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A NOTE ON A METRIC ASSOCIATED TO CERTAIN FINITE GROUPS

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Abstract: In this short note we introduce a new metric on certain finite groups. It leads to a class of groups for which the element orders satisfy an interesting inequality. This extends the class CP_2 studied in our previous paper [16].

1. Introduction

In group theory, and especially in geometric group theory, several metrics on a finite group G have been studied (see e.g. [2, 4, 6]). These are important because they give a way to measure the distance between any two elements of G. A new metric on certain finite groups G will be presented in the following.

Let $d : G \times G \longrightarrow \mathbb{N}$ be the function defined by

$$d(x, y) = o(xy^{-1}) - 1, \forall\, x, y \in G,$$

where $o(a)$ is the order of $a \in G$. Then d is a metric on G if and only if

$$(*) \qquad o(ab) < o(a) + o(b), \forall\, a, b \in G.$$

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Denote by CP_3 the class of finite groups G which satisfy $(*)$.

Remark that d becomes an ultrametric on G if and only if

$$o(ab) \leq \max\{o(a), o(b)\}, \forall\, a, b \in G,$$

that is G belongs to the class CP_2 studied in [16]. Consequently, CP_2 is properly contained in CP_3 (an example of a finite group in CP_3 but not in CP_2 is the symmetric group S_3). This implies that CP_3 contains all abelian p-groups, as well as the quaternion group Q_8 and the alternating group A_4, but at first sight it is difficult to describe all finite groups in this class. Their study is the main goal of our note.

Most of our notation is standard and will not be repeated here. Basic notions and results on group theory can be found in [7, 11, 12, 15].

2. Main results

First of all, we observe that CP_3 is closed under subgroups. On the other hand, since the cyclic group $\mathbb{Z}_6$ does not belong to CP_3 (it has two elements of orders 2 and 3 whose sum is of order 6), we infer that CP_3 is not closed under direct products or extensions.

Remarks.

1. We know that $CP_2 \subset CP_3$. Then, by Remark 1 of [16], we are able to indicate other three classes of finite p-groups, more large as the class of abelian p-groups, that are contained in CP_3: regular p-groups (see Theorem 3.14 of [15], II, page 47), p-groups whose subgroup lattices are modular (see Lemma 2.3.5 of [13]), and powerful p-groups for p odd (see the main theorem of [18]).

2. Q_8 is the smallest nonabelian p-group contained in CP_3, while the dihedral group D_8 is the smallest p-group not contained in CP_3. Note that all quaternion groups Q_{2^n}, $n \geq 4$, as well as all dihedral groups D_{2n}, $n \geq 4$, does not belong to CP_3.

3. The groups S_n with $n \geq 4$ are not contained in CP_3 (for this it is enough to observe that $S_4 \notin CP_3$: there are $a = (12)(34), b = (13) \in S_4$ such that $o(ab) = 4 \not< o(a) + o(b) = 4$). Similarly, the groups A_n with $n \geq 5$ are also not contained in CP_3.

Next we observe that the group $G = \langle a, b \mid a^4 = b^4 = 1,\ b^{-1}ab = a^{-1}\rangle$ belongs to CP_3, but its quotient $G/\langle b^2\rangle \cong D_8$ does not. This shows that CP_3 is also not closed under homomorphic images.

The following theorem gives a connection between CP_3 and the well-known class of CP-groups (see e.g. [1, 3, 5, 8, 9, 10, 14, 19]).

Theorem 2.1. CP_3 is properly contained in CP.

Proof. Assume that a group G in CP_3 contains an element x whose order is not a prime power. Then there are two powers a and b of x such that $o(a) = p$ and $o(b) = q$, where p and q are distinct primes. Since a and b commute, we have $o(ab) = pq$. By $(*)$ it follows that

$$pq < p + q,$$

a contradiction.

Clearly, S_4 is contained in CP, but not in CP_3. This shows that the inclusion $\mathrm{CP}_3 \subset \mathrm{CP}$ is proper. $\Diamond$

A similar argument as in the above proof leads to the following property of groups in CP_3.

Theorem 2.2. Any abelian subgroup of a group in CP_3 is a p-group. In particular, an abelian group is contained in CP_3 if and only if it is a p-group.

Our next result proves that the intersections of CP_2 and CP_3 with the class of p-groups are the same.

Theorem 2.3. A p-group G is contained in CP_3 if and only if it is contained in CP_2.

Proof. Assume that G belongs to CP_3 and let p^n be its order. We will prove that for every $i = 0, 1, ..., n$ the set $G_i = \{x \in G \mid o(x) \leq p^i\}$ is a normal subgroup of G. Let $x, y \in G_i$. Then $o(x), o(y) \leq p^i$ and therefore $o(xy) < 2p^i$ by $(*)$. On the other hand, we know that G belongs to CP and so $o(xy) = p^j$ for some non-negative integer j. Thus $p^j < 2p^i$, which leads to $j \leq i$, i.e. $xy \in G_i$. This proves that G_i is a subgroup of G. Moreover, G_i is normal in G because the order map is constant on each conjugacy class. Then Theorem A of [16] implies that G belongs to CP_2, as desired. $\Diamond$

By [14] we know that only eight nonabelian finite simple CP-groups exist: $\mathrm{PSL}(2, q)$ for $q = 4, 7, 8, 9, 17$, $\mathrm{PSL}(3, 4)$, $\mathrm{Sz}(8)$, and $\mathrm{Sz}(32)$. All these groups are not contained in CP_3, as shows the following theorem.

Theorem 2.4. CP_3 contains no nonabelian finite simple group.

Proof. Since the product of any two elements of order 2 of a group in CP_3 can have order at most 3, we infer that $\mathrm{PSL}(2, q)$ does not belong to CP_3 whenever $q \geq 4$ (note that $\mathrm{PSL}(2, 2) \cong S_3$ and $\mathrm{PSL}(2, 3) \cong A_4$ belong to CP_3). $\mathrm{PSL}(3, 4)$ has a subgroup isomorphic to $\mathrm{PSL}(2, 4) \cong A_5$, and consequently it also does not belong to CP_3. The same conclusion is obtained for the Suzuki groups $\mathrm{Sz}(8)$ and $\mathrm{Sz}(32)$ because they contain a subgroup isomorphic to D_{10}. $\diamondsuit$

Inspired by Theorem 2.4 and by Corollary E of [16] we came up with the following conjecture.

Conjecture 2.5. CP_3 is properly contained in the class of finite solvable groups.

Finally, we indicate two natural problems concerning the class of finite groups introduced in our paper.

Problem 1. Determine the intersection between CP_3 and CP_1.

Problem 2. Give a precise description of the structure of finite groups contained in CP_3.

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