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ON APPROXIMATE EULER DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract: The aim of this paper is to investigate the Hyers–Ulam stability of the linear differential equation $x^3y^{(3)}(x)+\alpha x^2y''(x)+\beta xy'(x)+\gamma y(x)=f(x)$, where $\alpha,\beta,\gamma\in\mathbb{R}$, $y\in C^3[a,b]$ and $f\in C[a,b]$ for $0<a<b<+\infty$ or $-\infty<a<b<0$.

1. Introduction

Let X be a normed space and let I be an open interval. We say the differential equation

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0y(t)+h(t)=0 \tag{1.1}$$

has the Hyers–Ulam stability, if for any function $f:I\to X$ satisfying the differential inequality

$$\|a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0y(t)+h(t)\|\leq\varepsilon$$

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for all $t \in I$ and for some $\varepsilon \geq 0$, there exists a solution $g : I \to X$ of (1.1) such that $\|f(t) - g(t)\| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression for ε only.

The stability problem of functional equations originated from a question of Ulam [14] concerning the stability of group homomorphisms. Hyers [4] solved the case of approximately additive mappings on Banach spaces. Thereafter, T. Aoki [2] and Th. M. Rassias [13] provided a generalization of the Hyers theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded (see also [3]).

The Hyers–Ulam stability of differential equations has been investigated by Alsina and Ger [1] (see also [11, 12]): If $\varepsilon > 0$ and a differentiable function $f : I \to \mathbb{R}$ satisfies the differential inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a differentiable function $f_0 : I \to \mathbb{R}$ satisfying $f_0'(t) = f_0(t)$ such that $|f(t) - f_0(t)| \leq 3\varepsilon$ for all $t \in I$. This result of Alsina and Ger has been generalized by some mathematicians (Ref. [5, 6, 7, 9, 10]).

Recently, Jung [8] has investigated the Hyers–Ulam stability of the second-order Euler differential equation $x^2y''(x) + \alpha xy'(x) + \beta y(x) = 0$.

The aim of this paper is to investigate the Hyers–Ulam stability of the third-order Euler differential equation

$$x^3y^{(3)}(x) + \alpha x^2y''(x) + \beta xy'(x) + \gamma y(x) = f(x) \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $y \in C^3[a, b]$ and $f \in C[a, b]$ for $0 < a < b < +\infty$ or $-\infty < a < b < 0$.

2. Hyers–Ulam stability of the differential equation (1.2)

In the following theorem, we prove the Hyers–Ulam stability of the differential equation (1.2). Throughout this section, a and b are real numbers with $0 < a < b < +\infty$ or $-\infty < a < b < 0$.

Theorem 2.1. *Let α, β and γ be real numbers. The differential equation $x^3y^{(3)}(x) + \alpha x^2y''(x) + \beta xy'(x) + \gamma y(x) = f(x)$ has the Hyers–Ulam stability, where $y \in C^3[a, b]$ and $f \in C[a, b]$.*

Proof. Suppose that $0 < a < b < +\infty$ and λ, μ and ν are the (real or complex) roots of $m^3 + (\alpha - 3)m^2 + (2 - \alpha + \beta)m + \gamma = 0$ with $p = \Re\lambda$,

$q = \Re\mu$ and $r = \Re\nu$. Let $\varepsilon > 0$ and $y \in C^3[a,b]$ such that

$$|x^3 y^{(3)}(x) + \alpha x^2 y''(x) + \beta x y'(x) + \gamma y(x) - f(x)| \le \varepsilon \tag{2.1}$$

for all $x \in [a,b]$. Let

$$g(x) = x^2 y''(x) + (\lambda + \alpha - 2)xy'(x) + (\lambda^2 + \alpha\lambda - 3\lambda + 2 - \alpha + \beta)y(x),$$

$$z(x) = g(b)b^{-\lambda}x^{\lambda} - x^{\lambda}\int_x^b t^{-\lambda-1}f(t)\,dt$$

for all $x \in [a,b]$. Then

$$xz'(x) = \lambda z(x) + f(x) \tag{2.2}$$

for all $x \in [a,b]$. It is clear that
$|xg'(x) - \lambda g(x) - f(x)| = |x^3 y^{(3)}(x) + \alpha x^2 y''(x) + \beta x y'(x) + \gamma y(x) - f(x)| \le \varepsilon$
for all $x \in [a,b]$. So

$$\begin{aligned}
|z(x) - g(x)| &= \left| g(b)b^{-\lambda}x^{\lambda} - x^{\lambda}\int_x^b t^{-\lambda-1}f(t)\,dt - g(x)\right| = \\
&= |x^{\lambda}|\left| g(b)b^{-\lambda} - g(x)x^{-\lambda} - \int_x^b t^{-\lambda-1}f(t)\,dt\right| = \\
&= x^p \left| \int_x^b [g(t)t^{-\lambda}]'dt - \int_x^b t^{-\lambda-1}f(t)\,dt \right| = \\
&= x^p \left| \int_x^b t^{-\lambda-1}[tg'(t) - \lambda g(t) - f(t)]dt \right| \le \\
&\le x^p \int_x^b |t|^{-\lambda-1}|tg'(t) - \lambda g(t) - f(t)|dt \le \\
&\le \varepsilon x^p \int_x^b t^{-p-1}\,dt
\end{aligned}$$

for all $x \in [a,b]$. Therefore

$$|z(x) - g(x)| \le \begin{cases} \frac{1-(\frac{a}{b})^p}{p}\varepsilon & \text{if } p \ne 0; \\ \ln(\frac{b}{a})\varepsilon & \text{if } p = 0 \end{cases} \tag{2.3}$$

for all $x \in [a,b]$. Let us consider $h(x) = xy'(x) - \mu y(x)$ and

$$k(x) = h(b)b^{-\nu}x^{\nu} - x^{\nu}\int_x^b t^{-\nu-1}z(t)\,dt,$$

$$u(x) = y(b)b^{-\mu}x^{\mu} - x^{\mu}\int_x^b t^{-\mu-1}k(t)\ dt$$

for all $x \in [a, b]$. Then $u \in C^3[a, b]$ and

$$xk'(x) = \nu k(x) + z(x), \quad xu'(x) = \mu u(x) + k(x), \tag{2.4}$$

$$\begin{aligned} xh'(x) - \nu h(x) &= x^2y'' + (1-\mu-\nu)xy' + \mu\nu y = \\ &= x^2y''(x) + (\lambda+\alpha-2)xy'(x) + \\ &\qquad + (\lambda^2 + \alpha\lambda - 3\lambda + 2 - \alpha + \beta)y(x) = \\ &= g(x). \end{aligned}$$

for all $x \in [a, b]$. Hence (2.4) implies that $x^2u''(x) + (1-\mu-\nu)xu'(x)+$ $+\mu\nu u(x) = z(x)$. This means

$$x^2u''(x)+(\lambda+\alpha-2)xu'(x)+(\lambda^2+\alpha\lambda-3\lambda+2-\alpha+\beta)u(x) = z(x) \tag{2.5}$$

for all $x \in [a, b]$. From the definitions of functions h and k, we have

$$\begin{aligned} |k(x) - h(x)| &= \Big|h(b)b^{-\nu}x^{\nu} - h(x) - x^{\nu}\int_x^b z(t)t^{-\nu-1}\ dt\Big| = \\ &= |x^{\nu}|\Big|h(b)b^{-\nu} - h(x)x^{-\nu} - \int_x^b z(t)t^{-\nu-1}\ dt\Big| = \\ &= x^r\left|\int_x^b [h(t)t^{-\nu}]'\ dt - \int_x^b z(t)t^{-\nu-1}\ dt\right| = \\ &= x^r\left|\int_x^b t^{-\nu-1}[th'(t) - \nu h(t) - z(t)]dt\right| \le \\ &\le x^r\int_x^b |t^{-\nu-1}||th'(t) - \nu h(t) - z(t)|dt \le \\ &\le x^r\int_x^b t^{-r-1}|th'(t) - \nu h(t) - z(t)|dt = \\ &= x^r\int_x^b t^{-r-1}|g(t) - z(t)|dt \end{aligned}$$

for all $x \in [a, b]$. It follows from (2.3) that

$$|k(x) - h(x)| \le \begin{cases} \frac{[1-(\frac{a}{b})^p][1-(\frac{a}{b})^r]}{rp}\varepsilon & \text{if } p, r \ne 0; \\ \frac{[1-(\frac{a}{b})^r]\ln(\frac{b}{a})}{r}\varepsilon & \text{if } r \ne 0,\ p = 0; \\ \frac{[1-(\frac{a}{b})^p]\ln(\frac{b}{a})}{p}\varepsilon & \text{if } p \ne 0,\ r = 0; \\ \varepsilon \ln^2(\frac{b}{a}) & \text{if } r, p = 0 \end{cases} \tag{2.6}$$

for all $x \in [a, b]$. Using (2.2) and (2.5), we get that

$$x^3u^{(3)}(x) + \alpha x^2u''(x) + \beta xu'(x) + \gamma u(x) = f(x)$$

for all $x \in [a, b]$. We also have

$$\begin{aligned} |y(x) - u(x)| &= \left| y(x) - y(b)b^{-\mu}x^{\mu} + x^{\mu} \int_x^b t^{-\mu-1}k(t)\ dt \right| = \\ &= |x^{\mu}| \left| y(x)x^{-\mu} - y(b)b^{-\mu} + \int_x^b t^{-\mu-1}k(t)\ dt \right| = \\ &= x^q \left| \int_x^b [y(t)t^{-\mu}]'\ dt - \int_x^b t^{-\mu-1}k(t)\ dt \right| = \\ &= x^q \left| \int_x^b t^{-\mu-1}[ty'(t) - \mu y(t) - k(t)]dt \right| \le \\ &\le x^q \int_x^b |t^{-\mu-1}||ty'(t) - \mu y(t) - k(t)|dt \le \\ &\le x^q \int_x^b t^{-q-1}|ty'(t) - \mu y(t) - k(t)|dt \\ &= x^q \int_x^b t^{-q-1}|h(t) - k(t)|dt \end{aligned}$$

for all $x \in [a, b]$. It follows from (2.6) that

$$|y(x)-u(x)| \leq \begin{cases} \frac{[1-(\frac{a}{b})^p][1-(\frac{a}{b})^q][1-(\frac{a}{b})^r]}{pqr}\varepsilon & \text{if } p,r,q \neq 0; \\ \frac{[1-(\frac{a}{b})^p][1-(\frac{a}{b})^r]\ln(\frac{b}{a})}{pr}\varepsilon & \text{if } p,r \neq 0,\ q=0; \\ \frac{[1-(\frac{a}{b})^p][1-(\frac{a}{b})^q]\ln(\frac{b}{a})}{pq}\varepsilon & \text{if } p,q \neq 0,\ r=0; \\ \frac{[1-(\frac{a}{b})^r][1-(\frac{a}{b})^q]\ln(\frac{b}{a})}{rq}\varepsilon & \text{if } r,q \neq 0,\ p=0; \\ \frac{[1-(\frac{a}{b})^p]\ln^2(\frac{b}{a})}{p}\varepsilon & \text{if } p \neq 0,\ r,q=0; \\ \frac{[1-(\frac{a}{b})^q]\ln^2(\frac{b}{a})}{q}\varepsilon & \text{if } q \neq 0,\ p,r=0; \\ \frac{[1-(\frac{a}{b})^r]\ln^2(\frac{b}{a})}{r}\varepsilon & \text{if } r \neq 0,\ p,q=0; \\ \varepsilon \ln^3(\frac{b}{a}) & \text{if } p,q,r=0 \end{cases}$$

for all $x \in [a,b]$. This completes the proof for the case $0 < a < b < +\infty$.

Now, suppose that $-\infty < a < b < 0$ and λ, μ and ν are the (real or complex) roots of $m^3+(3-\alpha)m^2+(2-\alpha+\beta)m-\gamma=0$. Let $\lambda = p+i\tilde{p}$, $\mu = q+i\tilde{q}$ and $\nu = r+i\tilde{r}$. Suppose that $\varepsilon > 0$ and $y \in C^3[a,b]$ satisfies (2.1). Let

$$g(x) = x^2y''(x) + (\lambda-\alpha-2)xy'(x) + (\lambda^2-\alpha\lambda+3\lambda+2-\alpha+\beta)y(x),$$
$$z(x) = g(b)b^{\lambda}x^{-\lambda} - x^{-\lambda}\int_x^b t^{\lambda-1}f(t)\ dt$$

for all $x \in [a,b]$. Then

$$xz'(x) = -\lambda z(x) + f(x), \tag{2.7}$$

and
$|xg'(x)+\lambda g(x)-f(x)| = |x^3y^{(3)}(x)+\alpha x^2y''(x)+\beta xy'(x)+\gamma y(x)-f(x)| \leq \varepsilon$
for all $x \in [a,b]$. So we have

$$|z(x)-g(x)| = \left|g(b)b^{\lambda}x^{-\lambda} - x^{-\lambda}\int_x^b t^{\lambda-1}f(t)\ dt - g(x)\right| =$$
$$= |x^{-\lambda}|\left|g(b)b^{\lambda} - g(x)x^{\lambda} - \int_x^b t^{\lambda-1}f(t)\ dt\right| =$$

$$= e^{\tilde{p}\pi}|x|^{-p} \left| \int_x^b [g(t)t^{\lambda}]' dt - \int_x^b t^{\lambda-1} f(t)\ dt \right| =$$
$$= e^{\tilde{p}\pi}|x|^{-p} \left| \int_x^b t^{\lambda-1}[tg'(t) + \lambda g(t) - f(t)] dt \right| \leq$$
$$\leq e^{\tilde{p}\pi}|x|^{-p} \int_x^b |t^{\lambda-1}||tg'(t) + \lambda g(t) - f(t)| dt \leq$$
$$\leq \varepsilon |x|^{-p} \int_x^b |t|^{p-1}\ dt$$

for all $x \in [a, b]$. Therefore

$$(2.8) \qquad |z(x) - g(x)| \leq \begin{cases} \frac{1-(\frac{b}{a})^p}{p}\varepsilon & \text{if } p \neq 0; \\ \varepsilon \ln(\frac{a}{b}) & \text{if } p = 0 \end{cases}$$

for all $x \in [a, b]$. Let $h(x) = xy'(x) + \mu y(x)$ and

$$k(x) = h(b) b^{\nu} x^{-\nu} - x^{-\nu} \int_x^b t^{\nu-1} z(t)\ dt,$$
$$u(x) = y(b) b^{\mu} x^{-\mu} - x^{-\mu} \int_x^b t^{\mu-1} k(t)\ dt$$

for all $x \in [a, b]$. Then $u \in C^3[a, b]$ and

$$(2.9) \qquad xk'(x) = -\nu k(x) + z(x), \quad xu'(x) = -\mu u(x) + k(x),$$

$$\begin{aligned} xh'(x) + \nu h(x) &= x^2 y'' + (1 + \mu + \nu) xy' + \mu\nu y = \\ &= x^2 y''(x) + (\alpha - \lambda - 2) xy'(x) + \\ &\qquad + (\lambda^2 - \alpha\lambda + 3\lambda + 2 - \alpha + \beta) y(x) = \\ &= g(x). \end{aligned}$$

for all $x \in [a, b]$. It follows from (2.9) that $x^2 u''(x) + (1 + \mu + \nu) xu'(x) +$ $+\mu\nu u(x) = z(x)$. This means

$$(2.10) \quad x^2 u''(x) + (\alpha - \lambda - 2) xu'(x) + (\lambda^2 - \alpha\lambda + 3\lambda + 2 - \alpha + \beta) u(x) = z(x)$$

for all $x \in [a, b]$. Hence (2.7) and (2.10) imply

$$x^3 u^{(3)}(x) + \alpha x^2 u''(x) + \beta x u'(x) + \gamma u(x) = f(x)$$

for all $x \in [a,b]$. From the definitions of functions h and k, we have

$$\begin{aligned}|k(x)-h(x)| &= \Big|h(b)b^{\nu}x^{-\nu}-h(x)-x^{-\nu}\int_x^b t^{\nu-1}z(t)\ dt\Big| = \\ &= |x^{-\nu}|\Big|h(b)b^{\nu}-h(x)x^{\nu}-\int_x^b t^{\nu-1}z(t)\ dt\Big| = \\ &= e^{\tilde{r}\pi}|x|^{-r}\left|\int_x^b [h(t)t^{\nu}]'dt-\int_x^b t^{\nu-1}z(t)\ dt\right| = \\ &= e^{\tilde{r}\pi}|x|^{-r}\left|\int_x^b t^{\nu-1}[th'(t)+\nu h(t)-z(t)]dt\right| \le \\ &\le e^{\tilde{r}\pi}|x|^{-r}\int_x^b |t^{\nu-1}||th'(t)+\nu h(t)-z(t)|dt = \\ &= |x|^{-r}\int_x^b |t|^{r-1}|th'(t)+\nu h(t)-z(t)|dt \le \\ &\le |x|^{-r}\int_x^b |t|^{r-1}|g(t)-z(t)|dt\end{aligned}$$

for all $x \in [a,b]$. It follows from (2.8) that

$$|k(x)-h(x)| \le \begin{cases} \frac{[1-(\frac{b}{a})^p][1-(\frac{b}{a})^r]}{rp}\varepsilon & \text{if } p,r\neq 0; \\ \frac{[1-(\frac{b}{a})^r]\ln(\frac{a}{b})}{r}\varepsilon & \text{if } r\neq 0,\ p=0; \\ \frac{[1-(\frac{b}{a})^p]\ln(\frac{a}{b})}{p}\varepsilon & \text{if } p\neq 0,\ r=0; \\ \varepsilon\ln^2(\frac{a}{b}) & \text{if } r,p=0 \end{cases} \tag{2.11}$$

for all $x \in [a,b]$. We also have

$$\begin{aligned}|y(x)-u(x)| &= \Big|y(x)-y(b)b^{\mu}x^{-\mu}+x^{-\mu}\int_x^b t^{\mu-1}k(t)\ dt\Big| = \\ &= |x^{-\mu}|\Big|y(x)x^{\mu}-y(b)b^{\mu}+\int_x^b t^{\mu-1}k(t)\ dt\Big| = \\ &= e^{\tilde{q}\pi}|x|^{-q}\left|\int_x^b [y(t)t^{\mu}]'dt-\int_x^b t^{\mu-1}k(t)\ dt\right| = \\ &= e^{\tilde{q}\pi}|x|^{-q}\left|\int_x^b t^{\mu-1}[ty'(t)+\mu y(t)-k(t)]dt\right| \le\end{aligned}$$

$$\leq e^{\tilde{q}\pi}|x|^{-q}\int_x^b |t^{\mu-1}||ty'(t)+\mu y(t)-k(t)|dt =$$

$$= |x|^{-q}\int_x^b |t|^{q-1}|h(t)-k(t)|dt$$

for all $x \in [a,b]$. It follows from (2.11) that

$$|y(x)-u(x)| \leq \begin{cases} \frac{[1-(\frac{b}{a})^p][1-(\frac{b}{a})^q][1-(\frac{b}{a})^r]}{pqr}\varepsilon & \text{if } p,r,q \neq 0; \\ \frac{[1-(\frac{b}{a})^p][1-(\frac{b}{a})^r]\ln(\frac{a}{b})}{pr}\varepsilon & \text{if } p,r \neq 0,\ q=0; \\ \frac{[1-(\frac{b}{a})^p][1-(\frac{b}{a})^q]\ln(\frac{a}{b})}{pq}\varepsilon & \text{if } p,q \neq 0,\ r=0; \\ \frac{[1-(\frac{b}{a})^r][1-(\frac{b}{a})^q]\ln(\frac{a}{b})}{rq}\varepsilon & \text{if } r,q \neq 0,\ p=0; \\ \frac{[1-(\frac{b}{a})^p]\ln^2(\frac{b}{a})}{p}\varepsilon & \text{if } p \neq 0,\ r,q=0; \\ \frac{[1-(\frac{b}{a})^q]\ln^2(\frac{b}{a})}{q}\varepsilon & \text{if } q \neq 0,\ p,r=0; \\ \frac{[1-(\frac{b}{a})^r]\ln^2(\frac{a}{b})}{r}\varepsilon & \text{if } r \neq 0,\ p,q=0; \\ \varepsilon\ln^3(\frac{a}{b}) & \text{if } p,q,r=0 \end{cases}$$

for all $x \in [a,b]$. This completes the proof. $\diamondsuit$

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