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1-PRIMITIVE NEAR-RINGS

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Abstract: We combine the concept of sandwich near-rings with that of centralizer near-rings to get a classification of zero symmetric 1-primitive near-rings as dense subnear-rings of centralizer near-rings with sandwich multiplication. This result generalizes the well known density theorem for zero symmetric 2-primitive near-rings with identity to the much bigger class of zero symmetric 1-primitive near-rings not necessarily having an identity.

1. Introduction

We consider right near-rings, this means the right distributive law holds, but not necessarily the left distributive law. The notation is that of [2]. Primitive near-rings play the same role in the structure theory of near-rings as primitive rings do in ring theory. When a primitive near-ring happens to be a ring, then it is a primitive ring in the usual sense. However, in near-ring theory there exist several types of primitivity depending on the type of simplicity of a near-ring group. A complete description of so called 2-primitive near-rings with identity is available. Such near-rings are dense subnear-rings of special types of centralizer near-rings (see [2] for a thorough discussion). If one studies primitive near-rings without necessarily having an identity, then still results are available but much more technical in detail in comparison to the 2-

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primitive case with identity. Combination of the concepts of centralizer near-rings and sandwich near-rings were used in [4] and in [5] to describe 1-primitive near-rings which do not necessarily have an identity. In this paper we will extend and simplify the results obtained in [4] and [5] and we will also consider 2-primitive near-rings as a special case of 1-primitive near-rings.

We will now briefly give the definitions for zero symmetric primitive near-rings to settle our notation which we will keep throughout the paper.

Let N be a zero symmetric near-ring. Let Γ be an N-group of the near-ring N. An N-ideal I of Γ is a normal subgroup of the group $(\Gamma, +)$ such that $\forall n \in N \forall \gamma \in \Gamma \forall i \in I : n(\gamma + i) - n\gamma \in I$. When N is itself considered as an N-group, then an N-ideal of N is just a left ideal of the near-ring. An N-group Γ of the near-ring N is of type 0 if $\Gamma \neq \{0\}$, if there are no non-trivial N-ideals in Γ, so Γ is a simple N-group, and if there is an element $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Such an element γ will be called a generator of the N-group Γ. The N-group Γ is of type 1 if it is of type 0 and moreover we have $N\gamma = \Gamma$ or $N\gamma = \{0\}$ for any $\gamma \in \Gamma$. Let K be a subgroup of the N-group Γ. K is called an N-subgroup of Γ if $NK \subseteq K$. The N-group Γ is called N-group of type 2 if $N\Gamma \neq \{0\}$ and there are no non-trivial N-subgroups in Γ. It is easy to see that an N-group of type 2 is also of type 1. In case N has an identity element, an N-group of type 1 is also of type 2 (see [2], Prop. 3.4 and Prop. 3.7).

Given an N-group Γ and a subset $\triangle \subseteq \Gamma$, then $(0 : \triangle) = \{n \in \in N | \forall \gamma \in \triangle : n\gamma = 0\}$ will be called the annihilator of $\triangle$. Γ will be called faithful if $(0 : \Gamma) = \{0\}$.

A near-ring N is called 1-primitive if it acts on a faithful N-group Γ of type 1. In such a situation we will say that the near-ring acts 1-primitively on the N-group Γ.

It is a well known fact that any zero symmetric and 1-primitive near-ring with identity which is not a ring is dense (i.e. equal to, in the finite case) in a centralizer near-ring $M_S(\Gamma) := \{f : \Gamma \to \Gamma | \forall \gamma \in \Gamma \ \forall s \in \in S : s(f(\gamma)) = f(s(\gamma)) \text{ and } f(0) = 0\}$, where $(\Gamma, +)$ is a group, 0 denoting its neutral element w.r.t. $+$, S is a fixedpointfree automorphism group of Γ and the near-ring operations are the pointwise addition of functions and function composition (see [2], Th. 4.52 for a detailed discussion).

In case a 1-primitive near-ring contains no identity element, the situation gets more complicated and one can use sandwich near-rings to get a classification of 1-primitive near-rings as certain kind of centralizer near-rings. We introduce the concept of a sandwich near-ring in the next definition which we will need in the next sections. The operation symbol $\circ$ stands for composition of functions.

Definition 1.1. Let $(\Gamma, +)$ be a group, $X \subseteq \Gamma$ a subset of Γ containing the zero 0 of $(\Gamma, +)$ and $\phi : \Gamma \longrightarrow X$ a map such that $\phi(0) = 0$. Define the following operation $\circ'$ on Γ^X (Γ^X denoting the set of all functions mapping from X to Γ): $f \circ' g := f \circ \phi \circ g$ for $f, g \in \Gamma^X$. Then $(\Gamma^X, +, \circ')$ is a near-ring, where the zero preserving functions form a zero symmetric subnear-ring which we denote as $M_0(X, \Gamma, \phi)$. Thus, $M_0(X, \Gamma, \phi) := = \{f : X \to \Gamma | f(0) = 0\}$ where the near-ring operations are the pointwise addition of functions $+$ and $\circ'$. $M_0(X, \Gamma, \phi)$ is called a sandwich near-ring and ϕ is called the sandwich function.

Note that if $\Gamma = X$ and $\phi = id$ then $M_0(X, \Gamma, \phi) = M_0(\Gamma)$, the near-ring of all zero preserving functions of a group. We give two non-trivial examples of sandwich near-rings in the following. They will also serve as examples explaining the concepts of primitivity again by doing some concrete calculations. Let $N := \{f \in M_0(\mathbb{Z}_4) | f(2) = f(3) = 0\}$. With respect to pointwise addition of functions and function composition, N is a zero symmetric near-ring which acts faithfully on the N-group $\mathbb{Z}_4$. For $\gamma \in \{0, 2, 3\}$ we have $N\gamma = \{0\}$ and if $\gamma = 1$ we have $N\gamma = \mathbb{Z}_4$. $\{0, 2\}$ is not an N-ideal of the N-group. To see this, let $f \in N$ be such that $f(1) = 3$. Then, $f(1 + 2) - f(1) = 1 \notin \{0, 2\}$. So, N acts 1-primitively on $\mathbb{Z}_4$. Clearly, $\{0, 2\}$ is an N-subgroup of the N-group $\mathbb{Z}_4$. Thus, N does not act 2-primitively on $\mathbb{Z}_4$. Note that N cannot be isomorphic to a centralizer near-ring since it is missing an identity element. Now let $X := = \{0, 1\}$ and $\phi : \mathbb{Z}_4 \to X$ such that $\phi(0) = \phi(2) = \phi(3) = 0$ and $\phi(1) = 1$. Then, $M_0(X, \mathbb{Z}_4, \phi)$ is a sandwich near-ring. For $f \in M_0(X, \mathbb{Z}_4, \phi)$ we have $f(0) = 0$ and $f(1) \in \{0, 1, 2, 3\}$. Let $g_1, g_2 \in M_0(X, \mathbb{Z}_4, \phi)$ such that $g_1(1) = 1$ and $g_2(1) = 2$. Then multiplication $\circ'$ is done as $g_1 \circ' g_2 = = g_1 \circ \phi \circ g_2$. So, $g_1 \circ' g_2(1) = g_1(\phi(g_2(1))) = g_1(\phi(2)) = g_1(0) = 0$. Thus $g_1 \circ' g_2$ is the zero function. Let $f \in N$ and let $\psi_f : X \to \mathbb{Z}_4, x \mapsto f(x)$. The function $h : N \to M_0(X, \mathbb{Z}_4, \phi), f \mapsto \psi_f$ is easily seen to be a near-ring isomorphism.

Similary, if we let $N_1 := \{f \in M_0(\mathbb{Z}_4) | f(3) = 0\}$ then one sees that N_1 is 2-primitive on $\mathbb{Z}_4$ because $\{0, 2\}$, the only non-trivial subgroup of $\mathbb{Z}_4$, is not an N_1-subgroup. Let $X = \{0, 1, 2\}$ and $\phi : \mathbb{Z}_4 \to X$ such that $\phi(0) = \phi(3) = 0$, $\phi(1) = 1$ and $\phi(2) = 2$. Then, $M_0(X, \mathbb{Z}_4, \phi)$ is a sandwich near-ring. For $f \in M_0(X, \mathbb{Z}_4, \phi)$ we have $f(0) = 0$, $f(1) \in \in \{0, 1, 2, 3\}$ and $f(2) \in \{0, 1, 2, 3\}$. Let $f \in N_1$ and let $\psi_f : X \to \mathbb{Z}_4$, $x \mapsto f(x)$. As in the example before the function $h : N \to M_0(X, \mathbb{Z}_4, \phi)$, $f \mapsto \psi_f$ is again seen to be a near-ring isomorphism.

Combinations of the concepts of centralizer near-rings and sandwich near-rings were used in [4] and in [5] to describe 1-primitive near-rings which do not necessarily have an identity. In [4], 1-primitive near-rings are described as dense subnear-rings of near-rings of the type of $M_0(X, \Gamma, \phi, S) := \{f : X \longrightarrow \Gamma \mid f(0) = 0 \text{ and } \forall s \in S\, \forall x \in X : f(s(x)) = = s(f(x))\}$, see Def. 2.1, where S is an automorphism group of Γ acting without fixed points on the set $X \setminus \{0\}$ and the near-ring operations are pointwise $+$ of functions and sandwich multiplication, but the result and proof in [4] requires that the primitive near-ring has a multiplicative right identity. We will see in this paper that this restriction is not needed. Thus we can generalize the result of [4] to near-rings not necessarily having a multiplicative right identity and obtain a description of all zero symmetric 1-primitive near-rings as well as 2-primitive near-rings as dense subnear-rings of sandwich centralizer near-rings. This construction simplifies the construction of [5].

In [5] also no multiplicative right identity is required and the 1-primitive near-ring is described as dense subnear-ring of a sandwich near-ring $M(X, N, \phi, \psi, B, C) := \{f : X \longrightarrow N \mid \forall s \in S\, \forall x \in X : f(s(x)) = = \psi(s)(f(x))\}$, where X is a non-empty set, $(N, +)$ a group, ϕ the sandwich function, B a subgroup of $\mathrm{Aut}(N, +)$, S a group of permutations on X and $\psi \in \mathrm{Hom}(S, B)$. This construction is more technical than that in [4] and that we will use in our approach in this paper and the functions in $M(X, N, \phi, \psi, B, C)$ are not centralized by elements of S. For the details of the construction we refer the interested reader to [5].

The idea of combining the concepts of centralizer near-rings and sandwich near-rings used in this paper allows us to explicitly describe and construct the sandwich function ϕ which determines the multiplication in the primitive near-ring. This will be done in the last section of this paper and will give us the possibility to construct zero symmetric 1-primitive near-rings without necessarily having an identity systematically. Examples to demonstrate this construction are included.

2. Sandwich centralizer near-rings

The following definition introduces certain types of sandwich near-rings which were used by the author in [3] and [4] to describe near-rings with a multiplicative right identity.

Definition 2.1. Let $(\Gamma, +)$ be a group, $X \subseteq \Gamma$ a subset of Γ containing the zero 0 of $(\Gamma, +)$ and $\phi : \Gamma \longrightarrow X$ a map such that $\phi(0) = 0$. Let $S \subseteq$ $\subseteq \mathrm{End}(\Gamma, +)$, S not empty, be such that $\forall s \in S, \forall \gamma \in \Gamma : \phi(s(\gamma)) = s(\phi(\gamma))$ and such that $S(X) \subseteq X$. Then $M_0(X, \Gamma, \phi, S) := \{f : X \longrightarrow \Gamma \mid f(0) = 0$ and $\forall s \in S, x \in X : f(s(x)) = s(f(x))\}$ is a zero symmetric subnear-ring of $M_0(X, \Gamma, \phi)$ as defined in Def. 1.1, which we call a sandwich centralizer near-ring.

It is straightforward to see that $M_0(X, \Gamma, \phi, S)$ is indeed a zero symmetric subnear-ring of $M_0(X, \Gamma, \phi)$ where the zero of $M_0(X, \Gamma, \phi, S)$ is the zero function $\overline{0}$ on X. Since $S(X) \subseteq X$, $M_0(X, \Gamma, \phi, S)$ is not empty, since $\overline{0}$ is contained in $M_0(X, \Gamma, \phi, S)$. Note that the function $id : X \to \Gamma, x \mapsto x$ is contained in $M_0(X, \Gamma, \phi, S)$ and serves as a multiplicative right identity of the near-ring. As shown in [3], any zero symmetric near-ring with a multiplicative right identity is isomorphic to a sandwich centralizer near-ring $M_0(X, \Gamma, \phi, S)$ with suitable X, Γ, ϕ, S.

3. The equivalence relation $\sim$

Given a 1-primitive near-ring and an N-group Γ of type 1 we now introduce an equivalence relation $\sim$ on Γ. What we need for the proof of our theorems in the next section is a special type of system of representatives w.r.t. $\sim$, being invariant under the N-automorphisms of Γ. The existence of such a representative system will be guaranteed in Lemma 3.3.

Definition 3.1. Let N be a near-ring and let Γ be an N-group. Let $\gamma_1, \gamma_2 \in \Gamma$. Define $\gamma_1 \sim \gamma_2$ iff $\forall n \in N : n\gamma_1 = n\gamma_2$.

It is easy to see that $\sim$ is an equivalence relation. To introduce another notation, we mention that when we have a function f with domain D and $M \subseteq D$, then $f_{|M}$ means the restriction of the function to the set M.

In the proof of Lemma 3.3 we will use that given an N-automorphism s of an N-group Γ, then also the inverse function s^{-1} is an N-automorphism of the N-group Γ. This is straightforward to see as is shown in the next proposition.

Proposition 3.2. *Let N be a zero symmetric near-ring and Γ an N-group. Let $s \in \mathrm{Aut}_N(\Gamma, +)$. Then also the inverse function $s^{-1} \in \mathrm{Aut}_N(\Gamma)$.*

Proof. Let $s \in \mathrm{Aut}_N(\Gamma)$. Clearly, $s^{-1} \in \mathrm{Aut}(\Gamma, +)$. Let $n \in N$ and $\gamma \in \Gamma$. Then, $n\gamma = n(s(s^{-1}(\gamma))) = s(n(s^{-1}(\gamma)))$. Thus, $s^{-1}(n\gamma) = n(s^{-1}(\gamma))$. So we see that also s^{-1} is an N-automorphism. $\Diamond$

Lemma 3.3. *Let N be a zero symmetric near-ring and Γ be an N-group of type 1. Let $S := \mathrm{Aut}_N(\Gamma, +)$. Then there is a set of representatives X of the equivalence relation $\sim$ such that $S(X) \subseteq X$ and $0 \in X$.*

Proof. Let $\theta_1 := \{\gamma \in \Gamma | N\gamma = \Gamma\}$ be the set of generators and $\theta_0 := = \{\gamma \in \Gamma | N\gamma = \{0\}\}$ be the set of non-generators of Γ. Since Γ is an N-group of type 1, $\Gamma = \theta_0 \cup \theta_1$. Let D be a set of representatives w.r.t. $\sim$. We let 0 be the representative for the equivalence class of 0 and therefore, for any $\delta \in \theta_0$, $\delta \sim 0$. Hence, $D = X_1 \cup \{0\}$, with $X_1 \subseteq \theta_1$. Let f be the function which maps every element in θ_1 to its representative in X_1 w.r.t. $\sim$. Let $\gamma \in \theta_1$ and let $S(\gamma) := \{s(\gamma) | s \in S\}$ be the orbit of γ under the action of S on Γ. It is easy to see that $S(\gamma) \subseteq \theta_1$. Let $s_1, s_2 \in S$ and suppose that $s_1(\gamma) \sim s_2(\gamma)$. Then, for all $n \in N$, $s_1(n\gamma) = ns_1(\gamma) = ns_2(\gamma) = s_2(n\gamma)$. Since $\gamma \in \theta_1$ we see that for all $\delta \in \Gamma$, $s_1(\delta) = s_2(\delta)$ and therefore, $s_1 = s_2$. This implies that the restriction $f_{|S(\gamma)}$ of f to the orbit $S(\gamma)$ is an injective map. Let $K := \{\cup_{\gamma \in J} S(\gamma) | J \subseteq \theta_1$ and $f_{|\cup_{\gamma \in J} S(\gamma)}$ is injective$\}$. As we have just shown, f is injective on any single orbit $S(\gamma)$, $\gamma \in \theta_1$. Consequently, K is not the empty set. K is ordered w.r.t. set inclusion $\subseteq$. Let I be an index set such that for $i \in I$, $C_i \in K$ and $(C_i)_{i \in I}$ forms a chain in K. Let $M := \cup_{i \in I} C_i$. So, $M = \cup_{i \in I}(\cup_{\gamma \in M_i} S(\gamma))$ where for $i \in I$, $M_i \subseteq \theta_1$ are suitable sets such that $(C_i)_{i \in I}$ forms a chain. So, the set M is a union of unions of orbits and consequently, M is a union of orbits of elements from θ_1. If we can show that f is injective on M, then $M \in K$. Suppose f is not injective on M. So there are $x, y \in M$, $x \neq y$ such that $f(x) = f(y)$. Thus, there are $j, l \in I$ such that $x \in C_j$ and $y \in C_l$. Since $(C_i)_{i \in I}$ forms a chain, we either have $C_j \subseteq C_l$ or $C_l \subseteq C_j$. So, either $x \in C_l$ and $y \in C_l$ or $x \in C_j$ and $y \in C_j$. f is injective on C_j as well as on C_l and consequently we have $f(x) \neq f(y)$ which is a contradiction to the assumption that f is not injective on M. Thus, $M \in K$ and M is an upper bound for the chain $(C_i)_{i \in I}$. By Zorn's Lemma, K contains a maximal element R, say. We claim that $R \cup \{0\}$ is a set of representatives w.r.t. $\sim$ which is invariant under the action of N-automorphisms of Γ.

Note that as an element of K, R is a union of orbits of S, so $S(R) \subseteq R$. Since f is injective on R, any two elements of R are in different equivalence classes. Suppose there is an element $\alpha \in \theta_1$ such that $\alpha \not\sim r$ for any $r \in R$. Let $s \in S$ and suppose there is an element $r \in R$ such that $s(\alpha) \sim r$. Since s is an N-automorphism, also the inverse function s^{-1} is an N-automorphism and contained in S (see Prop. 3.2). Thus, for all $n \in N$, $s(n\alpha) = ns(\alpha) = nr$ and therefore $n\alpha = ns^{-1}(r)$, so $\alpha \sim s^{-1}(r) \in R$. This is a contradiction to the assumption $\alpha \not\sim r$ for any $r \in R$ and so we see that for any $s \in S$ and any $r \in R$, $s(\alpha) \not\sim r$. But then, f is injective on $R \cup S(\alpha)$, so $R \cup S(\alpha) \in K$. Clearly, $S(\alpha) \not\subseteq R$ and so R is properly contained in $R \cup S(\alpha)$. This contradicts the maximality of R. Since we also have $\delta \sim 0$ for any element $\delta \in \theta_0$ we see that $X := R \cup \{0\}$ is a set of representatives w.r.t $\sim$. Since $S(0) = 0$ and $S(R) \subseteq R$, $S(X) \subseteq X$. $\Diamond$

We keep the notation of Lemma 3.3 to give some comments. We have seen in the proof of the lemma, that given an element $\gamma \in \theta_1$, then $S(\gamma)$ is a set of $\sim$ inequivalent elements. It is easy to see that $S(\theta_1) \subseteq \theta_1$. Consequently, the result of Lemma 3.3 is immediate and we would not have to apply Zorn's Lemma if the N-group Γ has finitely many orbits w.r.t. the action of S on Γ, in particular this is the case when Γ is finite.

4. Density theorems

We will now prove two theorems which are our main theorems of this paper. First we show that up to isomorphism zero symmetric and 1-primitive near-rings show up as dense subnear-rings of sandwich centralizer near-rings with special conditions on X, Γ, ϕ and S. Following this theorem we then can easily prove a similar result for 2-primitive near-rings. We should make clear what density means (see also [2], Prop. 4.26) and fix some more notation.

Definition 4.1. F is a dense subnear-ring of $M_0(X, \Gamma, \phi, S)$ if and only if $\forall s \in \mathbb{N}\ \forall x_1, \ldots, x_s \in X\ \forall g \in M_0(X, \Gamma, \phi, S) \exists f \in F$: $f(x_i) = g(x_i)$ for all $i \in \{1, \ldots, s\}$.

Also, we need the concept of fixedpointfreeness.

Definition 4.2. Let S be a group of automorphisms of a group Γ. Let $M \subseteq \Gamma \setminus \{0\}$ such that $S(M) \subseteq M$. S is called fixedpointfree on M if for $s \in S$ and $m \in M$, $s(m) = m$ implies $s = id$, id being the identity function. S is called a fixedpointfree automorphism group of Γ if it acts fixedpointfree on $\Gamma \setminus \{0\}$.

Let $(\Gamma, +)$ be a group. If I is a normal subgroup of Γ we will denote this as $I \lhd \Gamma$. For $\delta \in \Gamma$, $\delta + I$ is the coset of δ. $\emptyset$ stands for the empty set.

As already pointed out in the introduction, 1-primitive near-rings which are rings are primitive rings in the ring theoretical sense (see [2], Prop. 4.8). So, we restrict our discussion to non-rings. We are now ready to formulate our first theorem.

Theorem 4.3. *Let N be a zero symmetric near-ring which is not a ring. Then the following are equivalent:*

(1) *N is 1-primitive.*

(2) *There exist*

(a) *a group $(\Gamma, +)$,*
(b) *a set $X = \{0\} \cup X_1 \subseteq \Gamma$, $X_1 \neq \emptyset$, $0 \notin X_1$ and 0 being the zero of Γ,*
(c) *$S \leq \mathrm{Aut}(\Gamma, +)$, with $S(X) \subseteq X$ and S acting without fixed points on X_1,*
(d) *a function $\phi : \Gamma \to X$ with $\phi_{|X} = id$, $\phi(0) = 0$ and such that*
$$\forall \gamma \in \Gamma\ \forall s \in S : \phi(s(\gamma)) = s(\phi(\gamma)),$$

such that N is isomorphic to a dense subnear-ring M_S of $M_0(X, \Gamma, \phi, S)$ where X, Γ, ϕ, S additionally satisfy the following property (P):
Let $\Gamma_0 := \{\gamma \in \Gamma | \phi(\gamma) = 0\}$ and $C := \{I \lhd \Gamma | I \subseteq \Gamma_0$ and $\Gamma_0 = \cup_{\delta \in \Gamma_0} \delta + I$ and $\forall \gamma \in \Gamma \setminus \Gamma_0 \forall i \in I : S(\phi(\gamma + i)) = S(\phi(\gamma))\}$. Then $I \in C \Rightarrow (I = \{0\}$ or $\exists i \in I \exists \gamma_1 \in \Gamma \setminus \Gamma_0 \exists s \in S \exists \gamma \in \Gamma : \phi(\gamma_1 + i) = = s(\phi(\gamma_1))$ and $s(\gamma) - \gamma \notin I)$.

Proof. (1) $\Rightarrow$ (2): Let N act 1-primitively on the N-group Γ. Let $\theta_1 := = \{\gamma \in \Gamma | N\gamma = \Gamma\}$ be the set of generators and $\theta_0 := \{\gamma \in \Gamma | N\gamma = \{0\}\}$ be the set of non-generators of Γ. By 1-primitivity of N, $\Gamma = \theta_1 \cup \theta_0$. Let $S = \mathrm{Aut}_N(\Gamma, +)$. On Γ we define the equivalence relation $\sim$ as in

Def. 3.1 by $\gamma_1 \sim \gamma_2$ iff for all $n \in N$, $n\gamma_1 = n\gamma_2$. According to Lemma 3.3 we choose a set of representatives X of the equivalence relation $\sim$ in a way that $S(X) \subseteq X$ and 0 is the representative of the equivalence class of 0. Note that any element in θ_0 is equivalent to 0 w.r.t. $\sim$. Thus, $X = X_1 \cup \{0\}$, where $X_1 \subseteq \theta_1$. Note that $X_1 \neq \emptyset$ because as an N-group of type 1, Γ has a generator.

Let $\phi : \Gamma \to X, \gamma \mapsto x$ where $\gamma \sim x$. Then $\phi_{|X} = id$ and $\phi(0) = 0$ because the representative of the zero equivalence class was taken to be zero.

Next we show that for all $\gamma \in \Gamma$ and $s \in \mathrm{Aut}_N(\Gamma, +)$ we have $s(\phi(\gamma)) = \phi(s(\gamma))$. To show this, we first prove that $s(\phi(\gamma)) \sim s(\gamma)$ for any $\gamma \in \Gamma$. Let $n \in N$. Then, $ns(\phi(\gamma)) = s(n\phi(\gamma))$. Now, $\phi(\gamma) \sim \gamma$, so $s(n\phi(\gamma)) = s(n\gamma) = ns(\gamma)$. This shows that $s(\phi(\gamma)) \sim s(\gamma)$. Consequently, by the definition of ϕ, $\phi(s(\phi(\gamma))) = \phi(s(\gamma))$. Since $S(X) \subseteq X$ and $\phi_{|X} = id$ we have that $\phi(s(\phi(\gamma))) = s(\phi(\gamma))$ which proves the desired property.

Next we prove fixedpointfreeness of S on X_1. Note that $S(X_1) \subseteq X_1$ because $S(X) \subseteq X$ and S is a group of automorphisms, so only the zero 0 in X is mapped to zero. Let $\gamma \in X_1 \subseteq \theta_1$. Then $N\gamma = \Gamma$. Suppose that $s(\gamma) = \gamma$. Therefore, for all $n \in N$ we get $ns(\gamma) = s(n\gamma) = n\gamma$. Thus, for all $\delta \in \Gamma$ we have $s(\delta) = \delta$ and $s = id$.

Now we show how to embed N into $M_0(X, \Gamma, \phi, S)$. For every $n \in N$, let f_n be the function $f_n : X \longrightarrow \Gamma, x \mapsto nx$. We now prove that the mapping $h : n \mapsto f_n$ is an embedding of N into $M_0(X, \Gamma, \phi, S)$.

First we show that for $n \in N$, $f_n \in M_0(X, \Gamma, \phi, S)$, so for all $s \in S$, for all $x \in X$ we must have $s(f_n(x)) = f_n(s(x))$. Since s is an N-automorphism we get $f_n(s(x)) = n(s(x)) = s(nx) = s(f_n(x))$ and consequently, $f_n \in M_0(X, \Gamma, \phi, S)$ since also $f_n(0) = 0$. So, h maps N into $M_0(X, \Gamma, \phi, S)$.

h is a near-ring homomorphism: Let j and k be arbitrary elements of N. Then $h(j + k) = f_{(j+k)} = f_j + f_k$ by right distributivity of N. Let $x \in X$. Then $h(jk)(x) = f_{jk}(x) = (jk)x$. On the other hand, $h(j) \circ' h(k) = f_j \circ \phi \circ f_k$. So, for every $x \in X$, $f_j \circ \phi \circ f_k(x) = j(\phi(kx))$. By definition of ϕ we know that $\phi(kx) \sim kx$ and consequently, $j(\phi(kx)) = = j(kx) = (jk)x$. This shows that $h(j) \circ' h(k) = f_{jk} = h(jk)$.

h is injective: Since h is a near-ring homomorphism, it suffices to show that the kernel of h is zero. Suppose there exists an element $j \in N$ such that f_j is the zero function. This means that $f_j(x) = 0$ for all $x \in X$.

Since X is a set of representatives w.r.t. $\sim$, this implies $j\Gamma = \{0\}$. By faithfulness of Γ we get $j = 0$. Hence, h is injective and this finally proves that h is an embedding.

So we can embed N into the near-ring $M_0(X, \Gamma, \phi, S)$ and we let $h(N) =: M_S$. Consequently, $N \cong M_S$ and it remains to show that M_S is a dense subnear-ring of $M_0(X, \Gamma, \phi, S)$ where X, Γ, ϕ, S additionally satisfy the property (P).

Suppose $x_1 \in X_1$ and $x_2 \in X_1$ are from different orbits of S acting on X_1 and suppose x_1 and x_2 have the same annihilator. Then, $Nx_1 = = Nx_2 = \Gamma$ and $s : \Gamma \longrightarrow \Gamma, nx_1 \mapsto nx_2$ is a well defined N-automorphism of Γ, which is straightforward to see. Since $x_1 \in \theta_1$, there is an element $k \in N$ such that $kx_1 = x_1$. Consequently, $s(x_1) = kx_2$. For any $n \in N$ we have $nx_1 = nkx_1$ and therefore $n - nk \in (0 : x_1) = (0 : x_2)$. It follows that $nx_2 = nkx_2$ for all $n \in N$ which means that $s(x_1) = kx_2 \sim x_2$. Since $s(x_1) \in X_1$, $s(x_1) = \phi(s(x_1)) = x_2$ which contradicts the assumption that $x_1 \in X_1$ and $x_2 \in X_1$ are from different orbits of S acting on X_1. So, elements from different orbits of S acting on X_1 must have different annihilators.

Take $x_1, \ldots, x_l \in X_1$, $l \in \mathbb{N}$, from finitely many different orbits of S acting on X_1. Then all elements in $\{x_1, \ldots, x_l\}$ have different annihilators as we have just shown. Since N is not a ring we can apply Th. 4.30 of [2] to get that for all $\gamma_1, \ldots, \gamma_l \in \Gamma$ there exists some $n \in N$ such that $nx_i = \gamma_i$ for all $i \in \{1, \ldots, l\}$. Hence, for all $i \in \{1, \ldots, l\}$, $h(n)(x_i) = = f_n(x_i) = nx_i = \gamma_i$.

We now have to show that

$\forall l \in \mathbb{N}\ \forall x_1, \ldots, x_l \in X\ \forall f \in M_0(X, \Gamma, \phi, S) \exists m \in M_S$: $m(x_i) = f(x_i)$ for all $i \in \{1, \ldots, l\}$.

For any function $f \in M_0(X, \Gamma, \phi, S)$ and any function $m \in M_S$ we have $f(0) = m(0) = 0$. So it suffices to consider the case that for $l \in \mathbb{N}$, $x_1, \ldots, x_l \in X_1$. Let $f \in M_0(X, \Gamma, \phi, S)$. Let $v \in \mathbb{N}$ and let $z_1, \ldots, z_v$ be a set of orbit representatives for the elements $x_1, \ldots, x_l \in X_1$ under the action of S on X_1. Thus, $z_1, \ldots, z_v$ have different annihilators and so there is an element $m \in N$ such that $f_m(z_i) = f(z_i)$ for $i \in \{1, \ldots, v\}$. For $k \in \{1, \ldots, l\}$ there exists a unique $j \in \{1, \ldots, v\}$ such that $x_k \in \in S(z_j)$. Then $x_k = s(z_j)$ for some, by fixedpointfreeness of S on X_1, unique s and so, $f_m(x_k) = f_m(s(z_j)) = s(f_m(z_j)) = s(f(z_j)) = = f(s(z_j)) = f(x_k)$. So, f_m and f are equal functions when restricted to the set $\{x_1, \ldots, x_l\}$ and also $f_m \in M_S$. This proves density of M_S in

$M_0(X, \Gamma, \phi, S)$.

Finally we have to show that X, Γ, ϕ, S satisfy the property (P). Let $I \in C$. Assume that $I \neq \{0\}$. $I \neq \Gamma$ since $\Gamma_0 \neq \Gamma$, so I is a non-trivial normal subgroup of Γ. Suppose property (P) does not hold, so assume that for all $i \in I$, for all $\gamma_1 \in \Gamma \setminus \Gamma_0$, for all $s \in S$ and for all $\gamma \in \Gamma$ either $\phi(\gamma_1 + i) \neq s(\phi(\gamma_1))$ holds or $s(\gamma) - \gamma \in I$ holds. Since $I \in C$, for all $i \in I$ and for all $\gamma_1 \in \Gamma \setminus \Gamma_0$ there is an $s \in S$ such that $\phi(\gamma_1 + i) = s(\phi(\gamma_1))$ and therefore, for this $s \in S$, $s(\gamma) - \gamma \in I$ for all $\gamma \in \Gamma$.

Let $n \in N$ and $\gamma_0 \in \Gamma_0$. Thus, $\gamma_0 + I \subseteq \Gamma_0$ because by definition of the elements in C, Γ_0 is a union of cosets of I. Consequently, for all $j \in I$, $\gamma_0 + + j \in \gamma_0 + I \subseteq \Gamma_0$. Thus, $n(\gamma_0 + j) - n\gamma_0 = n\phi(\gamma_0 + j) - n\phi(\gamma_0) = 0 - 0 \in I$. Let $\gamma_1 \in \Gamma \setminus \Gamma_0$ and $i \in I$. Consequently, $\phi(\gamma_1 + i) = s(\phi(\gamma_1))$ for some $s \in S$ and so, since property (P) is assumed not to hold, $s(\gamma) - \gamma \in I$ for all $\gamma \in \Gamma$. So, for all $n \in N$, $n(\gamma_1 + i) - n\gamma_1 = n\phi(\gamma_1 + i) - n\phi(\gamma_1) = = ns(\phi(\gamma_1)) - n\phi(\gamma_1) = s(n\phi(\gamma_1)) - n\phi(\gamma_1) \in I$. This shows that I is a non-trivial and proper N-ideal of Γ, contradicting that N is 1-primitive on Γ. Hence, property (P) must hold.

(2) $\Rightarrow$ (1): We have to show that $N \cong M_S$ is a 1-primitive near-ring. Γ is an M_S-group in a natural way by defining the action $\odot$ of M_S on Γ as $m \odot \gamma := m(\phi(\gamma))$ for $m \in M_S$ and $\gamma \in \Gamma$. Since $\phi_{|X} = id$ we have $X = \phi(\Gamma)$, so Γ is a faithful M_S-group.

Let $\gamma \in \Gamma$ and suppose $\phi(\gamma) = 0$, so $\gamma \in \Gamma_0$. Then clearly $M_S \odot \gamma = = \{0\}$. On the other hand there exist elements $\gamma \in \Gamma$, such that $0 \neq \neq \phi(\gamma) \in X_1$ because $\phi_{|X} = id$ and $X_1 \neq \emptyset$. Let γ be such that $\phi(\gamma) \in X_1$, so $\gamma \in \Gamma \setminus \Gamma_0$. Let $\delta \in \Gamma$. We now define a function $f : X \longrightarrow \Gamma$ in the following way: $f(\phi(\gamma)) := \delta$. Let $s \in S$. Since S acts without fixedpoints on X_1, we can well define $f(s(\phi(\gamma))) := s(f(\phi(\gamma))) = s(\delta)$ and $f(X \setminus S(\phi(\gamma))) := \{0\}$. From the definition of f we see that $f \in M_0(X, \Gamma, \phi, S)$, so by density of M_S, there is a function $m \in M_S$ with $m(\phi(\gamma)) = f(\phi(\gamma)) = \delta$. Since $\delta \in \Gamma$ was chosen arbitrary, this shows that $M_S \odot \gamma = \Gamma$. Consequently, for $\delta \in \Gamma_0$ we have $M_S \odot \delta = \{0\}$ and for $\gamma \in \Gamma \setminus \Gamma_0$ we have $M_S \odot \gamma = \Gamma$.

We now show that there are no non-trivial M_S-ideals in Γ. Suppose that I is a non-trivial M_S-ideal of Γ. Then I is a non-trivial normal subgroup of $(\Gamma, +)$, and for all $m \in M_S$, $\gamma \in \Gamma$ and $i \in I$ we have that $m \odot (\gamma + i) - m \odot \gamma = m(\phi(\gamma + i)) - m(\phi(\gamma)) \in I$. Since $m(\phi(0)) = 0$ for all $m \in M_S$, I being an M_S-ideal implies that $m(\phi(i)) \in I$ for all $i \in I$ and all $m \in M_S$. This implies that $I \subseteq \Gamma_0$. Let $\delta \in \Gamma_0$, $i \in I$. Then, for all $m \in M_S$, $m(\phi(\delta+i)) - m(\phi(\delta)) = m(\phi(\delta+i)) \in I$. This shows that $\delta + i \in \Gamma_0$ and so, Γ_0 is a union of cosets of I.

Assume that I is not contained in the set C. Thus, there exists an element $\gamma \in \Gamma \setminus \Gamma_0$ and an element $i \in I$ such that $S(\phi(\gamma+i)) \neq S(\phi(\gamma))$. Since S is a group, this implies that for all $s \in S$, $\phi(\gamma+i) \neq s(\phi(\gamma))$. This means that $\phi(\gamma)$ and $\phi(\gamma+i)$ are in different orbits of S acting on X. Suppose that $\gamma + i \in \Gamma_0$. Since I is an M_S-ideal, this implies that $m(\phi(\gamma)) \in I$ for all $m \in M_S$, hence we must have $\phi(\gamma) = 0$, a contradiction to $\gamma \in \Gamma \setminus \Gamma_0$. So we have that $\phi(\gamma+i) \neq 0$ as well as $\phi(\gamma) \neq 0$ and we now define two functions $f_1 : X \longrightarrow \Gamma$ and $f_2 : X \longrightarrow \Gamma$.

Let $f_1(\phi(\gamma)) := \delta_1 \in I$, for $s \in S$ let $f_1(s(\phi(\gamma))) := s(f_1(\phi(\gamma)))$ and $f_1(X \setminus S(\phi(\gamma))) := \{0\}$. Let $f_2(\phi(\gamma+i)) := \delta_2 \notin I$, for $s \in S$ let $f_2(s(\phi(\gamma+i))) := s(f_2(\phi(\gamma+i)))$ and $f_2(X \setminus S(\phi(\gamma+i))) := \{0\}$. f_1 and f_2 are well defined because of fixedpointfreeness of S on X_1. It is a routine check to see that f_1 and f_2 are elements in $M_0(X, \Gamma, \phi, S)$. Since for all $s \in S$, $\phi(\gamma+i) \neq s(\phi(\gamma))$ we have that $f_1(\phi(\gamma+i)) = 0$ as well as $f_2(\phi(\gamma)) = 0$.

We now have that

$$\begin{aligned}
&(f_1+f_2)(\phi(\gamma+i)) - (f_1+f_2)(\phi(\gamma)) = \\
&= f_1(\phi(\gamma+i)) + f_2(\phi(\gamma+i)) - f_2(\phi(\gamma)) - f_1(\phi(\gamma)) = \\
&= 0 + \delta_2 - 0 - \delta_1 \notin I.
\end{aligned}$$

By density of M_S in $M_0(X, \Gamma, \phi, S)$, there is an element $m \in M_S$ such that $m(\phi(\gamma+i)) = (f_1+f_2)(\phi(\gamma+i))$ and $m(\phi(\gamma)) = (f_1+f_2)(\phi(\gamma))$. Consequently, I is not an M_S-ideal.

Assuming that I is not contained in the set C contradicts our assumption that I is an M_S-ideal. So, we now assume that I is contained in C. Consequently, by property (P), there exists $i \in I$, there exists $\gamma_1 \in \Gamma \setminus \Gamma_0$, there exists $s \in S$ and there exists $\gamma \in \Gamma$ such that $\phi(\gamma_1 + i) = s(\phi(\gamma_1))$ and $s(\gamma) - \gamma \notin I$. Since $\gamma_1 \in \Gamma \setminus \Gamma_0$, $M_S \odot \gamma_1 = \Gamma$, so there exists an element $m \in M_S$ such that $m(\phi(\gamma_1)) = \gamma$. Consequently, $m(\phi(\gamma_1+i)) - m(\phi(\gamma_1)) = m(s(\phi(\gamma_1))) - m(\phi(\gamma_1)) = = s(m(\phi(\gamma_1))) - m(\phi(\gamma_1)) = s(\gamma) - \gamma \notin I$. This is again a contradiction to the assumption that I is an M_S-ideal.

This shows that there exist no non-trivial M_S-ideals in Γ, so M_S is 1-primitive on Γ. $\Diamond$

Property (P) of Th. 4.3 is a technical condition which excludes subgroups of Γ to be M_S-ideals, in the language of Th. 4.3. Since any 2-

primitive near-ring is also 1-primitive, Th. 4.3 applies to zero symmetric and 2-primitive near-rings also. When considering 2-primitive near-rings the technical condition of property (P) can be much more simplified. This leads to an especially simple version of the theorem.

Theorem 4.4. *Let N be a zero symmetric near-ring which is not a ring. Then the following are equivalent:*

(1) *N is 2-primitive.*

(2) *There exist*

  (a) *a group $(\Gamma, +)$,*
  (b) *a set $X = \{0\} \cup X_1 \subseteq \Gamma$, $X_1 \neq \emptyset$, $0 \notin X_1$ and 0 being the zero of Γ,*
  (c) *$S \leq \mathrm{Aut}(\Gamma, +)$, with $S(X) \subseteq X$ and S acting without fixed points on X_1,*
  (d) *a function $\phi : \Gamma \to X$ with $\phi_{|X} = id$, $\phi(0) = 0$ and such that*
$$\forall \gamma \in \Gamma \ \forall s \in S : \phi(s(\gamma)) = s(\phi(\gamma)),$$

  such that N is isomorphic to a dense subnear-ring M_S of $M_0(X, \Gamma, \phi, S)$ where $\Gamma_0 := \{\gamma \in \Gamma | \phi(\gamma) = 0\}$ does not contain any non-trivial subgroups of Γ.

Proof. (1) $\Rightarrow$ (2) : Since N is 2-primitive it is also 1-primitive and hence Th. 4.3 and its proof of (1) $\Rightarrow$ (2) applies. So, let Γ be the N-group of type 2 the near-ring acts on 2-primitively. Let ϕ be as in the proof of (1) $\Rightarrow$ (2) of Th. 4.3. It only remains to show that Γ_0 does not contain any non-trivial subgroups of Γ. Suppose $K \subseteq \Gamma_0$ is a subgroup of Γ. Since $\phi(K) = \{0\}$ we know from the definition of ϕ that $K \subseteq \theta_0 = \{\gamma \in \Gamma | N\gamma = \{0\}\}$. Thus, $NK = \{0\} \subseteq K$ and K is an N-subgroup of Γ. It follows from 2-primitivity of N that $K = \{0\}$.

(2) $\Rightarrow$ (1) : As in the proof of (2) $\Rightarrow$ (1) of Th. 4.3 one shows that M_S acts faithfully on Γ with the action $\odot$ and such that for $\delta \in \Gamma_0$ we have $M_S \odot \delta = \{0\}$ and for $\gamma \in \Gamma \setminus \Gamma_0$ we have $M_S \odot \gamma = \Gamma$. Also, $\Gamma \setminus \Gamma_0 \neq \emptyset$. Suppose that K is an M_S-subgroup of Γ and $K \neq \Gamma$. It follows that $K \subseteq \Gamma_0$. By assumption, Γ_0 does not contain any non-trivial subgroups of Γ. Hence, $K = \{0\}$. Thus, Γ contains no non-trivial M_S-subgroups and M_S is 2-primitive on Γ. $\Diamond$

5. Construction of ϕ and examples

We keep the notation of the proof of Th. 4.3 throughout the whole section. In order to construct 1-primitive near-rings as sandwich centralizer near-rings with the help of Th. 4.3 one must be assured that X, Γ, ϕ, S satisfy property (P). In case $X = \Gamma$ and $\phi = id$ we have that $\Gamma_0 = \{0\}$. In this case property (P) is trivially fulfilled because only the trivial group $\{0\}$ is contained in C. In fact, in this case $M_0(X, \Gamma, \phi, S) = = M_S(\Gamma) := \{f : \Gamma \to \Gamma \mid \forall \gamma \in \Gamma \forall s \in S : s(f(\gamma)) = f(s(\gamma)) \text{ and } f(0) = 0\}$. So, $M_0(X, \Gamma, \phi, S) = M_S(\Gamma)$ is a 1-primitive near-ring with identity element (and thus 2-primitive). The fact that all zero symmetric 1-primitive near-rings with identity element which are not rings show up as dense subnear-rings of near-rings of the type $M_S(\Gamma)$ with S a group of fixed-pointfree automorphisms acting on Γ is certainly the most well known density theorem for primitive near-rings (see [2], Th. 4.52). This result is also covered by Th. 4.3, Th. 4.4 respectively, because in case of a near-ring with identity, ϕ as constructed in the proof of (1) $\Rightarrow$ (2) of Th. 4.3 is just the identity function. So, $X = \Gamma$, $\Gamma_0 = \{0\}$ and $M_0(X, \Gamma, \phi, S) = M_S(\Gamma)$. S is acting without fixed points on $\Gamma \setminus \{0\}$ because of condition (2c) in Th. 4.3.

We need not only consider near-rings with identity to obtain situations when property (P) is easily fulfilled. Property (P) is obviously fulfilled when C only contains the trivial group $\{0\}$. C will only contain the trivial group for example when Γ_0 is not a union of cosets of some non-trivial normal subgroup of Γ. Hence, probably the easiest way to obtain 1-primitive near-rings is to take $S = \{id\}$ and any function ϕ mapping from Γ to a subset X of Γ containing the zero 0 such that Γ_0 is not a union of cosets of some non-trivial normal subgroup of Γ. Note that the examples after Def. 1.1 are of that type.

To be in a position to construct 1-primitive near-rings or 2-primitive near-rings with the help of our main theorems and $S \neq \{id\}$ one has to find a suitable sandwich function ϕ which commutes with the automorphisms in S and S has to act without fixed points on the set $X_1 = X \setminus \{0\}$. If one has found such a function ϕ and the group S, then primitivity of the near-ring $M_0(X, \Gamma, \phi, S)$ only depends on the subgroups contained in Γ_0. In the following two propositions we show how to construct the sandwich function ϕ with the desired properties and show that any such ϕ can be constructed this way.

Proposition 5.1. *Let* $(\Gamma, +)$ *be a group and* $S \leq \mathrm{Aut}(\Gamma, +)$*. Let* $G \subseteq \subseteq \Gamma \setminus \{0\}$ *such that* $S(G) \subseteq G$ *and* S *acts without fixed points on* G*. Let* $\{e_i | i \in I\}$*,* I *a suitable index set, be a complete set of orbit representatives of the orbits of* S *acting on* G*. So,* $G = \cup_{i \in I} S(e_i)$*. Let* $\emptyset \neq J$*,* $J \subseteq I$*. Let* $X_1 := \cup_{j \in J} S(e_j)$ *and* $X := \{0\} \cup X_1$*. Let* $K := I \setminus J$*. If* $\emptyset \neq K$*, let* $f : \{e_k | k \in K\} \longrightarrow \cup_{j \in J} S(e_j)$ *be a function. Define* $\phi : \Gamma \longrightarrow X$ *as*

$$\phi(\gamma) := \begin{cases} 0 & \textit{if } \gamma \in \Gamma \setminus G, \\ \gamma & \textit{if } \gamma \in \cup_{j \in J} S(e_j), \\ s(f(e_k)) & \textit{if } K \textit{ is not empty and } \gamma = s(e_k) \in \cup_{k \in K} S(e_k). \end{cases}$$

Then, ϕ *is a well defined function such that* $\phi_{|X} = id$ *and* $\forall \gamma \in \Gamma\ \forall s \in \in S : \phi(s(\gamma)) = s(\phi(\gamma))$*. Furthermore,* S *acts without fixed points on* X_1 *and* $S(X_1) \subseteq X_1$*.*

Proof. Since $X_1 \subseteq G$, S acts without fixed points on X_1. Also, since X_1 is a union of orbits of S, $S(X_1) \subseteq X_1$. Suppose $K := I \setminus J$ is not the empty set. Let $f : \{e_k | k \in K\} \longrightarrow \cup_{j \in J} S(e_j)$ be a function. Let $\gamma \in \cup_{k \in K} S(e_k)$. By fixedpointfreeness of S on G there is a unique $s \in S$ and a unique $k \in K$ such that $\gamma = s(e_k)$. Then the definition $\phi(\gamma) := s(f(e_k))$ makes ϕ a well defined function. Since $0 \in \Gamma \setminus \cup_{i \in I} S(e_i)$ we have $\phi(0) = 0$ and so, $\phi_{|X} = id$. It remains to show that for all $\gamma \in \Gamma$ and all $s \in S$, $s(\phi(\gamma)) = \phi(s(\gamma))$. Let $\gamma \in \Gamma \setminus \cup_{i \in I} S(e_i)$ and $s \in S$. Then, $s(\phi(\gamma)) = s(0) = 0$ and since $s(\gamma) \in \Gamma \setminus \cup_{i \in I} S(e_i)$ we also have $\phi(s(\gamma)) = 0$. Let $\gamma \in \cup_{j \in J} S(e_j)$ and $s \in S$. Then, $s(\phi(\gamma)) = s(\gamma)$. On the other hand, $s(\gamma) \in \cup_{j \in J} S(e_j)$ and so we also have $\phi(s(\gamma)) = s(\gamma)$ by definition of ϕ. Suppose K is not empty. Let $\gamma \in \cup_{k \in K} S(e_k)$. So, $\gamma = s_1(e_k)$ for a unique $s_1 \in S$. Let $s \in S$. Then, $s(s_1(e_k)) \in \cup_{k \in K} S(e_k)$. Then, $\phi(s(\gamma)) = \phi(s(s_1(e_k))) = s(s_1(f(e_k)))$. On the other hand, we also have $s(\phi(\gamma)) = s(\phi(s_1(e_k))) = s(s_1(f(e_k)))$. This finally shows that $\phi(s(\gamma)) = s(\phi(\gamma))$ for all $\gamma \in \Gamma$. $\diamondsuit$

We give a concrete example to demonstrate the construction process of the function ϕ in Prop. 5.1. We use the notation of Prop. 5.1. Let $(\Gamma, +) := (\mathbb{Z}_7, +)$ and $S := \{id, -id\}$ where $-id : \mathbb{Z}_7 \to \mathbb{Z}_7, x \mapsto -x$. S is a fixedpointfree automorphism group of $(\mathbb{Z}_7, +)$. Let $G := \{1, 6, 2, 5\}$ be the union of the orbits of 6 and 5. Let $e_1 := 6$ and $e_2 := 5$, so $I := \{1, 2\}$. Let $J := \{1\}$ and $K := \{2\}$. Thus, $X_1 := \{1, 6\}$, the orbit of 6, and $X := \{0, 1, 6\}$. Let $f : \{5\} \to \{1, 6\}, f(5) = 1$ (here we could also define $f(5) = 6$ resulting in a different ϕ). Then $\phi : \mathbb{Z}_7 \to X$ is defined as follows: $0 = \phi(0) = \phi(3) = \phi(4)$, $\phi(1) = 1$, $\phi(6) = 6$. Since

$2 = -id(e_2) = -id(5)$ and $5 = id(e_2) = id(5)$ we get $\phi(2) = -id(f(5)) = = -id(1) = 6$ and $\phi(5) = id(f(5)) = 1$. Clearly, $\phi_{|X} = id$ and for all $\gamma \in \mathbb{Z}_7$ we have $\phi(-\gamma) = -\phi(\gamma)$ as is easily seen. Thus, ϕ has all the desired properties as claimed in Prop. 5.1. The sandwich centralizer near-ring $M_0(X, \Gamma, \phi, S)$ constructed using these groups S and Γ and this set X and function ϕ fulfilles Th. 4.4 because $(\Gamma, +)$ is a simple group. Thus, $M_0(X, \Gamma, \phi, S)$ is a 2-primitive near-ring without an identity element.

The next proposition shows that any sandwich function of the type we require in Th. 4.3 and Th. 4.4 is of the form as constructed in Prop. 5.1.

Proposition 5.2. *Let* $(\Gamma, +)$ *be a group and* $S \leq \mathrm{Aut}(\Gamma, +)$. *Let* $\emptyset \neq \neq X_1 \subseteq \Gamma \setminus \{0\}$ *such that* $S(X_1) \subseteq X_1$ *and* S *acts without fixed points on* X_1. *Let* $X := \{0\} \cup X_1$. *Let* $\phi : \Gamma \longrightarrow X$ *be a function such that* $\phi_{|X} = id$ *and such that* $\forall \gamma \in \Gamma\ \forall s \in S : \phi(s(\gamma)) = s(\phi(\gamma))$. *Let* $\Gamma_0 := \{\gamma \in \Gamma | \phi(\gamma) = 0\}$. *Then the following hold:*

(1) $G := \Gamma \setminus \Gamma_0$ *is a non-empty set such that* $S(G) \subseteq G$ *and* S *acts without fixed points on* G. *Thus,* $\phi(\gamma) = 0$ *if* $\gamma \in \Gamma \setminus G$.

(2) *Let* $\{e_i | i \in I\}$, I *a suitable index set, be a complete set of orbit representatives of the orbits of* S *acting on* G. *Then there is a non-empty subset* $J \subseteq I$ *such that* $\{e_j | j \in J\}$ *is a complete set of orbit representatives of the orbits of* S *acting on* X_1. *Thus,* $X = \{0\} \cup_{j \in J} S(e_j)$ *and* $\phi(\gamma) = \gamma$ *if* $\gamma \in \cup_{j \in J} S(e_j)$.

(3) *If* $K := I \setminus J$ *is not the empty set, then there is a function* $f : \{e_k | k \in K\} \longrightarrow \cup_{j \in J} S(e_j)$ *such that* $\phi(s(e_k)) = s(f(e_k))$ *for all* $s \in S$ *and all* $k \in K$.

Proof. G is not empty since $X_1 \subseteq G$. Let $g \in G$ and suppose that $\phi(s(g)) = 0$. Then, $s(\phi(g)) = 0$ and since s is an automorphism, $\phi(g) = 0$ which is a contradiction to the definition of G. So, $S(G) \subseteq G$. Let $g \in G$. Then, $\phi(g) \neq 0$ and consequently, $\phi(g) \in X_1$. Suppose $s(g) = g$, for some non-identity automorphism $s \in S$. Then, $s(\phi(g)) = \phi(s(g)) = \phi(g)$. Since $\phi(g) \in X_1$, this contradicts fixedpointfreeness of S on X_1. So, S acts without fixed points on G. Let $\{e_i | i \in I\}$, I a suitable index set, be a complete set of orbit representatives of the orbits of S acting on G. Since $S(X_1) \subseteq X_1$ we know that X_1 is a union of orbits of S, so there is a subset $J \subseteq I$ such that $X_1 = \cup_{j \in J} S(e_j)$. Let $K := I \setminus J$. Suppose K is not the empty set and let $k \in K$ and $s \in S$. Then,

$\phi(s(e_k)) = s(\phi(e_k))$. $\phi(e_k) \in \cup_{j\in J} S(e_j)$ and so we can define the function $f : \{e_k | k \in K\} \longrightarrow \cup_{j\in J} S(e_j), e_k \mapsto \phi(e_k)$. $\diamondsuit$

The construction of the sandwich function ϕ becomes especially simple if we let $I = J$, so $G = X_1$ in the language of Prop. 5.1. Let S be a group of automorphisms acting on Γ and let $\{e_l | l \in L\}$, L a suitable index set, be the set of orbit representatives of the action of S on Γ. Let $\emptyset \neq I \subseteq L$ such that for all $i \in I$, S acts without fixedpoints on $S(e_i)$. We now let $I = J$, so $G = X_1 = \cup_{i\in I} S(e_i)$. So, $K = \emptyset$ and the construction of ϕ is very easy. For $\gamma \in \Gamma \setminus G$ we have $\phi(\gamma) = 0$ and for $\gamma \in G = X_1$, $\phi(\gamma) = \gamma$. We are now in a position to construct $N := M_0(X, \Gamma, \phi, S)$. If N is primitive depends on the subgroups contained in $\Gamma \setminus G$ because in the language of Th. 4.3, $\Gamma \setminus G = \Gamma_0$. In particular, N will be 1-primitive if $\Gamma \setminus G$ is not the union of cosets of any non-trivial subgroup of Γ. Then, property (P) of Th. 4.3 is obviously fulfilled since the set C only contains the trivial group $\{0\}$. For example one can take S as a group of fixedpointfree automorphisms of a group Γ and let G be a union of non-zero orbits of S acting on Γ such that the size of $\Gamma \setminus G$ is not divisible by the order of any non-trivial subgroup contained in Γ. By letting $I = J$, so $G = X_1$ in the language of Prop. 5.1 and constructing ϕ as in Prop. 5.1, we can be assured that $M_0(X, \Gamma, \phi, S)$ is a 1-primitive near-ring. Note that if we have S as a group of fixedpointfree automorphism on Γ, then we may also choose $X_1 = \Gamma \setminus \{0\}$ and obtain $M_0(X, \Gamma, \phi, S) = M_S(\Gamma)$.

The following easy to establish proposition shows how big those primitive sandwich centralizer near-rings will be. Their size depends on the number of orbits of S acting on X_1. For a set M we let $|M|$ be its cardinality.

Proposition 5.3. *Let $N := M_0(X, \Gamma, \phi, S)$ be a finite 1-primitive sandwich centralizer near-ring with X, Γ, ϕ, S fulfilling the assumptions of Th. 4.3. Let k be the number of orbits of S when acting on X_1. Then,* $|N| = |\Gamma|^k$.

Proof. Let $\{e_i | i \in I\}$, I a suitable index set, be a complete set of orbit representatives of the orbits of S acting on X_1. Let $k = |I|$. Define a function $f : X \longrightarrow \Gamma$ in the following way: $f(0) = 0$, $f(e_i) := \gamma_i$ for $i \in I$, $\gamma_i \in \Gamma$ and $f(s(e_i)) := s(f(e_i))$. Since S acts without fixedpoints on X_1 we see that f is well defined and $f \in M_0(X, \Gamma, \phi, S)$. Conversely, any function in $M_0(X, \Gamma, \phi, S)$ is completely determined once one knows its function values on some set of orbit representatives. From that we see that the size of N is $|\Gamma|^k$. $\diamondsuit$

We will use the construction of ϕ we obtained in Prop. 5.1 to give another example how Th. 4.3 can be used to construct 1-primitive near-rings with $S \neq \{id\}$. We let $(\Gamma, +) := (\mathbb{Z}_{pq}, +)$ where p and q are two prime numbers such that p does not divide $q-1$ and q does not divide $p-1$ and $p \neq q$. We take $S = \mathrm{Aut}(\Gamma, +)$. Any group automorphism s of $\mathbb{Z}_{pq}$ is of the form $s : \Gamma \longrightarrow \Gamma, x \mapsto x \cdot a$, where $a \in \Gamma = \mathbb{Z}_{pq}$ is coprime to pq and $\cdot$ is the usual multiplication in $\mathbb{Z}_{pq}$. So, S has $(p-1)(q-1)$ elements, and consequently the orbit $S(1)$ of the number $1 \in \Gamma$ has $(p-1)(q-1)$ elements. Suppose that $s_1, s_2 \in S$, and $s_1(s_2(1)) = s_2(1)$. Then, $s_2^{-1}(s_1(s_2(1))) = 1$. Since S and $S(1)$ has the same number of elements, $s_2^{-1} \circ s_1 \circ s_2 = id$ and consequently, $s_1 = id$. This means that S acts without fixedpoints on $S(1)$ (S itself is not fixedpointfree on $\mathbb{Z}_{pq}$). We now let, in the notation of Prop. 5.1, $G = X_1 := S(1)$ and $X := \{0\} \cup X_1$ and define the function $\phi : \Gamma \longrightarrow X$ with $\phi_{|X} = id$ as $\phi(\gamma) = 0$ if $\gamma \in \Gamma \setminus G$ and for $\gamma \in G$, $\phi(\gamma) = \gamma$. So, by Prop. 5.1 we have for all $\gamma \in \Gamma$ and all $s \in S$, $\phi(s(\gamma)) = s(\phi(\gamma))$. Consequently we can build the sandwich centralizer near-ring $M_0(X, \Gamma, \phi, S)$ and it remains to show that property (P) of Th. 4.3 is fulfilled. Suppose there is a non-trivial subgroup I of Γ such that $\Gamma_0 = \Gamma \setminus G$ is a union of cosets of I and therefore also G is a union of cosets of I. As a proper subgroup of Γ, I can only have order p or order q. Suppose I has order p. Then, p divides the number of elements in G which is $(p-1)(q-1)$. Since p is a prime number, p must divide $q-1$. But this is not the case by choice of the prime numbers p and q. The same argument holds if I is assumed to have order q. This shows that $\Gamma_0 = \Gamma \setminus G$ is not a union of cosets of some non-trivial subgroup of Γ. Hence, property (P) is fulfilled and $M_0(X, \Gamma, \phi, S)$ is 1-primitive on Γ. Note that Γ_0 contains all the elements of Γ which do not have coprime order to pq. Thus, any element in Γ_0 generates a subgroup of Γ which is again contained in Γ_0. Since $\Gamma_0 \neq \{0\}$, $M_0(X, \Gamma, \phi, S)$ is not 2-primitive on Γ by Th. 4.4. The size of $M_0(X, \Gamma, \phi, S)$ is pq by Corollary 5.3.

Whenever X_1 is just one orbit of S as in the example, then by Corollary 5.3, $M_0(X, \Gamma, \phi, S)$ has size Γ. If S is a fixedpointfree automorphism group of Γ containing at least two non-identity automorphisms and if Γ is finite, then $M_0(X, \Gamma, \phi, S)$ is a so called planar near-ring by Th. 4.5 of [3]. Planar near-rings are rich in applications, see [1]. Using our main Th. 4.3 and Th. 4.4 and the method of constructing ϕ according to Prop. 5.1 one could now systematically investigate primitive near-rings acting on special types of groups Γ. This seems to be an interesting topic for further research but does not lie within the scope of this paper.

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