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REDUCTION OF SRIVASTAVA–DAOUST $\mathcal{S}$ FUNCTION OF TWO VARIABLES

Tibor K. **Pogány**

Faculty of Maritime Studies, University of Rijeka, Rijeka, Croatia

Arjun K. **Rathie**

Department of Mathematics, Central University of Kerala, Kasaragod, Kerala State, India

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Abstract: In this article we establish four reduction formulae for four members of the Srivastava–Daoust $\mathcal{S}$ function class in some special cases of their parameter array using the Beta-transform of certain Preece's, that is Bailey's identities treated previously by Kummer's I type transformation theorem.

1. Introduction

In the focus of our investigations are higher transcendental functions of hypergeometric type which building blocks are the generalized Pochhammer symbol or shifted factorial defined in terms of the Euler's Gamma function

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & b=0,\, a \in \mathbb{C}\setminus\{0\} \\ a(a+1)\cdots(a+b-1) & b\in\mathbb{N},\, a\in\mathbb{C} \end{cases},$$

E-mail addresses: poganj@pfri.hr, akrathie@gmail.com

using by convention that $(0)_0 = 1$. The below considered specific functions we define in the form of (multiple) power series, since all derivations include manipulations of coefficients in their power series representations.

The Srivastava–Daoust generalization of the Lauricella hypergeometric function F_D in n variables defined by [7, p. 454]

(1.1)

$$\mathcal{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left(\begin{array}{c}[(a):\theta',\cdots,\theta^{(n)}]:[(b'):\varphi'];\cdots;[(b^{(n)}):\varphi^{(n)}]\\ \\ [(c):\psi';\cdots;\psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}]\end{array}\middle|\begin{array}{c}x_1\\ \vdots\\ x_n\end{array}\right) =$$

$$= \sum_{m_1,\cdots,m_n\geq 0}\frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta^{(n)}_j}\prod\limits_{j=1}^{B'}(b'_j)_{m_1\varphi'_j}\cdots\prod\limits_{j=1}^{B^{(n)}}(b^{(n)}_j)_{m_n\varphi^{(n)}_j}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi^{(n)}_j}\prod\limits_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod\limits_{j=1}^{D^{(n)}}(d^{(n)}_j)_{m_n\delta^{(n)}_j}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

where the parameters satisfy

$$\theta'_1,\cdots,\theta'_A,\cdots,\delta^{(n)}_1,\cdots,\delta^{(n)}_{D^{(n)}} > 0.$$

For convenience, we write (a) to denote the sequence of A parameters $a_1,\cdots,a_A$, with similar interpretations for $(b'),\cdots,(d^{(n)})$. Empty products should be interpreted as unity. Srivastava and Daoust [9, pp. 157–158] reported that the series in (1.1) converges absolutely

(i) for all $x_1,\cdots,x_n\in\mathbb{C}$ when

$$\Delta_\ell = 1+\sum_{j=1}^{C}\psi^{(\ell)}_j+\sum_{j=1}^{D^{(\ell)}}\delta^{(\ell)}_j-\sum_{j=1}^{A}\theta^{(\ell)}_j-\sum_{j=1}^{B^{(\ell)}}\varphi^{(\ell)}_j > 0, \qquad \ell=\overline{1,n};$$

(ii) for $|x_\ell| < \eta_\ell$ when $\Delta_\ell = 0$, $\ell=\overline{1,n}$, where

$$\eta_\ell := \min_{\mu_1,\cdots,\mu_n>0}\left\{\mu_\ell^{1+\sum\limits_{j=1}^{D^{(\ell)}}\delta^{(\ell)}_j-\sum\limits_{j=1}^{B^{(\ell)}}\varphi^{(\ell)}_j}\ \frac{\prod\limits_{j=1}^{C}\left(\sum\limits_{\ell=1}^{n}\mu_\ell\psi^{(\ell)}_j\right)^{\psi^{(\ell)}_j}\prod\limits_{j=1}^{D^{(\ell)}}\left(\delta^{(\ell)}_j\right)^{\delta^{(\ell)}_j}}{\prod\limits_{j=1}^{A}\left(\sum\limits_{\ell=1}^{n}\mu_\ell\theta^{(\ell)}_j\right)^{\theta^{(\ell)}_j}\prod\limits_{j=1}^{B^{(\ell)}}\left(\varphi^{(\ell)}_j\right)^{\varphi^{(\ell)}_j}}\right\}.$$

When all $\Delta_\ell < 0$, $\mathcal{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}(x_1,\cdots,x_n)$ diverges except at the origin, that is, this series is formal.

In turn, specifying the parameter-array

$$(\theta'_1,\cdots,\theta'_A,\cdots,\delta^{(n)}_1,\cdots,\delta^{(n)}_{D^{(n)}}) = (1,\cdots,1),$$

we call the resulting $\mathcal{S}$ function a Kampé de Fériet generalized hypergeometric function, signifying it as F. The related convergence conditions follow from **(i)** and **(ii)**.

The further set of conditions for convergence of the series $\mathcal{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}$ is given in [9]. We remark at this point that the Srivastava–Daoust $\mathcal{S}$ generalized Lauricella hypergeometric function for $n=2$ reduces to $\mathcal{S}^{A:B;B'}_{C:D;D'}$, the Srivastava–Daoust generalized Kampé de Fériet hypergeometric function of two variables initially introduced in [7, 8]. A detailed account of the above function can be found in the article [9] and in the monograph [10].

Next,

(1.2)

$$ {}_p\Psi^*_q\left[\begin{matrix}(a,A)_p\\(b,B)_q\end{matrix}\,\Big|\,z\right] = {}_p\Psi^*_q\left[\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_q,B_q)\end{matrix}\,\Big|\,z\right] = \sum_{n\ge0}\frac{\prod_{j=1}^p (a_j)_{A_jn}}{\prod_{j=1}^q (b_j)_{B_jn}}\,\frac{z^n}{n!}$$

stands for the *unified variant of the Fox–Wright generalized hypergeometric function* with p upper and q lower parameters; $(a,A)_p$ denotes the parameter p-tuple $(a_1,A_1),\cdots,(a_p,A_p)$ and $a_j\in\mathbb{C}$, $b_i\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $A_j,B_i>0$ for all $j=\overline{1,p}, i=\overline{1,q}$, while the series converges for suitably bounded values of $|z|$ when

$$\Delta_0 := 1+\sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0\,.$$

In the case $\Delta_0 = 0$, the convergence holds in the open disk

$$|z| < \beta = \prod_{j=1}^q B_j^{B_j}\cdot\prod_{j=1}^p A_j^{-A_j}\,.$$

Let us point out that the original definition of the Fox–Wright function ${}_p\Psi_q[z]$ (consult monographs [3, 4]) contains Gamma functions instead of the here used generalized Pochhammer symbols. However, these two functions differ only up to constant multiplying factor, that is

$$ {}_p\Psi_q\Big[\begin{matrix} (a,A)_p \\ (b,B)_q \end{matrix} \Big| z \Big] = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)} \, {}_p\Psi_q^*\Big[\begin{matrix} (a,A)_p \\ (b,B)_q \end{matrix} \Big| z \Big] . $$

The unification's motivation is clear – for $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, ${}_p\Psi_q^*[z]$ one reduces exactly to the generalized hypergeometric function ${}_pF_q[z]$.

The main goal of our investigation is to express $\mathcal{S}^{1:0;0}_{0:0;0}(x,y)$ in terms of Fox–Wright Ψ-function (Th. 2.1), then to present $F^{1:1;1}_{1:1;1}(x,x)$ *via* $\mathcal{S}^{1:0;2}_{1:0;3}(x, \frac14 x^2)$ (Th. 2.2 and Th. 2.3) and in the form of a linear combination of two contiguous $\mathcal{S}^{1:0;2}_{1:0;3}(x, \frac14 x^2)$ expressions (Th. 2.4).

2. The results

Denote $\mathbb{D} = \{z \colon |z| < 1\}$ the open unit disk.

Theorem 2.1. *For all* $0 \le \alpha < 1$ *and all* $(x,y) \in \mathbb{C} \times \mathbb{D}$ *we have*

$$ {}_1\Psi_0\Big[\begin{matrix} (1,\alpha) \\ - \end{matrix} \Big| \frac{x}{(1-y)^\alpha} \Big] = 1 + \alpha x \, \mathcal{S}^{1:0;0}_{0:0;0}\Big(\begin{matrix} [\alpha;\alpha;1] : -; - \\ - \quad : -; - \end{matrix} \Big| \begin{matrix} x \\ y \end{matrix} \Big). \tag{2.3} $$

Proof. Let $\alpha > 0$ be given and let D_α denotes the set of all $(x,y) \in \mathbb{C}^2$ for which the double series

$$ \sum_{n,m\ge 0} \Gamma(m+\alpha n) \, \frac{x^n}{n!} \frac{y^m}{m!} $$

converges absolutely. The power series $\sum_{m\ge0} \Gamma(m+\alpha) \frac{x^m}{m!}$ have convergence radius equal to 1, so $D_\alpha \subseteq \mathbb{C} \times \mathbb{D}$. Having in mind that

$$ \frac{\Gamma(1+\beta)}{(1-y)^\beta} = \beta \sum_{m\ge0} \frac{\Gamma(m+\beta)}{m!} \, y^m \, , \qquad \beta > 0, y \in \mathbb{D} \tag{2.4} $$

substituting $\beta = \alpha n, \alpha \in [0,1)$, employing (2.4) we get

$$ \begin{aligned} {}_1\Psi_0\Big[\begin{matrix} (1,\alpha) \\ - \end{matrix} \Big| \frac{x}{(1-y)^\alpha} \Big] &= \sum_{n\ge0} \frac{\Gamma(1+\alpha n)\, x^n}{n!\,(1-y)^{\alpha n}} = \\ &= 1 + \alpha \sum_{n\ge1} \frac{x^n}{(n-1)!} \sum_{m\ge0} \Gamma(m+\alpha n) \frac{y^m}{m!} = \end{aligned} $$

$$= 1 + \alpha x \sum_{n,m\geq 0} \Gamma(m+\alpha(n+1)) \frac{x^n}{n!}\frac{y^m}{m!} =$$

$$= 1 + \alpha x\, \mathcal{S}^{1:0;0}_{0:0;0}\Big(\begin{matrix} [\alpha;\alpha;1] : -; - \\ - \quad : -; - \end{matrix} \,\Big|\, \begin{matrix} x \\ y \end{matrix} \Big).$$

Concluding $D_\alpha = \mathbb{C}\times\mathbb{D}$, the rest is obvious. $\Diamond$

Let us recall now a Bailey's result on the product of two confluent hypergeometric functions [1], also see for similar fashion results and extensions [1, 6] and the further references therein. For all $2\alpha, 2\beta \notin \mathbb{Z}_0^- := \{0,-1,-2,\cdots\}$ there holds

$$(2.5)\quad {}_1F_1\Big[\begin{matrix} \alpha \\ 2\alpha \end{matrix} \Big| x \Big] \cdot {}_1F_1\Big[\begin{matrix} \beta \\ 2\beta \end{matrix} \Big| -x \Big] = {}_2F_3\Big[\begin{matrix} \frac12(\alpha+\beta),\, \frac12(\alpha+\beta+1) \\ \alpha+\frac12,\, \beta+\frac12,\, \alpha+\beta \end{matrix} \Big| \frac{x^2}{4} \Big].$$

Applying this Bailey's result we derive our first *master reduction formula.*

Theorem 2.2. *For all* $2\alpha, 2\beta \notin \mathbb{Z}_0^-$, $\Re(e) > \Re(d) > 0$ *and for all* $x \in \mathbb{C}$ *we have*

(2.6)

$$F^{1:1;1}_{1:1;1}\Big[\begin{matrix} d: & \alpha; & \beta \\ e: & 2\alpha; & 2\beta \end{matrix} \Big| \begin{matrix} x \\ x \end{matrix} \Big] = \frac{2^{\alpha+\beta-1}\,\Gamma(e)}{\sqrt{\pi}\,\Gamma(d)} \cdot \Gamma(\alpha+\tfrac12)\Gamma(\beta+\tfrac12) \times$$

$$\times\, \mathcal{S}^{1:0;2}_{1:0;3}\Big(\begin{matrix} [d:1;2] : -; [\frac12(\alpha+\beta):1],\ [\frac12(\alpha+\beta+1):1] \\ [e:1;2] : -; [\alpha+\frac12:1], [\beta+\frac12:1], [\alpha+\beta:1] \end{matrix} \Big| \begin{matrix} x \\ \frac14 x^2 \end{matrix} \Big).$$

Proof. We start with the Bailey's identity (2.5); using Kummer's first transformation theorem, (2.5) becomes

$${}_1F_1\Big[\begin{matrix} \alpha \\ 2\alpha \end{matrix} \Big| x \Big] \cdot {}_1F_1\Big[\begin{matrix} \beta \\ 2\beta \end{matrix} \Big| x \Big] = \mathrm{e}^x \cdot {}_2F_3\Big[\begin{matrix} \frac12(\alpha+\beta),\, \frac12(\alpha+\beta+1) \\ \alpha+\frac12,\, \beta+\frac12,\, \alpha+\beta \end{matrix} \Big| \frac{x^2}{4} \Big].$$

Replace x by xt above and then multiply both sides of the last relation by $t^{d-1}(1-t)^{e-d-1}$ (supposing that $\Re(e) > \Re(d) > 0$) and integrating with respect to t between the limits 0 and 1 (that is, applying the so-called Beta-transform method) and simplyfing, we arrive at the stated *master* transformation formula (2.6).

Because $\Delta_1 = 1, \Delta_2 = 2$, according to convergence condition **(i)**, the double series $\mathcal{S}^{1:0;2}_{1:0;3}$ converges for all $x \in \mathbb{C}$. $\Diamond$

Theorem 2.3. *For all* $\rho > \alpha > 0$, $\Re(e) > \Re(d) > 0$*and for all* $x \in \mathbb{C}$ *we have*

$$(2.7)\quad F^{1:1;1}_{1:1;1}\Big[\begin{matrix} d: & \alpha; & \rho-\alpha \\ e: & \rho; & \rho \end{matrix} \Big| \begin{matrix} x \\ x \end{matrix} \Big] = \frac{\Gamma(e)}{\Gamma(d)} \cdot \frac{\Gamma(\rho)\,\Gamma(\frac12\rho)\Gamma(\frac12\rho+\frac12)}{\Gamma(\alpha)\,\Gamma(\rho-\alpha)} \times$$

$$\times\, \mathcal{S}^{1:0;2}_{1:0;3}\left(\begin{matrix} [d:1;2]: & -; & [\alpha:1], \quad [\rho-\alpha:1] \\ [e:1;2]: & -; & [\rho:1],[\tfrac12\rho:1],[\tfrac12\rho+\tfrac12:1] \end{matrix} \,\middle|\, \begin{matrix} x \\ \tfrac14 x^2 \end{matrix} \right).$$

Proof. Observe the following identity due to Ramanujan [2]

$${}_1F_1\left[\begin{matrix}\alpha\\ \rho\end{matrix}\,\middle|\,x\right]\cdot {}_1F_1\left[\begin{matrix}\alpha\\ \rho\end{matrix}\,\middle|\,-x\right] = {}_2F_3\left[\begin{matrix}\alpha, \quad \rho-\alpha\\ \rho,\ \tfrac12\rho,\ \tfrac12\rho+\tfrac12\end{matrix}\,\middle|\,\frac{x^2}{4}\right].$$

The Kummer's first transformation gives

$$\text{(2.8)} \qquad {}_1F_1\left[\begin{matrix}\alpha\\ \rho\end{matrix}\,\middle|\,x\right]\cdot {}_1F_1\left[\begin{matrix}\rho-\alpha\\ \rho\end{matrix}\,\middle|\,x\right] = \mathrm{e}^x\cdot {}_2F_3\left[\begin{matrix}\alpha, \quad \rho-\alpha\\ \rho,\ \tfrac12\rho,\ \tfrac12\rho+\tfrac12\end{matrix}\,\middle|\,\frac{x^2}{4}\right].$$

Replacing $x \mapsto xt$, then applying the Beta-transform to (2.8) with respect to $t \in (0,1)$, we arrive at the stated relation (2.7).

Since the convergence factors $\Delta_1 = 1, \Delta_2 = 2$ are the same as in the previous proof, we conclude that the series in (2.7) converge in the whole complex plane. $\Diamond$

Theorem 2.4. *For all $\rho \in (-1,3), \alpha > 0$ such that $|\rho - 2\alpha| < 1$ and $\Re(e) > \Re(d) > 0$ we have for any finite $x \in \mathbb{C}$ that*

(2.9)

$$F^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & 1-\alpha \\ e: & \rho; & 2-\rho\end{matrix}\,\middle|\,\begin{matrix}x\\ x\end{matrix}\right] = \sqrt{\pi}\,\frac{\Gamma(e)}{\Gamma(d)}\left\{ \frac{\Gamma(\frac12\rho+\frac12)\Gamma(\frac32-\frac12\rho)}{\Gamma(\alpha-\frac12\rho+\frac12)\,\Gamma(\frac12\rho-\alpha+\frac12)}\times\right.$$

$$\times\, \mathcal{S}^{1:0;2}_{1:0;3}\left(\begin{matrix} [d:1;2]: & -; & [\alpha-\frac12\rho+\frac12:1],\,[\frac12\rho-\alpha+\frac12:1] \\ [e:1;2]: & -; & [\frac12:1],[\frac32-\frac12\rho:1],[\frac12\rho+\frac12:1] \end{matrix} \,\middle|\, \begin{matrix} x \\ \tfrac14 x^2 \end{matrix} \right)+$$

$$+\,\frac{(2\alpha-\rho)(1-\rho)}{2\rho(2-\rho)}\cdot \frac{\Gamma(\frac12\rho+1)\Gamma(2-\frac12\rho)}{\Gamma(\alpha-\frac12\rho+1)\,\Gamma(\frac12\rho-\alpha+1)}\times$$

$$\left.\times\, \mathcal{S}^{1:0;2}_{1:0;3}\left(\begin{matrix} [d+1:1;2]: & -; & [\alpha-\frac12\rho+1:1],\,[\frac12\rho-\alpha+1:1] \\ [e+1:1;2]: & -; & [\frac32:1],[2-\frac12\rho:1],[\frac12\rho+1:1] \end{matrix} \,\middle|\, \begin{matrix} x \\ \tfrac14 x^2 \end{matrix} \right)\right\}.$$

Proof. Now, employing the identity by Preece [5], which holds for all $\rho \notin \mathbb{Z}\setminus\{1\}$:

$${}_1F_1\left[\begin{matrix}\alpha\\ \rho\end{matrix}\middle|x\right]\cdot {}_1F_1\left[\begin{matrix}\alpha-\rho+1\\ 2-\rho\end{matrix}\middle|-x\right] = {}_2F_3\left[\begin{matrix}\alpha-\frac12\rho+\frac12, & \frac12\rho-\alpha+\frac12\\ \frac12,\ \frac12\rho+\frac12,\ \frac32-\frac12\rho\end{matrix}\middle|\frac{x^2}{4}\right]+$$

$$+\,\frac{(2\alpha-\rho)(1-\rho)\,x}{\rho(2-\rho)}\cdot {}_2F_3\left[\begin{matrix}\alpha-\frac12\rho+1, & \frac12\rho-\alpha+1\\ \frac32,\ \frac12\rho+1,\ 2-\frac12\rho\end{matrix}\middle|\frac{x^2}{4}\right],$$

it becomes by the Kummer's first transformation

$$
{}_1F_1\left[\begin{matrix}\alpha\\ \rho\end{matrix}\middle| x\right] \cdot {}_1F_1\left[\begin{matrix}1-\alpha\\ 2-\rho\end{matrix}\middle| x\right] = \mathrm{e}^x \cdot {}_2F_3\left[\begin{matrix}\alpha-\frac12\rho+\frac12,\ \ \frac12\rho-\alpha+\frac12\\ \frac12,\ \frac12\rho+\frac12,\ \frac32-\frac12\rho\end{matrix}\middle| \frac{x^2}{4}\right] +
$$

$$
+ \frac{(2\alpha-\rho)(1-\rho)\,x\,\mathrm{e}^x}{\rho(2-\rho)} \cdot {}_2F_3\left[\begin{matrix}\alpha-\frac12\rho+1,\ \ \frac12\rho-\alpha+1\\ \frac32,\ \frac12\rho+1,\ 2-\frac12\rho\end{matrix}\middle| \frac{x^2}{4}\right].
$$

Again, following the lines of previous two proofs with $x \mapsto xt$, then using the Beta-transform of both sides in the last display with respect to $t \in (0,1)$, we deduce the asserted identity. $\Diamond$

Remark 1. Since our master formulae (2.6), (2.7) and (2.9) are valid for all finite complex $x \in \mathbb{C}$, and it represent a scaled family of reduction formulae, specific values of arguments give as many new reduction expressions as desired.

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