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ON THE RIEMANN SUMMABILITY OF DOUBLE TRIGONOMETRIC SERIES

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Abstract: In this article we extend the concept of the Riemann summability (see in [10]) from single to double trigonometric series. We make a brief overview of bounded-regular linear transformations applied to double sequences. Using Robison's [7] results in the theory of regular transformations we give two new theorems which are extensions of Riemann's famous theorems (see also in [10, Vol. I, p. 319–320]).

1. Introduction: Riemann summability of single trigonometric series

We briefly summarize the basic definitions and results on the Riemann summability of single trigonometric series. We also note that the Riemann summability of trigonometric series has been investigated exhaustively in [9] by F. Weisz, see especially on p. 202.

Let $\{c_m : m \in \mathbb{Z}\}$ be a doubly infinite series of complex numbers, in symbols: $\{c_m\} \subset \mathbb{C}$. We consider the single trigonometric series

$$\sum_{m\in\mathbb{Z}} c_m e^{imx} \tag{1.1}$$

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with its symmetric partial sums

$$S_M(x) := \sum_{|m|\leqslant M} c_m e^{imx}, \quad M = 0, 1, 2, \dots .$$

The series is convergent if the sequence of its symmetric partial sums converges.

Integrating the series (1.1) formally twice, we obtain the series

$$c_0 \frac{x^2}{2} - \sum_{|m|\geqslant 1} c_m \frac{e^{imx}}{m^2} =: f(x). \tag{1.2}$$

If the sequence $\{c_m\}$ is bounded, then the series in (1.2) converges absolutely and uniformly. Consequently, the function f is defined at every $x \in \mathbb{R}$, and it is continuous.

It is readily seen that

$$\begin{aligned} \frac{\Delta^2 f(x; 2u)}{4u^2} &:= \frac{f(x+2u) + f(x-2u) - 2f(x)}{4u^2} = \\ &= c_0 + \sum_{|m|\geqslant 1} c_m e^{imx} \left(\frac{\sin mu}{mu}\right)^2, \quad u > 0. \end{aligned} \tag{1.3}$$

If the limit of $\Delta^2 f(x; 2u)/4u^2$ exists as $u \to 0$, then it is called the symmetric derivative of f at the point x, and it is denoted by $D^2 f(x)$. Now, if $D^2 f(x_0)$ exists, then the series (1.1) is said to be summable at the point x_0 by the Riemann method of summation, or briefly: Riemann summable to the sum $D^2 f(x_0)$.

The next two theorems were proved by Riemann [6] (see also in [10, Vol. I, pp. 319–320]).

Theorem 1.1. *Suppose that* $\{c_m\} \subset \mathbb{C}$ *is such that*

$$\lim_{|m|\to\infty} c_m = 0. \tag{1.4}$$

If the series (1.1) *converges at some point* x *to a finite sum* S*, then it is also Riemann summable to* S*.*

Theorem 1.2. *If condition* (1.4) *is satisfied, then uniformly in* x *we have*

$$\frac{\Delta^2 f(x; 2u)}{4u} = c_0 u + \sum_{|m|\geqslant 1} c_m e^{imx} \frac{\sin^2 mu}{m^2 u} \to 0 \quad \textit{as} \quad u \to 0.$$

2. New results: Riemann summability of double trigonometric series

Let $\{c_{m,n} : (m,n) \in \mathbb{Z}^2\}$ be a double sequence of complex numbers, in symbols: $\{c_{m,n}\} \subset \mathbb{C}$. We consider the double trigonometric series

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_{m,n} e^{i(mx+ny)} \tag{2.1}$$

with the symmetric rectangular partial sums

$$S_{M,N}(x,y) := \sum_{|m|\leqslant M}\sum_{|n|\leqslant N} c_{m,n} e^{i(mx+ny)}, \quad M,N = 0,1,2,\dots. \tag{2.2}$$

We recall that the double series (2.1) is said to converge in Pringsheim's sense to the finite sum S at some point (x_0, y_0) if for every $\varepsilon > 0$ there exists a natural number M_0 such that

$$|S_{M,N}(x_0,y_0) - S| < \varepsilon \quad \text{if} \quad M,N > M_0.$$

This notion of convergence was introduced by Pringsheim [5]. The convergence of a double series in Pringsheim's sense does not imply the boundedness of its terms, and also does not involve the convergence of any of its row or column series defined respectively by

$$\sum_{m\in\mathbb{Z}} e^{imx}\left(c_{m,-n}e^{-iny} + c_{m,n}e^{iny}\right) \quad (n \in \mathbb{Z})$$

(2.3) and

$$\sum_{n\in\mathbb{Z}} e^{inx}\left(c_{-m,n}e^{-imx} + c_{m,n}e^{imx}\right) \quad (m \in \mathbb{Z}).$$

These are the reasons why Hardy [2] introduced a stronger notion of convergence, namely the regular convergence of double series. The series (2.1) is said to converge regularly to the sum S if it converges to S in Pringsheim's sense and each of its row and column series defined in (2.3) also converge as single series.

Móricz [3] showed that the regular convergence of the series (2.1) is equivalent with the following condition: for every $\varepsilon > 0$ there exists $M_1 \in \mathbb{N}$ such that

$$\left|\sum_{m_0\leqslant|m|\leqslant M}\ \sum_{n_0\leqslant|n|\leqslant N} c_{m,n}e^{i(mx+ny)}\right| < \varepsilon \text{ if } \max\{m_0,n_0\} > M_1 \tag{2.4}$$

$$\text{and } 0 \leqslant m_0 \leqslant M,\ 0 \leqslant n_0 \leqslant N.$$

Integrating the double series (2.1) formally twice with respect to both x and y, we obtain the double series

$$c_{0,0}\frac{x^2y^2}{4}-\frac{y^2}{2}\sum_{|m|\geqslant 1}c_{m,0}\frac{e^{imx}}{m^2}-\frac{x^2}{2}\sum_{|n|\geqslant 1}c_{0,n}\frac{e^{inx}}{n^2}+ + \sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}c_{m,n}\frac{e^{i(mx+ny)}}{m^2n^2} =: F(x,y). \tag{2.5}$$

If the sequence $\{c_{m,n}\}$ is bounded, then the double series in (2.5) converges absolutely and uniformly. Consequently, the function F is defined at every $(x,y)\in\mathbb{R}^2$, and it is continuous.

We introduce the notation (cf. the numerator in (1.3))

$$\begin{aligned}&\Delta^2F(x,y;2u,2v):=\\&:=F(x+2u,y+2v)+F(x-2u,y+2v)+F(x+2u,y-2v)+\\&\quad+F(x-2u,y-2v)-2F(x+2u,y)-2F(x,y+2v)-\\&\quad-2F(x-2u,y)-2F(x,y-2v)+4F(x,y),\quad u,v>0.\end{aligned}$$

It is easy to check that analogously to (1.3) we have

$$\frac{\Delta^2F(x,y;2u,2v)}{16u^2v^2}=c_{0,0}+\sum_{|m|\geqslant 1}c_{m,0}e^{imx}\left(\frac{\sin mu}{mu}\right)^2+\sum_{|n|\geqslant 1}c_{0,n}e^{iny}\left(\frac{\sin nv}{nv}\right)^2+ +\sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}c_{m,n}e^{i(mx+ny)}\left(\frac{\sin mu}{mu}\right)^2\left(\frac{\sin nv}{nv}\right)^2. \tag{2.6}$$

If the limit of $\Delta^2F\,(x,y;2u,2v)\,/16u^2v^2$ exists as $u,v\to 0$, then it may be called the second symmetric derivative of F at the point (x,y), and it may be denoted by $D^2F(x,y)$. Now, if $D^2F(x_0,y_0)$ exists, then the double series (2.1) is said to be summable at the point (x_0,y_0) by the Riemann method of summation, or briefly: Riemann summable to the sum $D^2F(x_0,y_0)$.

The next two theorems are counterparts of Riemann's First and Second Theorems.

Theorem 2.1. *Suppose that $\{c_{m,n}\}\subset\mathbb{C}$ is such that*

$$\lim_{|m|+|n|\to\infty}c_{m,n}=0. \tag{2.7}$$

If the double series (2.1) *converges regularly at some point* (x, y) *to a finite sum* S*, then it is also Riemann summable to* S*.*

Theorem 2.2. *If condition* (2.7) *is satisfied, then uniformly in* (x, y) *we have*

$$\frac{\Delta^2 F\,(x, y; 2u, 2v)}{16uv} \to 0 \quad \textit{as} \quad u, v \to 0.$$

3. Proofs of the new theorems

We recall that the proofs of Riemann's First and Second Theorems (see in [10, Vol. I, pp. 319–320]) based on the following two well-known methods:

(i) summation by parts;

(ii) checking the fulfillment of the Toeplitz conditions, which guarantee that a linear transformation of sequences be regular (see [8] and also [10, Vol. I, pp. 74–75]).

In order to present a linear transformation of double sequences let $\mathcal{A} := \left[a^{j,k}_{m,n} : m, n = 0, 1, 2, ...\right]$ be a doubly infinite matrix of real numbers for all $j, k = 1, 2, \ldots$. Given a double sequence $\{s_{m,n} : m, n = 0, 1, 2, \ldots\}$ of real or complex numbers, the sums

$$t_{j,k} := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a^{j,k}_{m,n} s_{m,n} \tag{3.1}$$

are called the $\mathcal{A}$-means of the sequence $\{s_{m,n}\}$, yield a method of summability (see in [1]). More exactly, the double sequence $\{s_{m,n}\}$ is said to be $\mathcal{A}$-summable to a finite limit s if the $\mathcal{A}$-means exist for all $j, k = 1, 2, \ldots$ in the sense of Pringsheim's converence:

$$\lim_{M,N \to \infty} \sum_{m=0}^{M} \sum_{n=0}^{N} a^{j,k}_{m,n} s_{m,n} = t_{j,k} \quad \text{and} \quad \lim_{j,k \to \infty} t_{j,k} = s.$$

Now, we say that a matrix $\mathcal{A}$ is bounded-regular if every bounded and convergent sequence $\{s_{m,n}\}$ is $\mathcal{A}$-summable to the same limit and the $\mathcal{A}$-means are also bounded. G. M. Robison [7] proved that the necessary and sufficient conditions for $\mathcal{A}$ to be bounded-regular are the following four ones:

(a) $\lim_{j,k\to\infty}\sum_{m=0}^{\infty}\left|a_{m,n}^{j,k}\right| = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots;$

(b) $\lim_{j,k\to\infty}\sum_{n=0}^{\infty}\left|a_{m,n}^{j,k}\right| = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots;$

(c) $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|a_{m,n}^{j,k}\right| \leqslant C < \infty \quad \text{for} \quad j, k = 1, 2, \ldots;$

(d) $\lim_{j,k\to\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{j,k} = 1.$

We note that the finiteness of the double sum in (c) implies the existence of the double sum in (d); and it also implies the convergence of the double series in (3.1) for every bounded and convergent double sequence $\{s_{m,n}\}$. Furthermore, in the special case when

$$\lim_{m,n\to\infty} s_{m,n} = 0,$$

condition (d) is not needed in order to conclude that

$$\lim_{j,k\to\infty} t_{j,k} = 0.$$

After these preliminaries we are ready to prove the theorems given in Sec. 2.

Proof of Theorem 2.1. Let $(x, y) \in \mathbb{R}^2$ be a point at which the series (2.1) converges to the limit S in Pringsheim's sense. By (2.6), it is enough to show that

$$\frac{\Delta^2 F\,(x, y; 2u_j, 2v_k)}{16u_j^2 v_k^2} \to S$$

holds for every $u_j, v_k \to 0$, $u_j, v_k > 0$. Set

$$g : [0,\infty) \to \mathbb{R}, \quad g(0) := 1 \quad \text{and} \quad g(x) = \left(\frac{\sin x}{x}\right)^2 \quad (x \neq 0).$$

This notation and (2.6) with $u = u_j$, $v = v_k$ give

$$\text{(3.2)} \qquad \frac{\Delta^2 F\,(x, y; 2u_j, 2v_k)}{16u_j^2 v_k^2} = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_{m,n} e^{i(mx+ny)} g(mu_j) g(nv_k).$$

Using the symmetric rectangular partial sums of (2.1) defined in (2.2) we have

$$c_{m,n}e^{i(mx+ny)} + c_{-m,-n}e^{-i(mx+ny)} + c_{-m,n}e^{i(-mx+ny)} + c_{m,-n}e^{i(mx-ny)} =$$
$$= S_{m,n} - S_{m-1,n} - S_{m,n-1} + S_{m-1,n-1}, \quad m, n \geqslant 1.$$

Consequently, the symmetric rectangular partial sums of the series in (3.2) can be represented as follows (for further details, see [4]):

$$\sum_{|m|\leqslant M}\sum_{|n|\leqslant N} c_{m,n}e^{i(mx+ny)}g(mu_j)g(nv_k) =$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} S_{m,n}\left[\, g\left(mu_j\right)g\left(nv_k\right) - g\left((m+1)u_j\right)g\left(nv_k\right) - \right.$$

$$\left. - g\left(mu_j\right)g\left((n+1)v_k\right) + g\left((m+1)u_j\right)g\left((n+1)v_k\right)\,\right] +$$

$$+ \sum_{m=0}^{M-1} S_{m,N}\left[g\left(mu_j\right)g\left(Nv_k\right) - g\left((m+1)u_j\right)g\left(Nv_k\right)\right] +$$

$$+ \sum_{n=0}^{N-1} S_{M,n}\left[g\left(Mu_j\right)g\left(nv_k\right) - g\left(Mu_j\right)g\left((n+1)v_k\right)\right] +$$

$$+ S_{M,N}\, g\left(Mu_j\right)g\left(Nv_k\right) =: S_1 + S_2 + S_3 + S_4,$$

say. Our first goal is to prove that

$$\frac{\Delta^2 F\left(x, y; 2u_j, 2v_k\right)}{16u_j^2v_k^2} = \lim_{M,N\to\infty} S_1 = \tag{3.3}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} S_{m,n}\big[g(mu_j)g(nv_k) - g((m+1)\,u_j)g(nv_k) -$$

$$- g(mu_j)g((n+1)v_k) + g((m+1)u_j)g((n+1)v_k)\big],$$

which can be obtained by showing that $S_2, S_3, S_4 \to 0$, as $M, N \to \infty$. First, we observe that (2.4) implies the boundedness of the symmetric rectangular partial sums $S_{m,n}$. In this case $S_4 \to 0$ is clearly satisfied. Second, we have

$$\sum_{m=0}^{\infty}\left|g\left(mu_j\right)g\left(Nv_k\right) - g\left((m+1)u_j\right)g\left(Nv_k\right)\right| = \tag{3.4}$$

$$= g\left(Nv_k\right)\sum_{m=0}^{\infty}\left|g\left(mu_j\right) - g\left((m+1)u_j\right)\right|.$$

Furthermore, we can estimate as follows

(3.5)

$$\sum_{m=0}^{\infty}\left|g\left(mu_j\right) - g\left((m+1)\,u_j\right)\right| = \sum_{m=0}^{\infty}\left|\int_{mu_j}^{(m+1)u_j} g'(t)dt\right| \leqslant \int_0^{\infty}\left|g'(t)\right|dt.$$

Applying l'Hospital's rule twice we find

$$g'(t) = 2 \cdot \frac{t \sin t \cos t - \sin^2 t}{t^3} \to 0 \quad (t \to 0), \tag{3.6}$$

thus we obtain

$$\int_0^\infty |g'(t)|\, dt \leqslant C_1 + 2 \cdot \int_1^\infty \frac{t+1}{t^3} dt =: C_2 (< \infty). \tag{3.7}$$

Since $g\left(Nv_k\right) \to 0$, by (3.4)–(3.7) we may conclude that $S_2 \to 0$. With minor modifications, from

$$\lim_{M \to \infty} \sum_{n=0}^{\infty} \left| g\left(Mu_j\right) g\left(nv_k\right) - g\left(Mu_j\right) g\left((n+1)v_k\right) \right| = 0,$$

we can also deduce that $S_3 \to 0$. Summarizing the above results, (3.3) is proved.

Now, we define the $\mathcal{A} = \left[a_{m,n}^{j,k}\right]$ matrix with

$$\begin{aligned} a_{m,n}^{j,k} :=& g(mu_j)g(nv_k) - g((m+1)\,u_j)g(nv_k) - \\ & - g(mu_j)g((n+1)\,v_k) + g((m+1)\,u_j)g((n+1)\,v_k). \end{aligned}$$

It is clearly seen from (3.3) that the bounded-regularity of the matrix $\mathcal{A}$ ensures the Riemann summability of the series (2.1) to S. Hence, to complete the proof we need to check whether this matrix $\mathcal{A}$ satisfies the (a)–(d) conditions of Robison's theorem. To verify (a) we see that

$$\begin{aligned} \sum_{m=0}^{\infty} \left| a_{m,n}^{j,k} \right| =& \sum_{m=0}^{\infty} \mid g(nv_k)\left(g(mu_j) - g((m+1)u_j)\right) - \\ & - g((n+1)\,v_k)\left(g(mu_j) + g((m+1)\,u_j)\right) \mid = \\ =& \sum_{m=0}^{\infty} \left| \left(g(mu_j) - g((m+1)\,u_j)\right)\left(g(nv_k) - g((n+1)v_k)\right) \right| = \\ =& \left| \left(\frac{\sin nv_k}{nv_k}\right)^2 - \left(\frac{\sin(n+1)v_k}{(n+1)v_k}\right)^2 \right| \cdot \sum_{m=0}^{\infty} \left| g(mu_j) - g((m+1)u_j) \right|. \end{aligned}$$

For any $n = 0, 1, 2, \dots$ we clearly have

$$\left(\frac{\sin nv_k}{nv_k}\right)^2 - \left(\frac{\sin(n+1)v_k}{(n+1)v_k}\right)^2 \to 0 \quad (k \to \infty).$$

Using our results (3.5)–(3.7) again, we get that condition (a) is satisfied. Condition (b) can be shown in the same way.

We may also prove condition (c) from (3.5)–(3.7) in the following way

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|a_{m,n}^{j,k}\right| =$$

$$= \sum_{m=0}^{\infty}\left|g\left(mu_j\right)-g\left(\left(m+1\right)u_j\right)\right| \cdot \sum_{n=0}^{\infty}\left|g\left(nv_k\right)-g\left(\left(n+1\right)v_k\right)\right| <$$

$$< 2C_2 =: C(<\infty) \quad \text{for every } \ j,k=1,2,\ldots.$$

Finally, to verify condition (d) we will show that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{j,k} = 1 \quad \text{for every } j,k=1,2,\ldots.$$

To prove this equality we use the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}^{j,k} = \sum_{m=0}^{\infty}\left(\left(\frac{\sin mu_j}{mu_j}\right)^2 - \left(\frac{\sin(m+1)u_j}{(m+1)u_j}\right)^2\right)\cdot$$

$$\cdot\sum_{n=0}^{\infty}\left(\left(\frac{\sin nv_k}{nv_k}\right)^2 - \left(\frac{\sin(n+1)v_k}{(n+1)v_k}\right)^2\right),$$

where both series have telescoping partial sums which converge to 1 independently of j and k.

Now, Robison's theorem completes the proof of our theorem. ◊

Proof of Theorem 2.2. Similarly to (2.6) we find that

$$\frac{\Delta^2 F\left(x,y;2u_j,2v_k\right)}{16u_jv_k} = u_jv_kc_{0,0} + v_k\sum_{|m|\geqslant 1}c_{m,0}e^{imx}\frac{\sin^2 mu_j}{m^2u_j} +$$

$$+ u_j\sum_{|n|\geqslant 1}c_{0,n}e^{iny}\frac{\sin^2 nv_k}{n^2v_k} + \sum_{|m|\geqslant 1}\sum_{|n|\geqslant 1}c_{m,n}e^{i(mx+ny)}\frac{\sin^2 mu_j}{m^2u_j}\frac{\sin^2 nv_k}{n^2v_k}$$

$$=: S_5+S_6+S_7+S_8,$$

say. We will show that $S_i \to 0$ $(i=5,6,7,8)$ as $u_j, v_k \to 0$ $(j,k\to\infty)$.

It is obvious that $S_5 \to 0$ as $j,k\to\infty$. We may also get $S_6, S_7 \to 0$ $(j,k\to\infty)$ due to Th. 1.2.

We rewrite S_8 into the following equivalent form

$$S_8 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\big(c_{m,n}e^{i(mx+ny)} + c_{-m,-n}e^{-i(mx+ny)} +$$

$$+ c_{-m,n}e^{i(-mx+ny)} + c_{m,-n}e^{i(mx-ny)}\big)\frac{\sin^2 mu_j}{m^2u_j}\frac{\sin^2 nv_k}{n^2v_k}.$$

Since the double sequence
$s_{m,n} := c_{m,n}e^{i(mx+ny)}+c_{-m,-n}e^{-i(mx+ny)}+c_{-m,n}e^{i(-mx+ny)}+c_{m,-n}e^{i(mx-ny)}$
tends to 0 as $m,n \to \infty$, arguing in the same way as in the proof of Th. 2.1 it is enough to prove that the matrix $\mathcal{A} = \left[a^{j,k}_{m,n}\right]$ defined by

$$a^{j,k}_{m,n} := \frac{\sin^2 mu_j}{m^2u_j}\frac{\sin^2 nv_k}{n^2v_k}$$

satisfies the conditions of bounded-regularity. As we have mentioned earlier, condition (d) is superfluous in the special case when our sequence tends to zero. In order to verify (a), we analyze the following series

$$\sum_{m=1}^{\infty}\left|a^{j,k}_{m,n}\right| = \frac{\sin^2 nv_k}{n^2v_k}\sum_{m=1}^{\infty}\frac{\sin^2 mu_j}{m^2u_j}. \tag{3.8}$$

Since

$$\frac{\sin^2 nv_k}{n^2v_k} \to 0 \quad \text{as } k \to \infty,$$

hence we need to show that the series on the right-hand side of (3.8) remains bounded as $j \to \infty$. Set

$$M = M(j) := \left[\frac{1}{u_j}\right] + 1,$$

and accordingly

$$\frac{1}{u_j} < M \leqslant \frac{1}{u_j} + 1.$$

After these observations we may estimate in the following:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{\sin^2 mu_j}{m^2u_j} &= \sum_{m=1}^{M}\frac{m^2u_j^2}{m^2u_j} + \sum_{m=M+1}^{\infty}\frac{1}{m^2u_j} < Mu_j + \frac{1}{u_jM} \leqslant \\ &\leqslant \left(\frac{1}{u_j}+1\right)u_j + \frac{1}{u_j\frac{1}{u_j}} = 2 + u_j < 2 + \max_j u_j.\end{aligned} \tag{3.9}$$

Thus, we justified that condition (a) is satisfied. Analogously, we may obtain that condition (b) is also satisfied. To check the fulfillment of condition (c) we may estimate in a similar way as in (3.9):

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|a^{j,k}_{m,n}\right| &= \sum_{m=1}^{\infty}\frac{\sin^2 mu_j}{m^2u_j}\sum_{n=1}^{\infty}\frac{\sin^2 nv_k}{n^2v_k} \leqslant \\ &\leqslant \left(2+\max_j u_j\right)\left(2+\max_k v_k\right) =: C(<\infty).\end{aligned}$$

Finally, applying Robison's theorem gives the proof of our statement. ◊

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