Mathematica Pannonica
25/1 (2014–2015), 3–15

A SPECIAL DYNAMIC SYSTEM WITH TWO TIME DELAYS

Akio **Matsumoto**

Department of Economics, Senior Researcher, Institute of Further Development of Dynamic Economic Research, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan

Ferenc **Szidarovszky**

Department of Applied Mathematics, University of Pécs, Ifjúság útja 6, H-7624, Pécs, Hungary

Received: January 28, 2014

MSC 2000: 34 K 20, 37 N 40, 91 B 62

Keywords: Time delays, stability, bifurcation.

Abstract: A special dynamic system is analyzed which describes Goodwin's business cycle model (Goodwin, 1951). In realistic economies there are time delays in both investment and consumption. The two time delays have a significant effect on the asymptotic behavior of the system. Without delay the system is locally asymptotically stable with reasonable parameter selection, however in the presence of delays stability might be lost. This paper gives a complete stability analysis of the delayed system by determining the stability switch curves and characterizing the directions of the stability switches based on the monotonic properties of the curves.

1. Introduction

Physical and economic systems often deal with delayed data, so the dynamic equations describing the motion or development of such

E-mail addresses: akiom@tamacc.chuo-u.ac.jp, szidarka@gmail.com

systems are usually delay differential equations. The asymptotical behavior of these systems became a central research topic recently. There are two different ways to model time delays (Cooke and Grossman, 1982). In applying the concept of *continuously distributed delays*, it is assumed that the length of the delay is uncertain following a particular distribution. Cushing (1977) provided a comprehensive summary of the relevant methodology with applications to population dynamics. If the length of the delay is known, then *fixed delays* are considered. Bellman and Cooke (1956) introduced the relevant methodology. The methods and stability conditions are model dependent, so researchers have examined particular model types and investigated their asymptotical behavior. The approach becomes much more complicated if multiple delays are present. The pioneering works of Hale (1979) and Hale and Huang (1993) can be considered as basic breakthrough in this area. The paper of Piotrowska (2007) examined some properties of the stability switch curves for important special models. A large number of works deal with delay differential equations in control engineering applications. Loiseau et al. (2009) presented a good collection of a wide variety of works in this area. For example Ochoa et al. (2009) used Lyapunov matrices and developed methods for their computation. Peet et al. (2009) uses SOS (Sum of Square) and a generalized Zhang's method for stability analysis. Breda et al. (2009) used TRACE-DDE, a contour plot method to construct the stability switch curve without analytic result. More recently Matsumoto and Szidarovszky (2013) gave a complete description of the stability switches and asymptotical properties of a certain class of dynamic systems arising in the study of dynamic oligopolies. However the same approach cannot be used in the case of different dynamic models such as Goodwin's business cycle model (Goodwin, 1951). In this paper we will examine the local asymptotical behavior of the corresponding two-delay model. The paper is organized as follows. The classical Goodwin model is introduced in Sec. 2, and its single-delay extension is discussed in Sec. 3, and then the general case is investigated, where stability switches are determined, and conditions for the local asymptotical stability of the delay system are derived. The last section concludes the paper and further research directions are outlined.

2. The model

Goodwin's classical model can be described by the following two-dimensional system:

$$\varepsilon\dot{y}(t) = \dot{k}(t) - (1-\alpha)y(t), \tag{1}$$
$$\dot{k}(t) = \varphi(\dot{y}(t))$$

where y is the national income, k is the capital stock, $\varphi(\dot{y})$ denotes the induced investment with $\varphi(0) = 0$ and α, ε are positive constants. By combining these equations a single-dimensional nonlinear equation is obtained:

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{2}$$

The local asymptotical stability of this system can be examined by linearization around the steady state $\bar{y} = 0$:

$$\varepsilon\dot{y}(t) - \nu\dot{y}(t) + (1-\alpha)y(t) = 0 \tag{3}$$

where $\nu = \varphi^{'}(0)$. From economic consideration we discuss the case when $\nu < \varepsilon$. By assuming delays in both investment and consumption this equation becomes a delay differential equation with two delays:

$$\varepsilon\dot{y}(t) - \nu\dot{y}(t-\theta) + (1-\alpha)y(t-\sigma) = 0. \tag{4}$$

By introducing the notation

$$a = \frac{\nu}{\varepsilon} \quad \text{and} \quad b = \frac{1-\alpha}{\varepsilon}$$

this equation simplifies as

$$\dot{y}(t) - a\dot{y}(t-\theta) + by(t-\sigma) = 0 \tag{5}$$

with characteristic equation

$$\lambda - a\lambda e^{-\theta\lambda} + be^{-\sigma\lambda} = 0. \tag{6}$$

The stability of system (5) can be examined by finding the locations of the characteristic roots.

3. The single-delay case

Assume first that $\theta = 0$, so equation (6) becomes

$$\lambda(1-a) + be^{-\sigma\lambda} = 0. \tag{7}$$

At $\sigma = 0$ the characteristic root is $-b/(1-a)$, so the system is stable if $a < 1$, which is the case, since $\nu < \varepsilon$. At any stability switch $\lambda = i\omega$, where we can assume that $\omega > 0$, since the conjugate of any characteristic root is also a characteristic root. By substitution into equation (7), we have

$$i\omega(1-a) + b(\cos\sigma\omega - i\sin\sigma\omega) = 0 \tag{8}$$

and separating the real and imaginary parts gives two equations for unknowns ω and σ as

$$\begin{aligned} b\cos\sigma\omega &= 0, \\ \omega(1-a) - b\sin\sigma\omega &= 0, \end{aligned} \tag{9}$$

from which we conclude that $\cos\sigma\omega = 0$ and $\sin\sigma\omega = 1$. So

$$\begin{aligned} \omega &= \frac{b}{1-a}, \\ \sigma &= \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for } k = 0, 1, 2, ..., \end{aligned}$$

that is, we have infinitely many potential stability switches. In order to see if there are actual stability switches we select σ as the bifurcation parameter and consider the characteristic root as function of σ, $\lambda = \lambda(\sigma)$. By implicitly differentiating equation (7) with respect to σ, we have

$$\frac{d\lambda}{d\sigma}(1-a) + be^{-\sigma\lambda}\left(-\lambda - \sigma\frac{d\lambda}{d\sigma}\right) = 0 \tag{10}$$

implying that

$$\begin{aligned} \frac{d\lambda}{d\sigma} &= \frac{\lambda be^{-\sigma\lambda}}{1-a-b\sigma e^{-\sigma\lambda}} = \\ &= -\frac{\lambda^2(1-a)}{1-a+\sigma\lambda(1-a)} = \\ &= -\frac{\lambda^2}{1+\sigma\lambda} \end{aligned} \tag{11}$$

where we used equation (7). If $\lambda = i\omega$, then

$$\frac{d\lambda}{d\sigma} = \frac{\omega^2}{1 + i\sigma\omega} \tag{12}$$

with real part

$$\text{Re}\left[\frac{d\lambda}{d\sigma}\right] = \frac{\omega^2}{1 + (\sigma\omega)^2} > 0. \tag{13}$$

Therefore by gradually increasing the value of σ from zero, at each potential stability switch a pair of characteristic roots changes its real part from negative to positive. So the system becomes unstable at the smallest such value,

$$\sigma_0 = \frac{1 - a}{b}\frac{\pi}{2}, \tag{14}$$

and the stability cannot be regained later. Hence we have the following result:

Proposition 1. *System (5) with $\theta = 0$ and $a < 1$ is locally asymptotically stable if $\sigma < \sigma_0$ and unstable for $\sigma > \sigma_0$. At $\sigma = \sigma_0$, Hopf bifurcation occurs giving the possibility of the birth of limit cycles.*

4. The general case

The characteristic equation of system (5) is considered now. We know that its characteristic roots have negative real parts if $\theta = 0$ and $\sigma < \sigma_0$. At any stability switch $\lambda = i\omega$ (with $\omega > 0$) and by substituting it into equation (6) we get

$$i\omega - ia\omega(\cos\theta\omega - i\sin\theta\omega) + b(\cos\sigma\omega - i\sin\sigma\omega) = 0. \tag{15}$$

By separating the real and imaginary parts we have two equations for three unknowns:

$$\begin{aligned} -a\omega\sin\theta\omega + b\cos\sigma\omega &= 0,\\ a\omega\cos\theta\omega + b\sin\sigma\omega &= \omega. \end{aligned} \tag{16}$$

By introducing the notation

$$x = \cos \theta\omega \text{ and } y = \sin \sigma\omega$$

and using the first equation of (16) we get

$$a\omega\sqrt{1-x^2} = b\sqrt{1-y^2} \tag{17}$$

so

$$a^2\omega^2 - b^2 = a^2\omega^2 x^2 - b^2 y^2. \tag{18}$$

From the second equation of (16) we have

$$y = \frac{\omega - a\omega x}{b}, \tag{19}$$

and by substituting it into (18),

$$a^2\omega^2 - b^2 = a^2\omega^2 x^2 - (\omega - a\omega x)^2 \tag{20}$$

implying that

$$\cos \theta\omega = x = \frac{(1+a^2)\omega^2 - b^2}{2a\omega^2} \tag{21}$$

and from (19),

$$\sin \sigma\omega = y = \frac{(1-a^2)\omega^2 + b^2}{2b\omega}. \tag{22}$$

Feasible solutions exist only if both x and y are in interval $[-1, 1]$ which can be reduced to

$$\frac{b}{1+a} \leq \omega \leq \frac{b}{1-a}. \tag{23}$$

From (16) it is clear that $\sin \theta\omega$ and $\cos \sigma\omega$ have the same sign, therefore we have two parametric curves describing the set of potential stability switches:

$$C_1(k,n) = \begin{cases} \sigma = \dfrac{1}{\omega}\left[\sin^{-1}\left(\dfrac{(1-a^2)\omega^2 + b^2}{2b\omega}\right) + 2k\pi\right] \\ \theta = \dfrac{1}{\omega}\left[\cos^{-1}\left(\dfrac{(1+a^2)\omega^2 - b^2}{2a\omega^2}\right) + 2n\pi\right] \end{cases} \tag{24}$$

and

$$(25)\qquad C_2(k,n) = \begin{cases} \sigma = \dfrac{1}{\omega}\left[\pi - \sin^{-1}\left(\dfrac{(1-a^2)\omega^2+b^2}{2b\omega}\right) + 2k\pi\right] \\ \\ \theta = \dfrac{1}{\omega}\left[2\pi - \cos^{-1}\left(\dfrac{(1+a^2)\omega^2-b^2}{2a\omega^2}\right) + 2n\pi\right] \end{cases}$$

with $k, n = 0, 1, 2, ...$ and

$$\omega \in \left[\frac{b}{1+a}, \frac{b}{1-a}\right].$$

Notice first that at $\omega = b/(1+a)$,

$$\frac{(1+a^2)\omega^2 - b^2}{2a\omega^2} = -1, \ \frac{(1-a^2)\omega^2+b^2}{2b\omega} = 1$$

and at $\omega = b/(1-a)$,

$$\frac{(1+a^2)\omega^2 - b^2}{2a\omega^2} = \frac{(1-a^2)\omega^2+b^2}{2b\omega} = 1.$$

Therefore the initial and end points of $C_1(k,n)$ are

$$I_1(k,n) = \frac{1+a}{b}\left(\frac{\pi}{2} + 2k\pi, \pi + 2n\pi\right),$$
$$E_1(k,n) = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi, 2n\pi\right)$$

and these for $C_2(k,n)$ are

$$I_2(k,n) = \frac{1+a}{b}\left(\frac{\pi}{2} + 2k\pi, \pi + 2n\pi\right),$$
$$E_2(k,n) = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi, 2\pi + 2n\pi\right).$$

Clearly $C_1(k,n)$ and $C_2(k,n)$ have the same initial point and $C_1(k,n+1)$ and $C_2(k,n)$ have identical endpoints. Fig. 1 shows these connecting curves for $k = 0$ and $n = 0, 1, 2, ...$ with the parameter specification of $\alpha = 17/20$, $\varepsilon = 3/4$, $\delta = 6/5$ and $\nu = 9/80$. The continuous curves show $C_1(0,n)$ and the dotted curves give $C_2(0,n)$. These curves are shifted to the right by increasing the value of k.

Notice that with fixed value of k, all initial points with different values of n have the same abscissas, and the same holds for the endpoints as well. The common abscissa values are

$$\sigma_I = \frac{1+a}{b}\left(\frac{\pi}{2} + 2k\pi\right) \quad \text{and} \quad \sigma_E = \frac{1-a}{b}\left(\frac{\pi}{2} + 2k\pi\right),$$

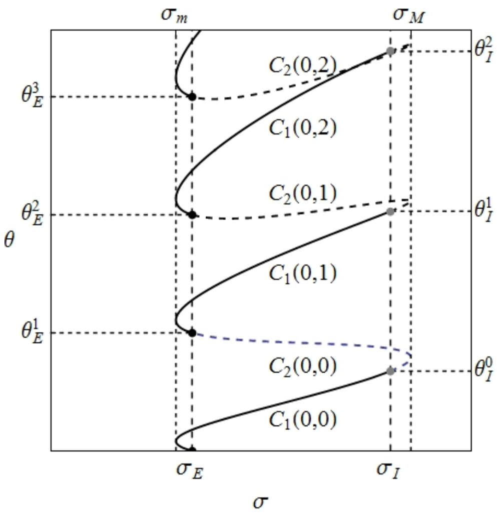

Figure 1. Shapes of curves $C_1(0,n)$ and $C_2(0,n)$ for $n = 0, 1, 2$

respectively. Notice that

$$E_1(0,0) = \left(\frac{1-a}{b}\frac{\pi}{2}, 0\right)$$

so from the previous section we know that the system is stable for

$$\theta = 0 \text{ and } \sigma < \frac{1-a}{b}\frac{\pi}{2}$$

which is the linear segment connecting the origin with $E_1(0,0)$. At the points of the horizontal axis being to the right of $E_1(0,0)$ the system is unstable. Select and fix a value of $\theta > 0$ and gradually increase the value of σ from zero. The resulting horizontal line will have infinitely many intersections with the curves $C_1(k,n)$ and $C_2(k,n)$. The directions of stability switches at the intersections can be determined by considering σ as the bifurcation parameter, and considering the characteristic roots as functions of σ, $\lambda = \lambda(\sigma)$. By implicitly differentiating equation (6) with respect to σ we get a simple equation for $d\lambda/d\sigma$:

$$\frac{d\lambda}{d\sigma} - a\frac{d\lambda}{d\sigma}e^{-\theta\lambda} - a\lambda e^{-\theta\lambda}\left(-\theta\frac{d\lambda}{d\sigma}\right) + be^{-\sigma\lambda}\left(-\lambda - \sigma\frac{d\lambda}{d\sigma}\right) = 0 \tag{26}$$

implying that

$$\frac{d\lambda}{d\sigma} = \frac{b\lambda e^{-\sigma\lambda}}{1 - ae^{-\theta\lambda} + a\lambda\theta e^{-\theta\lambda} - b\sigma e^{-\sigma\lambda}}. \tag{27}$$

From (6) we see that

$$ae^{-\theta\lambda} = \frac{1}{\lambda}\left(\lambda + be^{-\sigma\lambda}\right), \tag{28}$$

so

$$\frac{d\lambda}{d\sigma} = \frac{b\lambda^2}{\lambda^2\theta e^{\sigma\lambda} + (-b + b\lambda\theta - b\lambda\sigma)}. \tag{29}$$

At $\lambda = i\omega$ we have

$$\frac{d\lambda}{d\sigma} = \frac{-b\omega^2}{-\omega^2\theta(\cos\sigma\omega + i\sin\sigma\omega) + (-b + i\omega\theta b - i\omega b\sigma)} \tag{30}$$

with real part having the same sign as

$$b\omega^2(\omega^2\theta\cos\sigma\omega + b).$$

So at any point of the curve $C_1(k,n)$ or $C_2(k,n)$, instability is retained or stability is lost if $\omega^2\theta\cos\sigma\omega + b > 0$ and stability may be regained if $\omega^2\theta\cos\sigma\omega + b < 0$. Notice first that on $C_1(k,n)$,

$$\sigma\omega \in \left[2k\pi, \frac{\pi}{2} + 2k\pi\right],$$

so $\cos\sigma\omega > 0$ implying that on all intersections with $C_1(k,n)$ at least one pair of characteristic roots changes the sign of its real part from negative to positive. Consider next a curve $C_2(k,n)$. On this curve

$$\begin{aligned}\frac{\partial\theta}{\partial\omega} &= -\frac{1}{\omega^2}\left[2\pi - \cos^{-1}\left(\tfrac{(1+a^2)\omega^2 - b^2}{2a\omega^2}\right) + 2n\pi\right] + \\ &\quad + \frac{1}{\omega}\frac{1}{\sqrt{1 - \left(\tfrac{(1+a^2)\omega^2 - b^2}{2a\omega^2}\right)^2}}\frac{2b^2}{2a\omega^3} = \\ &= -\frac{1}{\omega^2}\omega\theta + \frac{1}{\omega}\frac{1}{-\sin\theta\omega}\frac{b^2}{a\omega^3}.\end{aligned} \tag{31}$$

From the first equation of (16), we have

$$\sin\theta\omega = \frac{b}{a\omega}\cos\sigma\theta \tag{32}$$

so

$$\begin{aligned}\frac{\partial\theta}{\partial\omega} &= -\frac{1}{\omega}\left(\theta + \frac{1}{\frac{b}{a\omega}\cos\sigma\theta}\frac{b^2}{a\omega^3}\right) = \\ &= -\frac{1}{\omega^3\cos\sigma\omega}\left(\theta\omega^2\cos\sigma\omega + b\right).\end{aligned} \tag{33}$$

Since $\cos\sigma\omega < 0$ on $C_2(k,n)$, we conclude that stability is lost or instability is retained when $\partial\theta/\partial\omega > 0$ and the stability might be regained if $\partial\theta/\partial\omega < 0$. The first case occurs when the curve $C_2(k,n)$ is increasing in θ from right to left and the second case occurs when the curve is decreasing in θ from right to left.

Next we show that at each intersection the characteristic roots are single. In contrary, assume that λ is a multiple characteristic root. Then it solves the characteristic equation and its derivative:

$$\lambda - a\lambda e^{-\theta\lambda} + be^{-\sigma\lambda} = 0 \tag{34}$$

and

$$1 - ae^{-\theta\lambda} + a\lambda\theta e^{-\theta\lambda} - b\sigma e^{-\sigma\lambda} = 0. \tag{35}$$

If $\lambda = i\omega$, then

$$i\omega - ia\omega(\cos\theta\omega - i\sin\theta\omega) + b(\cos\sigma\omega - i\sin\sigma\omega) = 0 \tag{36}$$

and

$$1 - a(\cos\theta\omega - i\sin\theta\omega) + ia\theta\omega(\cos\theta\omega - i\sin\theta\omega) - b\sigma(\cos\sigma\omega - i\sin\sigma\omega) = 0. \tag{37}$$

By separating the real and imaginary parts, four equations are obtained for the four unknowns, $\sin\theta\omega$, $\cos\theta\omega$, $\sin\sigma\omega$ and $\cos\sigma\omega$:

$$-a\omega\sin\theta\omega + b\cos\sigma\omega = 0, \tag{38}$$

$$\omega - a\omega\cos\theta\omega - b\sin\sigma\omega = 0, \tag{39}$$

$$1 - a\cos\theta\omega + a\theta\omega\sin\theta\omega - b\sigma\cos\sigma\omega = 0, \tag{40}$$

$$a\sin\theta\omega + a\theta\omega\cos\theta\omega + b\sigma\sin\sigma\omega = 0. \tag{41}$$

Simple calculation shows that the solution is the following:

$$\sin\theta\omega = -\frac{\theta\omega}{a\left(1+\omega^2(\sigma-\theta)^2\right)}, \quad \cos\theta\omega = \frac{1+\omega^2\sigma(\sigma-\theta)}{a\left(1+\omega^2(\sigma-\theta)^2\right)}, \tag{42}$$

$$\sin\sigma\omega = \frac{\omega^3\theta(\theta-\sigma)}{b\left(1+\omega^2(\sigma-\theta)^2\right)}, \quad \cos\sigma\omega = -\frac{\theta\omega^2}{b\left(1+\omega^2(\sigma-\theta)^2\right)}, \tag{43}$$

and now from (33) at these values,

$$\frac{\partial\theta}{\partial\omega} = \frac{\theta^2\omega^4 - b^2\left(1+\omega^2(\sigma-\theta)^2\right)}{\omega^3(\cos\sigma\omega)b\left(1+\omega^2(\sigma-\theta)^2\right)} = 0, \tag{44}$$

since from (43),

$$\begin{aligned} 1 &= \sin^2\sigma\omega + \cos^2\sigma\omega = \\ &= \frac{\theta^2\omega^4[1+\omega^2(\sigma-\theta)^2]}{b^2[1+\omega^2(\sigma-\theta)^2]^2} = \\ &= \frac{\theta^2\omega^4}{b^2\left(1+\omega^2(\sigma-\theta)^2\right)}. \end{aligned} \tag{45}$$

Consequently multiple characteristic roots are possible only at the extreme values of θ with respect to ω on $C_2(k,n)$. This is not an intersection since the horizontal line is tangent to the curve at the extreme points.

Let $SG(k,n)$ denote that part of $C_2(k,n)$ which decreases in θ from right to left and let $SL(k,n)$ denote the union of the rest of $C_2(k,n)$ and the entire curve $C_1(k,n)$. We can now summarize our results as follows:

Proposition 2. *The stability switch curves str the connecting segments,* $C_1(k,0)$, $C_2(k,0)$, $C_1(k,1)$, $C_2(k,2), \ldots$ *for all values of* $k \geq 0$. *With fixed* $\theta > 0$ *gradually increasing the value of* σ *from zero, at the intersecions with* $SG(k,n)$ *one pair of characteristic roots changes the sign of the real part from positive to negative, and at the intersections with* $SL(k,n)$ *one pair changes the sign of its real part from negative to positive. At the intersections Hopf bifurcation occurs giving the possibility of the birth of limit cycles.*

At any intersection with $SG(k,n)$ stability is regained if only one pair of characteristic roots had positive real part before. At the intersections with $SL(k,n)$, stability is lost if all pairs of characteristic roots had negative real parts before. Otherwise the system remains unstable after the intersection.

Fig. 2 with $\sigma_m \simeq 6.48$ and $\sigma_M \simeq 9.28$ shows again the continuous curves, $C_1(0,n)$ and $C_2(0,n)$ $(n = 0,1,2,3,\ldots)$ under the same specification of the parameters as before. The horizontal line shows the stability losses and gains. When we increase the value of σ along the horizontal line at $\bar{\theta}(=82)$, stability is lost at point A, regained at point B and lost again at point C. However system is unstable after point C. The stability

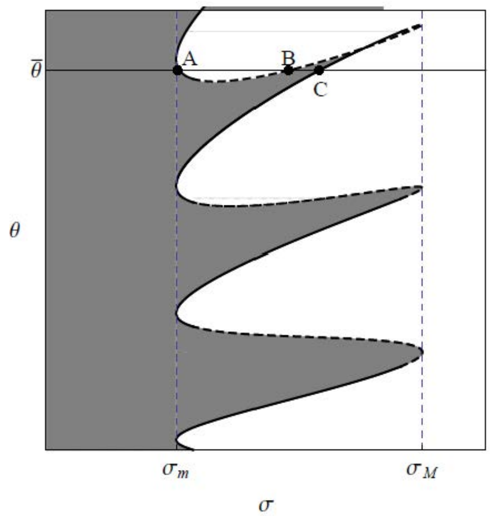

Figure 2. Stability switches

region is the gray region. If (σ, θ) is any point, then we have to consider the linear segment connecting points $(0, \theta)$ and (σ, θ) and count the number of intersections with stability loss (L) and number of intersections with stability gain (G). The point (θ, σ) is a stability point if $G \geq L$.

5. Conclusions

In this paper a special dynamic system with two delays was examined. The stability switch curves were determined and the directions of the stability switches were characterized by the monotonicity of the different segments of the curves. Small values of σ are harmless, since system is stable with any values $\theta > 0$. With large values of σ, the stability region is an irregular domain depending on both values of θ and σ.

This study discovered only local asymptotic stability. The global asymptotic behavior of the system in case of local instability is an interesting research issue which can be examined by computer simulation. This is our next project.

Acknowledgements. The authors are grateful to an anonymous referee for insightful comments and highly appreciate the financial supports from the MEXT-Supported Program for the Strategic Research Foundation at

Private Universities 2013–2017, the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) 24530202, 25380238 and 26380316). The usual disclaimers apply.

References

[1] BELLMAN, R. and COOKE, K. L.: *Differential-difference Equations*, Academic Press, New York, 1956.

[2] BREDA, D., MASET, S. and VERMIGLIO, R.: TRACE-DDE: A Tool for Robust Analysis and Characteristic Equations for Delay Differential Equations, in: *Topics in Time Delay Systems* (Loiseau, J. J. et al. editors), Lecture Notes in Control and Information Sciences 388: 145–155, Springer-Verlag, Berlin–Heidelberg, 2009.

[3] COOKE, K. L. and GROSSMAN, Z.: Discrete Delay, Distributed Delay and Stability Switches, *J. Math. Anal. Appl.* **86** (1982), 592–627.

[4] CUSHING, J. M.: *Integro-Differential Equations and Delay Models in Population Dynamics*, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[5] GOODWIN, R.: The Nonlinear Accelerator and the Persistence of Business Cycles, *Econometrica* **19** (1951), 1–17.

[6] HALE, J.: Nonlinear Oscillations in Equations with Delays, in: *Nonlinear Oscillations in Biology* (Happenstadt, L. C. editor), Lectures in Applied Mathematics, 17: 157–185, Am. Math. Soc., Providence, R. I., 1979.

[7] HALE, J. and HUANG, W.: Global Geometry of the Stable Regions for Two Delay Differential Equations, *J. Math. Anal. Appl.* **178** (1993), 344–362.

[8] LOISEAU, J. J., NUCGUEKS, W., NICULESCU, S.-I. and SIPAHI, R. (eds): *Topics in Time Delay Systems*, Lecture Notes in Control and Information Sciences 388, Springer-Verlag, Berlin–Heidelberg, 2009.

[9] MATSUMOTO, A. and SZIDAROVSZKY, F.: An Elementary Study of a Class of Dynamic Systems with Two Delays, *CUBO, A Mathematical Journal* **14** (2013), 103–113.

[10] OCHOA, G., VELAZQUES, J. E., KHARITONOV, V. and MONDIE, S.: Lyapunov Matrices for Neutral Type Time Delay Systems, in: *Topics in Time Delay Systems* (Loiseau, J. J. et al. editors), Lecture Notes in Control and Information Sciences 388: 145–155, Springer Verlag, Berlin–Heidelberg, 2009.

[11] PEET, M. M., BONNET, C. and ÖZBAY, H.: SOS Methods for Stability Analysis of Neutral Differential System, in: *Topics in Time Delays Systems* (Loiseau, J. J. et al. editors), Lecture Notes in Control and Information Sciences 388: 98–107, Springer-Verlag, Berlin–Heidelberg, 2009.

[12] PIOTROWSKA, M.: A Remark on the ODE with Two Discrete Delays, *J. Math. Anal. Appl.* **329** (2007), 664–674.