Mathematica Pannonica
25/2 (2014–2015), 63–74

A CLASS OF WEAKLY BERWALD FINSLER METRICS

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Received: February 27, 2016

MSC 2000: 53 B 40, 53 C 60

Keywords: (α, β)-metric, E-curvature, mean Landsberg curvature.

Abstract: The notion of weakly Berwald metric was introduced by Shen. The class of weakly Berwald Finsler metrics contains the class of Berwald metrics as a special case. In this paper, we find a condition for the existence of weakly Berwaldian (α, β)-metrics which are not Berwaldian.

1. Introduction

The geodesic curves of a Finsler metric $F = F(x, y)$ on a smooth manifold M are determined by the ODE $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. L. Berwald first discovered that the third order derivatives of spray coefficients give rise to an invariant for Finsler metrics. Indeed, the third order vertical derivatives of G^i yields the Berwald curvature

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = 0.$$

A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the Berwald curvature vanishes.

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It is remarkable that, Berwald called these spaces by "affinely connected spaces" in 1927. On a Berwald manifold (M, F), the parallel translation along any geodesic preserves the Minkowski functionals [7]. Thus, Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space. In [10], Z. Shen shows that any regular (α, β)-metrics are Berwald metric if and only if β is parallel with respect to α.

Taking a trace of Berwald curvature give us the mean Berwald curvature

$$E_{jk} := \frac{1}{2} B^m_{\ jkm}.$$

A Finsler metric F is called a weakly Berwald metric if $\mathbf{E} = 0$ [12]. In [1], Akbar-Zadeh used this non-Riemannian quantity when he characterized Finsler metrics of constant flag curvature among the Finsler metrics of scalar flag curvature. By definition, every Berwald metric is weakly Berwaldian, but the converse might not be true in general. For example, Bao-Shen constructed a family of non-Berwaldian Randers metrics on S^3 that are weakly Berwald [4].

In [3], Bácsó-Yoshikawa find the necessary and sufficient condition under which a Randers and Kropina metrics are weakly Berwaldian. In [8], Lee-Lee proved that the infinite series metric $F = \beta^2/(\beta - \alpha)$ is a weakly Berwald metric if and only if β is a killing 1-form with constant length with respect to α. In [13], Peyghan-Tabatabeifar with the author showed that the second approximate Matsumoto metric $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ is a weakly Berwald metric if and only if β is a killing 1-form with constant length with respect to α. In [5], Chen-He-Pan proved that every (α, β)-metric of non-Randers type is weak Berwald if and only if β is a killing 1-form with constant length with respect to α. In [2], Bácsó-Szilágyi give an example of weakly Berwald metric and find a sufficient condition for the existence of weakly Berwald Kropina metric. This motivates us to find some sufficient conditions under which an (α, β)-metric is weakly Berwaldian while is not Berwaldian. Thus the following natural question arises:

Under which conditions, an (α, β)-metric is weakly Berwaldian and not Berwaldian?

An (α, β)-metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, in which $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on M and $b := \|\beta_x\|_\alpha$ (for more details, see [6][11][13][14][15]). For an (α, β)-metric $F := \alpha\phi(s)$, define $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j := b^i s_{ij}, \quad s_0 := s_j y^j.$$

For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, let us define

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \\ \Phi &:= -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \\ \Psi &:= \frac{Q'}{2\Delta}, \quad \Omega := \frac{\Phi}{2\Delta^2}. \end{aligned}$$

In this paper, we find a condition under which an (α, β)-metric is weakly Berwaldian that is not Berwaldian.

Theorem 1.1. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Suppose that F satisfies

$$A_1b_ib_j + A_2(b_iy_j + b_jy_i) + A_3y_iy_j + A_4a_{ij} + A_5(s_ib_j + s_jb_i) + A_6(s_iy_j + s_jy_i) = 0, \tag{1.1}$$

where

$$\begin{aligned} A_1 &:= \frac{1}{2\alpha^2\Delta^2}\Big[\Phi Q'' + 2\Delta^2\Omega''Q + 4\Delta^2\Omega'Q' + 2\Delta^2\Psi''\Big]s_0, \\ A_2 &:= -\frac{1}{2\alpha^3\Delta^2}\Big[2\Delta^2\Psi'' + 2\Omega'\Delta^2 Q \\ &\quad + 2s\Delta^2\Omega''Q + 4s\Delta^2\Omega'Q' + 2s\Delta^2\Psi'' + \Phi Q' + s\Phi Q''\Big]s_0, \\ A_3 &:= \frac{1}{2\alpha^4\Delta^2}\Big[2s^2\Delta^2\Omega''Q + 6s\Delta^2\Psi' + 2s^2\Delta^2\Psi'' + 4s^2\Delta^2\Omega'Q' + s^2\Phi Q'' \\ &\quad + 6s\Omega'\Delta^2 Q + 3s\Phi Q'\Big]s_0, \\ A_4 &:= -\frac{1}{2\alpha^2\Delta^2}\Big[2\Delta^2\Psi' + 2\Omega'\Delta^2 Q + \Phi Q'\Big]ss_0, \\ A_5 &:= \frac{1}{2\alpha\Delta^2}\Big[2\Omega'\Delta^2 Q + 2\Delta^2\Psi' + \Phi Q'\Big], \\ A_6 &:= -\frac{s}{\alpha}A_5. \end{aligned}$$

Let β is a Killing 1-form and not closed. Then F is a weakly Berwaldian metric which is not Berwaldian.

Example 1.2. The Kropina metric $F=\alpha^2/\beta$ on an n-dimensional manifold M satisfies (3.15). The mean Berwald curvature of F is given by

$$E_{kl}=\frac{2(n+1)}{b^2\alpha^6}\Bigg[\Big(\alpha^2(b_k a_{l0}+b_l a_{k0}+a_{kl}\beta)-4a_{k0}a_{l0}\beta\Big)r_{00}-\alpha^2(b_k-2\beta a_{k0})r_{l0}-\alpha^2(b_l-2\beta a_{l0})r_{k0}+\beta r_{kl}\Bigg],$$

where $a_{k0}=a_{ki}y^i$. It is easy to see that, if β is a Killing 1-form, then $\mathbf{E}=0$.

2. Preliminaries

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by $\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i:=\frac{1}{4}g^{il}\Big[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k-\frac{\partial(F^2)}{\partial x^l}\Big],\qquad y\in T_xM.$$

The $\mathbf{G}$ is called the spray associated to (M,F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i+2G^i(\dot{c})=0$.

Define $\mathbf{B}_y:T_xM\otimes T_xM\otimes T_xM\to T_xM$ by $\mathbf{B}_y(u,v,w):=B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

$\mathbf{B}_y(u,v,w)$ is symmetric in u, v and w. From the homogeneity of spray coefficients, we have $\mathbf{B}_y(y,v,w)=0$. $\mathbf{B}$ is called the Berwald curvature. F is called a Berwald if $\mathbf{B}=\mathbf{0}$. In this case, G^i are quadratic in $y\in T_xM$ for all $x\in M$, i.e., there exists $\Gamma^i{}_{jk}=\Gamma^i{}_{jk}(x)$ such that $G^i=\Gamma^i{}_{jk}y^jy^k$.

Define the mean of Berwald curvature $\mathbf{B}_y$, by $\mathbf{E}_y:T_xM\otimes T_xM\to\mathbb{R}$, where

$$\mathbf{E}_y(u,v):=\frac{1}{2}\sum_{i=1}^{n}g^{ij}(y)g_y\Big(\mathbf{B}_y(u,v,e_i),e_j\Big).\tag{2.1}$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y\in TM\setminus\{0\}}$ is called the *mean Berwald curvature* or *E-curvature.* In local coordinates, $\mathbf{E}_y(u,v) := E_{ij}(y)u^iv^j$, where

$$E_{ij} := \frac{1}{2}B^m{}_{mij}.$$

By definition, $\mathbf{E}_y(u,v)$ is symmetric in u and v and we have $\mathbf{E}_y(y,v) = 0$. $\mathbf{E}$ is called the mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$. By definition, the following hold

$$E_{ij} = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m \partial y^i \partial y^j} = \frac{1}{2}\frac{\partial^2 G^m_m}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial^2 \Gamma^m_{ij}}{\partial y^m}.$$

By definition, every Berwald metric is a weakly Berwald metric. But the converse is not true.

3. Proof of Theorem 1.1

For an (α,β)-metric $F := \alpha\phi(s)$, define $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Put

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_j := b^i r_{ij}, \quad s_j := b^i s_{ij},$$

$$r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad s_{i0} := s_{ij}y^j, \ r_0 := r_jy^j, \ s_0 := s_jy^j.$$

Now, we are going to prove the Theorem 1.1. First, we find the formula of E-curvature of (α,β)-metrics. By definition, we have

$$G^i = G^i_\alpha + \alpha Q s^i_0 + (-2Q\alpha s_0 + r_{00})(\Theta\frac{y^i}{\alpha} + \Psi b^i), \tag{3.1}$$

where $G^i = G^i(x,y)$ and $G^i_\alpha = G^i_\alpha(x,y)$ denote the coefficients of F and α respectively in the same coordinate system, and

$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]}, \quad \Psi := \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

Thus, we remark the following which is proved in [10].

Lemma 3.1. Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric on an n-dimensional manifold M. Then the following holds

$$\frac{\partial G^m}{\partial y^m} = y^m\frac{\partial}{\partial x^m}(\ln\sigma_\alpha) + 2\Psi(r_0 + s_0) - \frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0). \tag{3.2}$$

After a long and tedious computations, we obtain the following.

Lemma 3.2. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Then the mean Berwald curvature of F is given by the following

$$\begin{aligned} E_{ij} = \;& C_1 b_i b_j + C_2(b_i y_j + b_j y_i) + C_3 y_i y_j + C_4 a_{ij} + C_5(r_{i0} b_j + r_{j0} b_i) \\ & + C_6(r_{i0} y_j + r_{j0} y_i) + C_7 r_{ij} + C_8(s_i b_j + s_j b_i) + C_9(s_i y_j + s_j y_i) \\ & + C_{10}(r_i b_j + r_j b_i) + C_{11}(r_i y_j + r_j y_i), \end{aligned} \tag{3.3}$$

where

$$C_1 := \frac{1}{2\alpha^3\Delta^2}\Big[\alpha\Big(\Phi Q'' + 2\Delta^2\Omega'' Q + 4\Delta^2\Omega' Q' + 2\Delta^2\Psi''\Big)s_0 + \Big(2\alpha\Delta^2\Psi'' - \Delta^2\Omega''\Big)r_0\Big],$$

$$C_2 := -\frac{1}{2\alpha^4\Delta^2}\Big[\alpha\Big(2\Delta^2\Psi'' + 2\Omega'\Delta^2 Q + 2sQ\Delta^2\Omega'' + 4\Delta^2\Omega' Q' s + 2s\Delta^2\Psi'' + Q'\Phi + sQ''\Phi\Big)s_0 + \Big(2\alpha\Delta^2\Psi' + 2s\alpha\Delta^2\Psi'' - 2\Omega'\Delta^2 - s\Delta^2\Omega''\Big)r_0\Big],$$

$$C_3 := \frac{1}{4\alpha^5\Delta^2}\Big[2s\alpha\Big(2s\Delta^2\Omega'' Q + 6\Omega'\Delta^2 Q + 4s\Delta^2\Omega' Q' + 2s\Delta^2\Psi'' + 6\Delta^2\Psi' + 3\Phi Q' + s\Phi Q''\Big)s_0 - \Big(2\Delta^2 s^2\Omega'' - 12\alpha\Delta^2\Psi' s - 4\alpha\Delta^2\Psi'' s^2 + 3\Phi + \Omega'\Delta^2 s\Big)r_0\Big],$$

$$C_4 := -\frac{1}{4\alpha^3\Delta^2}\Big[2\alpha\Big(2s\Delta^2\Psi' + 2sQ\Omega'\Delta^2 + s\Phi Q'\Big)s_0 - \Big(\Phi + 2s\Omega'\Delta^2 - 4s\alpha\Delta^2\Psi'\Big)r_0\Big],$$

$$C_5 := -\frac{\Omega'}{\alpha^2},$$

$$C_6 := \frac{2\Delta^2 s\Omega' + \Phi}{2\alpha^3\Delta^2},$$

$$
\begin{aligned}
C_7 &:= -\frac{\Phi}{2\alpha\Delta^2},\\
C_8 &:= \frac{1}{2\alpha\Delta^2}\{2\Omega'\Delta^2 Q + 2\Delta^2\Psi' + \Phi Q'\},\\
C_9 &:= -\frac{s}{\alpha}C_8,\\
C_{10} &:= \frac{\Psi'}{\alpha},\\
C_{11} &:= -\frac{s}{\alpha}C_{10}.
\end{aligned}
$$

Proof. The following hold

$$
\alpha_{y^i} = \alpha^{-1}y_i, \qquad s_{y^i} = \frac{b_i\alpha - \alpha_{y^i}\beta}{\alpha^2} = \alpha^{-2}(\alpha b_i - sy_i), \tag{3.4}
$$

where $s = \beta/\alpha$. Put

$$
\big(Q(s)\big)' := Q_{y^i}, \quad \big(\Phi(s)\big)' := \Phi_{y^i}, \quad \big(\Psi(s)\big)' := \Psi_{y^i},
$$

Then

$$
\big(Q(s)\big)' = s_{y^i}Q'(s) = \alpha^{-2}(\alpha b_i - sy_i)Q'(s). \tag{3.5}
$$

$$
\big(\Phi(s)\big)' = s_{y^i}\Phi'(s) = \alpha^{-2}(\alpha b_i - sy_i)\Phi'(s), \tag{3.6}
$$

$$
\big(\Psi(s)\big)' = s_{y^i}\Psi'(s) = \alpha^{-2}(\alpha b_i - sy_i)\Psi'(s). \tag{3.7}
$$

Let us put

$$
\big(Q(s)\big)'' := Q_{y^iy^j}, \quad \big(\Phi(s)\big)'' := \Phi_{y^iy^j}, \quad \big(\Psi(s)\big)'' := \Psi_{y^iy^j},
$$

The following hold

$$
\alpha_{y^iy^j} = \alpha^{-1}(a_{ij} - \alpha^{-2}y_iy_j). \tag{3.8}
$$

Then

$$
\begin{aligned}
s_{y^iy^j} &= \frac{2\alpha_i\alpha_j\beta - \alpha(\alpha_i b_j + \alpha_j b_i + \alpha_{ij}\beta)}{\alpha^3}\\
&= \alpha^{-4}\Big[3sy_iy_j - \alpha(y_ib_j + y_jb_i + a_{ij}\beta)\Big].
\end{aligned}
$$

Therefore

$$\begin{aligned}\big(Q(s)\big)'' &= s_{y^iy^j}Q'(s)+s_{y^i}Q''(s)=\\ &= \frac{3sy_iy_j-\alpha(y_ib_j+y_jb_i+a_{ij}\beta)}{\alpha^4}Q'+\frac{\alpha b_i-sy_i}{\alpha^2}Q'',\end{aligned}$$

$$\begin{aligned}\big(\Phi(s)\big)' &= s_{y^iy^j}\Phi'(s)+s_{y^i}\Phi''(s)=\\ &= \frac{3sy_iy_j-\alpha(y_ib_j+y_jb_i+a_{ij}\beta)}{\alpha^4}\Phi'+\frac{\alpha b_i-sy_i}{\alpha^2}\Phi'',\end{aligned}$$

$$\begin{aligned}\big(\Psi(s)\big)' &= s_{y^iy^j}\Psi'(s)+s_{y^i}\Psi''(s)=\\ &= \frac{3sy_iy_j-\alpha(y_ib_j+y_jb_i+a_{ij}\beta)}{\alpha^4}\Psi'+\frac{\alpha b_i-sy_i}{\alpha^2}\Psi''.\end{aligned}$$

By taking twice vertical derivation of (3.2) with respect to y^i and y^j and considering the above relations, we get (3.3). ♢

By Lemma 3.2, we get the following.

Corollary 3.3. Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on a manifold M. Suppose that β is Killing 1-form with constant length with respect to α, i.e., $r_{ij}=s_i=0$. Then F reduces to a weakly Berwald metric.

The formula of E-curvature of Randers metrics and Kropina metrics computed from the Lemma 3.2 coincides with the one computed in [3].

Proof of Theorem 1.1. Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M. Suppose that β is a Killing 1-form, i.e., $r_{ij}=0$. By putting it in (3.3) and considering (3.15), it follows that F is weakly Berwaldian. Now, we are going to show that the Berwald curvature is not necessarily vanishes. Since β is Killing, then (3.1) reduces to following

$$G^i=G^i_\alpha+\alpha Qs^i_0-2\alpha Qs_0\Big[\Theta\frac{y^i}{\alpha}+\Psi b^i\Big]. \tag{3.9}$$

Taking a vertical derivation of (3.9) with respect to y^j yields

$$G^i{}_j = (G^i_\alpha)_j + \alpha_j Q s^i_0 + \alpha Q_j s^i_0 + \alpha Q s^i_j - 2\Big[\alpha_j Q s_0 + \alpha Q_j s_0 + \alpha Q s_j\Big] \cdot\Big[\Theta\frac{y^i}{\alpha} + \Psi b^i\Big] - 2Q\alpha s_0\Big[\Theta_j\frac{y^i}{\alpha} + \Theta\frac{\delta^i_j\alpha - \alpha_j y^i}{\alpha^2} + \Psi_j b^i\Big]. \tag{3.10}$$

By taking a vertical derivation of (3.10) with respect to y^k, we have

$$\begin{aligned}\Gamma^i{}_{jk} = \bar{\Gamma}^i_{jk} &+ \alpha_{jk} Q s^i_0 + \alpha_j Q_k s^i_0 + \alpha_j Q s^i_k + \alpha_k Q_j s^i_0 + \alpha Q_{jk} s^i_0\\ &+\alpha Q_j s^i_k + (\alpha_k Q + \alpha Q_k)s^i_j - 2\Bigg[(\alpha_{jk}Q + \alpha_j Q_k + \alpha_k Q_j + \alpha Q_{jk})s_0\\ &+(\alpha_j Q + \alpha Q_j)s_k + (\alpha_k Q + \alpha Q_k)s_j\Bigg]\Bigg[\Theta\frac{y^i}{\alpha} + \Psi b^i\Bigg] - 2Q\alpha s_0\Bigg[\Theta_{jk}\frac{y^i}{\alpha}\\ &+\Theta_j\Big(\frac{\delta^i_k\alpha - \alpha_k y^i}{\alpha^2}\Big) + \Theta_k\Big(\frac{\delta^i_j\alpha - \alpha_j y^i}{\alpha^2}\Big) + \Theta\Big(\frac{\delta^i_j\alpha - \alpha_j y^i}{\alpha^2}\Big)_k + \Psi_{jk}b^i\Bigg]\\ &-2\Bigg[\alpha_j Q s_0 + \alpha Q_j s_0 + \alpha Q s_j\Bigg]\Bigg[\Theta_k\frac{y^i}{\alpha} + \Theta\frac{\delta^i_k\alpha - \alpha_k y^i}{\alpha^2} + \Psi_k b^i\Bigg]\\ &-2\Bigg[Q_k\alpha s_0 + Q\alpha_k s_0 + Q\alpha s_k\Bigg]\Bigg[\Theta_j\frac{y^i}{\alpha} + \Theta\frac{\delta^i_j\alpha - \alpha_j y^i}{\alpha^2} + \Psi_j b^i\Bigg].\end{aligned} \tag{3.11}$$

If β is not closed 1-from, then by (3.11) it follows that the Berwald curvature is not vanishing necessarily. This complete the proof. ◊

Remark 3.4. *Let (M, F) be a Finsler manifold. The third order derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ is the Cartan torsion $\mathbf{C}_y$ on T_xM. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. Taking a trace of $\mathbf{C}_y$ and $\mathbf{L}_y$ yield the mean Cartan torsion $\mathbf{I}_y$ and mean Landsberg curvature $\mathbf{J}_y$, respectively. A Finsler metric is called weakly Landsbergian if $\mathbf{J} = 0$. The formula for the mean Cartan torsion of an (α, β)-metric is given by following*

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\Bigg[(n+1)\frac{\phi'}{\phi} - (n-2)\frac{s\phi''}{\phi - s\phi'} - \frac{3s\phi'' - (b^2 - s^2)\phi'''}{(\phi - s\phi') + (b^2 - s^2)\phi''}\Bigg]. \tag{3.12}$$

By taking a horizontal derivation of (3.12) along Finslerian geodesic, we get the mean Landsberg curvature of an (α,β)-metric $F=\alpha\phi(s)$ as follows

$$
\begin{aligned}
J_i = & -\frac{1}{2\Delta\alpha^4}\Bigg[\frac{2\alpha^2}{b^2-s^2}\Big(\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big)(r_0+s_0)h_i \\
& +\frac{\alpha}{b^2-s^2}\Big(\Psi_1+s\frac{\Phi}{\Delta}\Big)\Big(r_{00}-2\alpha Q s_0\Big)h_i+\alpha\Big(\alpha^2\Delta s_{i0}-\alpha Q' s_0 h_i \\
& +\alpha Q(\alpha^2 s_i - y_i s_0)+\alpha^2(r_{i0}-2\alpha Q s_i)-(r_{00}-2\alpha Q s_0)y_i\Big)\frac{\Phi}{\Delta}\Bigg], \qquad (3.13)
\end{aligned}
$$

where $h_i:=\alpha b_i - s y_i$ (see [9]). Let β is a Killing. Then (3.13) reduces to following

$$
\begin{aligned}
J_i = & -\frac{1}{2\Delta\alpha^4}\Bigg[\frac{2\alpha^2}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big]s_0h_i \\
& -\frac{2\alpha^2 Q}{b^2-s^2}(\Psi_1+s\frac{\Phi}{\Delta})s_0h_i+\alpha\Big[\alpha^2\Delta s_{i0}-\alpha Q's_0h_i+\alpha Q(\alpha^2 s_i-y_is_0) \\
& -2\alpha^3 Q s_i+2\alpha Q s_0 y_i\Big]\frac{\Phi}{\Delta}\Bigg]. \qquad (3.14)
\end{aligned}
$$

By (3.14), if β is not closed 1-from then the mean Landsberg curvature is not vanishing necessarily.

By the same argument used in the Theorem 1.1, one can get the following.

Theorem 3.5. Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M. Suppose that F satisfies

$$
D_1b_ib_j+D_2(b_iy_j+b_jy_i)+D_3y_iy_j+D_4a_{ij}+D_5(s_ib_j+s_jb_i)+D_6(s_iy_j+s_jy_i)=0, \qquad (3.15)
$$

where

$$
\begin{aligned}
D_1 := & \ \frac{1}{2\alpha^3}\Big[2\alpha\Phi''-\Omega''\Big]r_0, \\
D_2 := & -\frac{1}{2\alpha^4}\Big[2\alpha\Psi'+2s\alpha\Psi''-2\Omega'-s\Omega''\Big]r_0,
\end{aligned}
$$

$$
\begin{aligned}
D_3 &:= -\frac{1}{4\alpha^5}\Big[2s^2\Omega'' - 12s\alpha\Psi' - 4s^2\alpha\Psi'' + 3\Phi + s\Omega'\Big]r_0,\\
D_4 &:= \frac{1}{4\alpha^3\Delta^2}\Big[\Phi + 2s\Omega'\Delta^2 - 4s\alpha\Delta^2\Psi'\Big]r_0,\\
D_5 &:= -\frac{1}{\alpha^2}\Omega',\\
D_6 &:= \frac{1}{2\alpha^3\Delta^2}\Big[2s\Omega'\Delta^2 + \Phi\Big],\\
D_7 &:= -\frac{1}{2\alpha\Delta^2}\Phi,\\
D_8 &:= \frac{1}{\alpha}\Psi',\\
D_9 &:= -\frac{s}{\alpha^2}\Psi'.
\end{aligned}
$$

Let β is a closed 1-form and not Killing. Then F is a weakly Berwaldian metric which is not Berwaldian.

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