.18}$$

By using computer algebra, relations (1.3) and (1.5) the proof immediately follows. Of course, instead of (γ_1, γ_2) any another ordered pair (φ_1, φ_2), (σ_1, σ_2), (τ_1, τ_2) can be treated.

From Corollary 2.18 can be concluded that there is almost complete analogy between properties of the functions γ_1 and γ_2 and the functions τ_1 and τ_2. It is because γ_1 and γ_2 map (R_0, r_0, d_0) onto (R_1, r_1, d_1) and (R_2, r_2, d_2), respectively, and τ_1, τ_2 map (d_0, r_0, R_0) onto (d_1, r_1, R_1) and (d_2, r_2, R_2), respectively; cf. with Conjecture 1.7 and compare relations (1.10).

Theorem 2.20. Let R_0, r_0, d_0 be as in Theorem 2.1 and assume that $g(R_0, r_0, d_0) = (\hat{R}_0, \hat{r}_0, \hat{d}_0)$. Then

$$\text{either} \quad \gamma_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0) \quad \text{or} \quad \gamma_2(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0).$$

Proof. The assertion follows from Theorem 2.1;

The relation $\gamma_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0)$ can be reduced into $\gamma_1\left(\hat{d}_0/\hat{R}_0\right) = d_0/R_0$, consult (1.11b). ◊

By similar way Theorem 2.3 implies the following result.

Theorem 2.21. Let R_0, r_0, d_0 be as in Theorem 2.1. Then

$$\text{either} \quad \tau_1(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (d_0, r_0, R_0) \quad \text{or} \quad \tau_2(\hat{d}_0, \hat{r}_0, \hat{R}_0) = (d_0, r_0, R_0).$$

Conjecture 2.22. Let $(R_0, r_0, d_0) \in \mathbb{R}^3_+$ and $F_n(R_0, r_0, d_0) = 0$ where $n \geq 3$ is an odd integer. If $\gamma_1(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0)$ then rotation numbers of $(\hat{R}_0, \hat{r}_0, \hat{d}_0)$ is even, but if $\gamma_2(\hat{R}_0, \hat{r}_0, \hat{d}_0) = (R_0, r_0, d_0)$ then rotation number of $(\hat{R}_0, \hat{r}_0, \hat{d}_0)$ is odd. Analogous holds for another ordered pairs (φ_1, φ_2), (σ_1, σ_2), (τ_1, τ_2).

3. About n–gons with incircle and those with excircle

In this section we expose some improvements and extension of results reported in [9].

First in brief on a phenomenon which refers to Fuss' relations for bicentric n-gons with excircle where $n \geq 3$ is an odd integer, that is, few words about the geometrical configurations determined by triple (R_0, r_0, d_0) and by (d_0, r_0, R_0) where $R_0, r_0, d_0 > 0$ and $F_n(R_0, r_0, d_0) = 0$, see Figures 5, 6 and 7. The Figure 5 is determined by (R_0, r_0, d_0) and Figure 6 is determined by triple (d_0, r_0, R_0) dual to the triple (R_0, r_0, d_0), that is, $(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0) = (d_0, r_0, R_0)$, compare (1.1). The first figure refers to bicentric polygons with incircle and the second refers to bicetric polygons with excircle. Each of them can be called dual to the other.

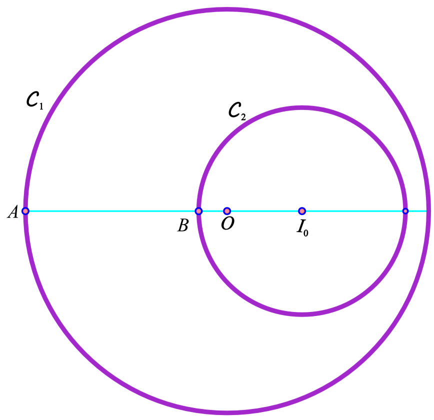

Figure 2: The center of $\mathcal{C}_1$ is point O, the center of $\mathcal{C}_2$ is point I_0, $|AO| = R_0$, $|BI_0| = r_0$, $|OI_0| = d_0$.

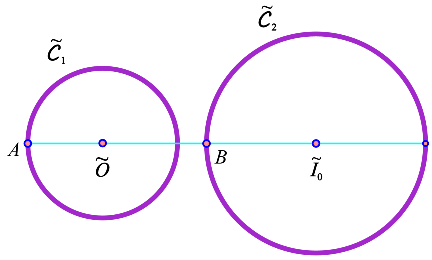

Figure 3: The center of $\tilde{\mathcal{C}}_1$ is denoted by $\tilde{O}$, the center of $\tilde{\mathcal{C}}_2$ is denoted by $\tilde{I}_0$, $\left|A\tilde{O}\right| = \tilde{R}_0 = d_0$, $\left|B\tilde{I}_0\right| = \tilde{r}_0 = r_0$, $\left|\tilde{O}\tilde{I}_0\right| = \tilde{d}_0 =_0$.

Theorem 3.1. Let $R_0, r_0, d_0 > 0$ which satisfy $R_0 > r_0 + d_0$ and $F_n(R_0, r_0, d_0) = 0$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\tilde{\mathcal{C}}_1$, $\tilde{\mathcal{C}}_2$ be positioned according to Figure 3 and Figure 3. Then for any fixed point $P \in \mathcal{C}_1$, there is exists a point $\tilde{P} \in \tilde{\mathcal{C}}_1$ for which holds

$$|PT| = \left|\tilde{P}\tilde{T}\right|, \tag{3.1}$$

where $|PT|$ is the tangent length drawn from P to $\mathcal{C}_2$ and $|\tilde{P}\tilde{T}|$ is the length of tangent drawn from $\tilde{P}$ to $\tilde{\mathcal{C}}_2$.

Proof. Let h denotes homotheticity of the plane which contains circle $\mathcal{C}_1$, such that center O of $\mathcal{C}_1$ is center of h and d_0/R_0 is coefficient of h. Thus h maps circle $\mathcal{C}_1$ onto circle $\mathcal{C}_1'$ which is congruent to the circle $\tilde{\mathcal{C}}_1$ (since $\frac{d_0}{R_0} \cdot R_0 = d_0$). (See Figure 4). The circle $\mathcal{C}_1'$ has radius $d_0 = \tilde{R}_0$ as the circle $\tilde{\mathcal{C}}_1$.

Accordingly, we introduce the rectangular coordinate system xOy in the plane which contains circle $\mathcal{C}_1$ such that center O of $\mathcal{C}_1$ is its origin and center I_0 of $\mathcal{C}_2$ lie on the positive part of x-axis. Thus, if P has coordinates (u_1, v_1) then point P' has coordinates (u', v') so that

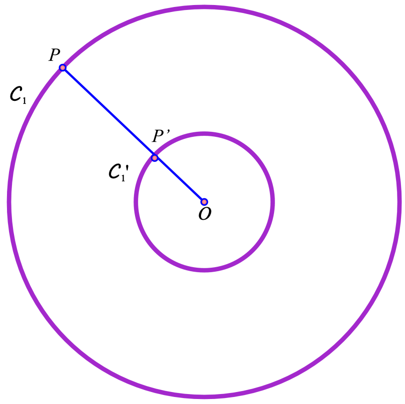

Figure 4:

$u' = \frac{d_0}{R_0}\, u$, $v' = \frac{d_0}{R_0}\, v$. Hence

$$P(u,v) \to P'\left(\frac{d_0}{R_0}\, u, \frac{d_0}{R_0}\, v\right). \tag{3.2a}$$

Also, define another rectangular coordinate system $\tilde{x}\tilde{O}\tilde{y}$ in the plane which contains circle $\tilde{\mathcal{C}}_1$ so, that center $\tilde{O}$ of $\tilde{\mathcal{C}}_1$ is the origin and positive part of $\tilde{x}$–axis contain the center $\tilde{I}_0$ of $\tilde{\mathcal{C}}_2$. Thus, there is a point $\tilde{P} \in \tilde{\mathcal{C}}_1$ with coordinates which are the same as those of point P', which means

$$P(u,v) \to \tilde{P}\left(\frac{d_0}{R_0}\, u, \frac{d_0}{R_0}\, v\right). \tag{3.2b}$$

Now we can write

$$|PT|^2 = (u - d_0)^2 + v^2 - r_0^2 = R_0^2 + d_0^2 - r_0^2 - 2d_0 u,$$
$$\left|\tilde{P}\tilde{T}\right|^2 = (u' - R_0)^2 + (v')^2 - r_0^2 = R_0^2 + d_0^2 - r_0^2 - 2d_0 u,$$

since $|PT|^2 = |PI_0|^2 - r_0^2$, $|\tilde{P}\tilde{T}|^2 = |\tilde{P}\tilde{I}_0|^2 - r_0^2$. ♢

Corollary 3.2. Let (R_0, r_0, d_0) be a positive triple for which $F_n^{(k)}(R_0, r_0, d_0) = 0$, where k is an element from the set of rotation numbers for n. Let $A_1 \cdots A_n$ be an n-gon inscribed in $\mathcal{C}_1$ and circumscribed around $\mathcal{C}_2$ such that the first vertex is P, that is, $A_1 = P$. Let

$t_1, \cdots, t_n$ be tangent lengths of the n-gon $A_1 \cdots A_n$. Then there is an n-gon $\tilde{A}_1 \cdots \tilde{A}_n$ inscribed in $\tilde{\mathcal{C}}_1$ and circumscribed around $\tilde{\mathcal{C}}_2$ such that $\tilde{t}_i = t_i,\ i = 1, \cdots, n$ that is

$$\left|\tilde{A}_i\tilde{T}_i\right| = |A_iT_i|\,, \qquad i = 1, \cdots, n.$$

Proof. For each vertex A_i of the n-gon $A_1 \cdots A_n$ by Theorem 3.1 there is vertex $\tilde{A}_i$ of the n-gon $\tilde{A}_1 \cdots \tilde{A}_n$ for which $\tilde{t}_i = t_i, i = 1, \cdots, n$. From this follows

$$\sum_{i=1}^{n} \arctan \frac{\tilde{t}_i}{r_0} = \sum_{i=1}^{n} \arctan \frac{t_i}{r_0} = k\pi.$$

Thus, the n-gon $\tilde{A}_1 \cdots \tilde{A}_n$ is inscribed in $\tilde{\mathcal{C}}_1$ and circumscribed to $\tilde{\mathcal{C}}_2$. $\Diamond$

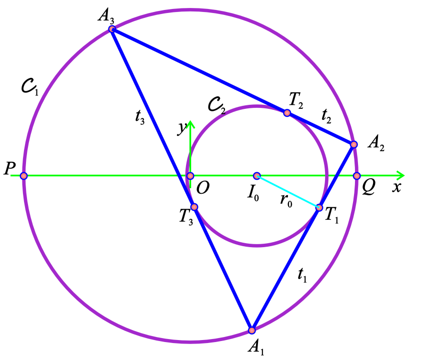

Figure 5: The triple $(R_0, r_0, d_0) = (5, 2.1, 2)$ is a solution of Fuss' relation $F_3(R, r, d) = 0$, that is, $R_0^2 - d_0^2 - 2R_0r_0 = 0$.

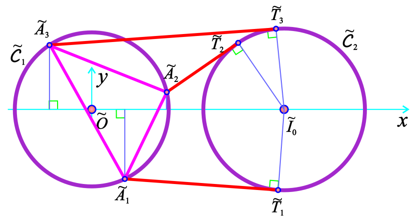

Figure 6: The triple $(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0) = (2, 2.1, 5)$ is a solution of Fuss' relation $d^2 - R^2 - 2dr = 0$ and it holds $|\tilde{A}_i\tilde{T}_i| = |A_iT_i|, i = 1, 2, 3$.

Example 3.3. Consider the case $n = 3$. Observe Figure 5, where $t_1 = |A_1T_1| = 4$, $t_2 = |A_2T_2| = 2.257285251$, $t_3 = |A_3T_3| = 5.973973936$. Using these tangent lengths we see that vertices of the triangle $\Delta A_1A_2A_3$ (related to coordinate system like the one in Theorem 3.1) are

$$A_1(2.1475, -4.51532429), \quad A_2(4.873665824, 1.116962318),$$

$$A_3(-2.74591147, 4.178512917).$$

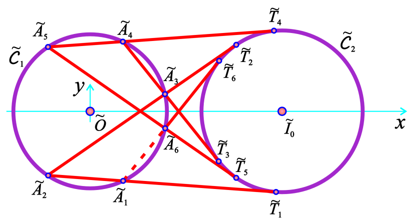

Figure 7: Triangles $\tilde{A}_1\tilde{A}_3\tilde{A}_5$ and $\tilde{A}_4\tilde{A}_6\tilde{A}_2$ are axial symmetric in relation to x-axis and triangle $\Delta\tilde{A}_1\tilde{A}_3\tilde{A}_5$ is congruent to the triangle $\Delta\tilde{A}_1\tilde{A}_2\tilde{A}_3$ shown at Figure 6.

So, for $t_1 = 4$ we have $t_1^2 = R_0^2 + d_0^2 - r_0^2 - 2d_0u_1$ from which follows $u_1 = 2.1475$, since $t_1 = 4$, $R_0 = 5$, $r_0 = 2.1$, $d_0 = 2$.

The vertices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ of the corresponding triangle referred to $(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0) = (2, 2.1, 5)$ are given by $\tilde{A}_i(\tilde{u}_i, \tilde{v}_i)$, $i = 1, 2, 3$, where

$$(\tilde{u}_1, \tilde{v}_i) = \left(\frac{2}{5}\, u_i, \frac{2}{5}\, v_i\right), \qquad i = 1, 2, 3. \tag{3.3}$$

Consider now Figure 6. The vertices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ of the triangle $\Delta\tilde{A}_1\tilde{A}_2\tilde{A}_3$ are given by (3.3).

Accordingly, using $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$ we derive tangent lengths and find that $\left|\tilde{A}_i\tilde{T}_i\right| = |A_iT_i|$, $i = 1, 2, 3$ Therefore from $\tilde{t}_i = t_i$, $i = 1, 2, 3$, bearing in mind that

$$\sum_{i=1}^{3} \arctan\frac{t_i}{r_0} = \pi,$$

we conclude

$$\arctan\frac{\tilde{t}_1}{r_0} + \arctan\frac{\tilde{t}_2}{r_0} + \arctan\frac{\tilde{t}_3}{r_0} = \pi.$$

The triangle $\Delta\tilde{A}_1\tilde{A}_2\tilde{A}_3$ shown at Figure 4(b) is similar to $\Delta A_1A_2A_3$ presented on Figure 5.

The triangles $\Delta\tilde{A}_1\tilde{A}_3\tilde{A}_5$ and $\Delta\tilde{A}_4\tilde{A}_6\tilde{A}_2$ presented in Figure 7 are axial symmetric with respect to x-axis. From this clearly follows that

$$\begin{aligned} |\tilde{A}_1\tilde{T}_1| &= |\tilde{A}_4\tilde{T}_4| = |A_1T_1|, \\ |\tilde{A}_2\tilde{T}_2| &= |\tilde{A}_5\tilde{T}_5| = |A_2T_2|, \\ |\tilde{A}_3\tilde{T}_3| &= |\tilde{A}_6\tilde{T}_6| = |A_3T_3|. \end{aligned} \tag{3.4}$$

Conjecture 3.4. Let $n \geq 3$ be odd integer and let $(R_0, r_0, d_0) \in \mathbb{R}^3_+$ so that $F_n(R_0, r_0, d_0) = 0$. Let C_1, C_2 and $\tilde{C}_1, \tilde{C}_2$ be like C_1, C_2 and $\tilde{C}_1, \tilde{C}_2$ at Figure 2. Then there exists $2n$-gon $\tilde{A}_1 \cdots \tilde{A}_{2n}$ inscribed in $\tilde{C}_1$ and circumscribed to $\tilde{C}_2$ that n-gons

$$\tilde{A}_1\tilde{A}_3 \ldots \tilde{A}_{2n-1}, \quad \tilde{A}_{1+n}\tilde{A}_{3+n} \ldots \tilde{A}_{n+n}\tilde{A}_2\tilde{A}_4\tilde{A}_6 \cdots \tilde{A}_{n-1}$$

are axially symmetric with respect to x-axis. Using those n-gons we get for a given odd $n \geq 3$ geometrical configuration completely analogous to the configurations presented at Figures 6 and 7.

In the case of even $n \geq 4$ only an n-gon $\tilde{A}_1 \ldots \tilde{A}_n$ exists, see Figure 8 and Figure 9 where $n = 4$.

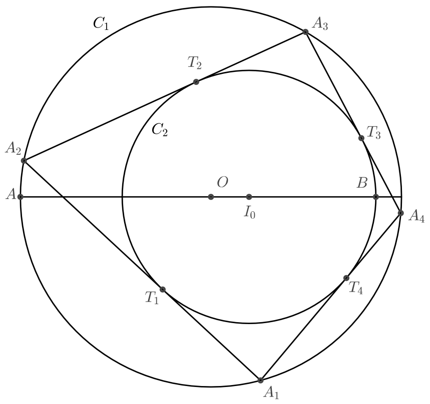

Figure 8: At this figure is $|AO| = R_0$, $|OI_0| = d_0$, $|I_0B| = r_0$, where $R_0 = 5$, $r_0 = 3.328201177\ldots$, $d_0 = 1$ and $F_n(R_0, r_0, d_0) = 0$

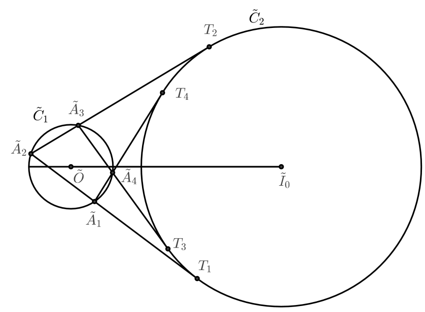

Figure 9: This Figure is dual to Figure 8

Definition 3.5. Let R, d be mutually interchanged in Fuss' relation $F_n(R, r, d) = 0$ for bicentric n-gons with incircle and denote the so obtained relation $\tilde{F}_n(d, r, R) = 0$.

Theorem 3.6. Assume $(R_0, r_0, d_0) \in \mathbb{R}^3_+$ and $F_n(R_0, r_0, d_0) = 0$. Then

$$F_n(R_0, r_0, d_0) = 0 \quad \Leftrightarrow \quad \tilde{F}_n(d_0, r_0, R_0) = 0.$$

Corollary 3.7. Specify $d = a, R = b$; $a > b > 0$ in relation $\tilde{F}_n(d, r, R) = 0$. Then there is $r > 0$ for which $\tilde{F}_n(a, r, b) = 0$ and $F_n(a, r, b) = 0$, that is (a, r, b) is also a solution of Fuss' relation $F_n(R, r, d) = 0$.

It is natural to call relations $F_n(R, r, d) = 0$ and $\tilde{F}_n(d, r, R) = 0$ *dual Fuss' relations for bicentric polygons*, the first one for bicentric polygons with incircle and the second one for bicentric polygons with excircle.

Now, we discuss certain relations for odd $n \geq 3$, see Conjecture 3.4. As an example see also Figure 5, Figure 6 and Figure 7. Here is $\tilde{t}_i = t_{i+3} = t_i$, $i = 1, 2, 3$, where $\tilde{t}_i$ refers to Figure 6, t_{i+3} refers to Figure 7 and t_i turns out to be referred to Figure 5. Triangles $\Delta\tilde{A}_1\tilde{A}_3\tilde{A}_5$ and $\Delta\tilde{A}_4\tilde{A}_6\tilde{A}_2$ are axially symmetric with respect to x-axis and the indices are related in the manner $4 = 1 + 3, 6 = 3 + 3, 5 + 3 = 8, 8 = 2 \cdot 3 + 2$.

The following question arises: which of the Figures 6 and 7 is more appropriate in the theory of bicentric polygons with excircle? The following facts suggest the answer.

($\mathbf{i}_1$) From each of the vertices A_1, A_2, A_3 drawn at Figure 5 there are two equal length tangents to the circle $\mathcal{C}_2$

($\mathbf{i}_2$) From each of the vertices $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ of $\Delta\tilde{A}_1\tilde{A}_2\tilde{A}_3$ drawn at Figure 6 there are two equal length tangents to the circle $\tilde{\mathcal{C}}_2$. These tangents are the same as the tangents drawn at Figure 7 using the well known algorithm.

The facts expressed by ($\mathbf{i}_1$) and ($\mathbf{i}_2$) suggest that the axial symmetric vertices in Figure 7 can be identified and instead of Figure 7 to use Figure 6. The Figure 6 is simpler and from it we easily get the other.

Next, we investigate Fuss' relations for bicentric n-gons with incircle for even $n \geq 4$.

Definition 3.8. Let R_0, r_0, d_0 be any given positive solution of Fuss' relation $F_n(R, r, d) = 0$ for even $n \geq 4$. Then $F_n(R, r, d) = 0$ will be called reduced Fuss' relation for bicentric n-gons with incircle if $F_n(R_0, r_0, d_0) = 0$, but is not $F_n(d_0, r_0, R_0) = 0$, where R, r, d in $F_n(R, r, d) = 0$ are replaced by d_0, r_0, R_0, respectively.

Conjecture 3.9. Any Fuss' relation $F_n(R, r, d) = 0$ for even $n \geq 4$ can be expressed as a reduced one. If Fuss' relation $F_n(R, r, d{=}0$ with n even is a polynomial in R, r, d, then this relation cannot be a reduced one.

Here are some examples where $n = 4, 6$, that is

$$F_4(R, r, d) = p^2 + q^2 - p^2q^2 = 0, \tag{3.5}$$
$$F_6(R, r, d) = 3p^4q^4 - 2p^4q^2 - p^4 - 2p^2q^4 + 2p^2q^2 - q^4 = 0, \tag{3.6}$$

where

$$p = \frac{R + d}{r}, \qquad q = \frac{R - d}{r}.$$

From (3.5) we deduce the reduced Fuss' relation for bicentric quadrilaterals with incircle in the form

$$pq - \sqrt{p^2 + q^2} = 0.$$

Of course, the related reduced Fuss' relation for bicentric quadrilaterals with excircle becomes

$$p(-q) - \sqrt{p^2 + q^2} = 0.$$

From (3.6) we get the reduced Fuss' relations for bicentric hexagons with incircle and those with excircle, which reads

$$pq\sqrt[4]{3} = \sqrt[4]{2p^4q^2 + p^4 + 2p^2q^4 - 2p^2q^2 + q^4},$$
$$p(-q)\sqrt[4]{3} = \sqrt[4]{2p^4q^2 + p^4 + 2p^2q^4 - 2p^2q^2 + q^4}.$$

Obviously, instead of (3.6) we can employ

$$2p^2q^2 = q^4 + 2p^2q^4 + p^4 + 2p^4q^2 - 3p^4q^4,$$

which results in reduced amount of calculations.

Analogously can be reached the reduced Fuss' relation $F_n(R, r, d) = 0$ and $\tilde{F}_n(R, r, d) = 0$ for $n = 8, 10, 12, \ldots, 20$, compare [11, pp. 161-166].

4. Conclusion

In studying bicentric n-gons with incircle and those with excircle it is highly powerful and useful to introduce the functions $\gamma_i, \varphi_i, \sigma_i, \tau_i,\ i = 1, 2$ which any of considered geometrical configurations map onto each other. So, for instance

$$\gamma_1(R_0, r_0, d_0) = (R_1, r_1, d_1), \quad \varphi_1(R_0, r_0, d_0) = (d_1, r_1, R_1),$$
$$\sigma_1(d_0, r_0, R_0) = (R_1, r_1, d_1), \quad \tau_1(d_0, r_0, R_0) = (d_1, r_1, R_1),$$

where the triples turn out to be the solutions of the corresponding Fuss' relations. Hence,

($\mathbf{j}_1$) Each function γ_1, γ_2 map geometrical configuration with incircle onto geometrical configurations with incircle.

($\mathbf{j}_2$) Both functions φ_1, φ_2 map geometrical configuration with incircle onto geometrical configurations with excircle.

($\mathbf{j}_3$) Each of the functions σ_1, σ_2 map geometrical configuration with excircle onto geometrical configurations with incircle.

($\mathbf{j}_4$) The functions τ_1, τ_2 map geometrical configuration with excircle onto geometrical configurations with excircle.

The results obtained are valuable contributions to the theory of bicentric polygons where conics are circles.

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