Mathematica Pannonica
24/2 (2013), 221–230

A NOTE ON STRONGLY f-REGULAR RINGS

D. I. C. **Mendes**

Department of Mathematics, University of Beira Interior, Covilhã, Portugal

Received: May 2013

MSC 2010: Primary 16 D 25, 16 E 50, 16 N 60; secondary 16 N 80, 16 R 20

Keywords: Strongly f-regular ring, strongly regular ring, prime ring, domain, Kurosh–Amitsur radical.

Abstract: A ring R is a strongly f-regular ring if, for every element a of R, a belongs to the principal ideal generated by a^2. The study of strongly f-regular chain rings and their additive groups is well known. We present characterizations of strongly f-regular rings, not necessarily satisfying the chain condition, and determine the additive groups of those which are either torsion or torsion-free. We also show that strong f-regularity is a hereditary radical property. Finally, we present characterizations of rings whose proper homomorphic images are strongly f-regular, and classify the additive groups of those which have nonzero characteristic or are torsion-free.

1. Introduction

All rings considered are associative and do not necessarily have identity. Following Blair [2], we say that a ring R is *f-regular* if $a \in (a)_R^2$ for each $a \in R$, where $(a)_R$ denotes the principal ideal of R generated by a. It is well known that a ring is f-regular if and only if it is fully idempotent (that is, every ideal is idempotent) and that the class of all such rings is a hereditary radical class. We study a subclass of the class of all f-regular rings, which also turns out to be a hereditary radical

E-mail address: imendes@ubi.pt

class. A ring R such that $a \in (a^2)_R$ for each $a \in R$ will be called strongly f-regular. This is equivalent to the requirement that $a \in (a^n)_R$ for each $a \in R$ and positive integer n. If every proper homomorphic image of R is strongly f-regular, then we say that R is a proper strongly f-regular ring. The main purpose of this note is to give characterizations of strongly f-regular rings, to classify proper strongly f-regular rings, and to determine the structure of the additive groups of these rings when they are torsion or torsion-free. In particular, we show that a necessary and sufficient condition for a ring R to be strongly f-regular is that every factor ring of R is reduced (that is, has no nonzero nilpotent elements). Conditions similar to this one have been studied by several authors. Courter [4] investigated those rings which have the property that every homomorphic image is semiprime (f-regular rings), Blair and Tsutsui ([3] , [15]) studied those rings with the property that every (proper) homomorphic image is prime and Hirano [9] those whose (proper) homomorphic images are domains. The latter author showed, in particular, that the class of rings which have the property that every homomorphic image is a domain, coincides with the class of strongly f-regular chain rings. The main results in this paper are analogous to those given by Hirano.

2. Strongly f-regular rings

We begin this section with a characterization of f-regular rings in terms of their prime factor rings. For this purpose, we consider the following conditions on a ring R:

($*$) the union of every chain of semiprime ideals of R is semiprime;

($\blacklozenge$) $(K+I) \cap (K+J) = K+(I \cap J)$ for all ideals K, I and J of R.

We point out that a similar characterization for left weakly regular rings (that is, rings in which every left ideal is idempotent) has been given in ([8], Prop. 2.4) and ([12], Th. 2), but we include the proof to facilitate the reading.

Theorem 1. *A ring R is f-regular if and only if R is semiprime, satisfies conditions ($*$) and ($\blacklozenge$) and each prime factor ring of R is f-regular.*

Proof. Suppose that R is f-regular. To show that R satisfies condition ($\blacklozenge$), let K, I and J be arbitrary ideals of R. Then $(K+I) \cap (K+J) =$ $= [(K+I) \cap (K+J)]^2 \subseteq (K+J)(K+I) \subseteq K + JI \subseteq K + (I \cap J) \subseteq$ $\subseteq (K+I) \cap (K+J)$. The remaining conditions are immediately verified.

Conversely, suppose that R is semiprime, satisfies conditions $(*)$ and $(\blacklozenge)$ and each prime factor ring of R is f-regular. If there exists an ideal K of R such that $K^2 \neq K$, then, by using $(*)$ and Zorn's Lemma, we can choose a semiprime ideal M of R which is maximal with respect to the property that $K^2 + M \neq K + M$, that is, $K \nsubseteq K^2 + M$. Then the ring $\overline{R} = R/M$ is semiprime but not prime and hence there exist nonzero ideals $\overline{A}$ and $\overline{B}$ of $\overline{R}$ such that $\overline{AB} = \overline{0} = \overline{BA}$. Therefore $\overline{B} \subseteq$ $\subseteq \operatorname{ann}\left(\overline{A}\right)$, where $\operatorname{ann}\left(\overline{A}\right)$ denotes the annihilator of $\overline{A}$ in $\overline{R}$. Moreover, $\overline{A} \subseteq \operatorname{ann}\left(\operatorname{ann}\left(\overline{A}\right)\right)$, the annihilator of $\operatorname{ann}\left(\overline{A}\right)$ in $\overline{R}$. It is easily seen that $\operatorname{ann}\left(\overline{A}\right)$ and $\operatorname{ann}\left(\operatorname{ann}\left(\overline{A}\right)\right)$ are nonzero semiprime ideals of $\overline{R}$, where $\operatorname{ann}\left(\overline{A}\right) = I/M$ and $\operatorname{ann}\left(\operatorname{ann}\left(\overline{A}\right)\right) = J/M$ for certain ideals I and J of R and $\operatorname{ann}\left(\overline{A}\right) \cap \operatorname{ann}\left(\operatorname{ann}\left(\overline{A}\right)\right) = \overline{0}$. Hence $I \cap J \subseteq M$, where I and J are semiprime ideals of R. By the choice of M, $K \subseteq K^2 + I$. Similarly, $K \subseteq K^2 + J$. Thus $K \subseteq (K^2 + I) \cap (K^2 + J) = K^2 + (I \cap J) \subseteq K^2 + M$; a contradiction. $\Diamond$

The next theorem gives several characterizations of strongly f-regular rings.

Theorem 2 (see [9], Th. 1)**.** *The following statements are equivalent:*

(i) *R is strongly f-regular;*

(ii) *$a \in Ra^2R$ for every $a \in R$;*

(iii) *$a \in Ra^nR$ for every $a \in R$ and positive integer n;*

(iv) *each nonzero factor ring of R is reduced;*

(v) *R cannot be homomorphically mapped onto a subdirectly irreducible ring having a nonzero nilpotent element in the heart;*

(vi) *every nonzero factor ring of R is a subdirect product of subdirectly irreducible domains;*

(vii) *R is f-regular and every prime factor ring of R is strongly f-regular;*

(viii) *R is f-regular and every prime factor ring of R is a domain;*

(ix) *R is semiprime, satisfies $(*)$ and $(\blacklozenge)$ and each prime factor ring of R is strongly f-regular.*

Proof. Clearly, statements (i), (ii) and (iii) are equivalent.

By a straightforward argument, we can show that (i) is equivalent to (iv).

It is obvious that (iv) implies (v).

Taking ([5], Prop. 1.1) into account and the fact that prime reduced rings are domains, we have that (v) implies (vi).

(vi) implies (iv). This implication follows from ([7], Th. 3.20.5).

It is clear that (i) implies (vii).

(vii) implies (viii). If a prime factor ring $\overline{R}$ of R is strongly f-regular, then it follows from above that $\overline{R}$ is reduced and, since a prime reduced ring is a domain, the result follows.

(viii) implies (iv). If R is f-regular and every prime factor ring of R is a domain, then $N\left(\overline{R}\right) = \beta\left(\overline{R}\right) = 0$ for every factor ring $\overline{R}$ of R ([14], Prop. 1.13), where $N\left(\overline{R}\right)$ and $\beta\left(\overline{R}\right)$ denote the set of all nilpotent elements of $\overline{R}$ and the prime radical of $\overline{R}$, respectively.

It follows from Th. 1 that (ix) is equivalent to (vii). $\Diamond$

Let us recall that a ring R is called *von Neumann regular* if $a \in aRa$ for each $a \in R$ and *strongly regular* if $a \in Ra^2$ for each $a \in R$. As is well known, a ring R is strongly regular if and only if R is von Neumann regular and reduced. It is easily deduced from the above theorem that simple domains and strongly regular rings are strongly f-regular. Moreover, every strongly f-regular ring is f-regular but the converse does not hold in general. Indeed, any non-reduced von Neumann regular ring is f-regular but not strongly f-regular. In fact, if R is von Neumann regular, then from the equivalence of statements (i) and (iv) in Th. 2, R is strongly f-regular if and only if R is strongly regular. A ring is called *weakly right duo* if for each $a \in R$ there exists a positive integer n such that $a^n R$ is an ideal of R. In weakly right duo rings with identity, the concepts of strong f-regularity and strong regularity are equivalent, by ([11], Prop. 2.5). The existence of further classes of strongly f-regular rings may be deduced from the fact that direct sums and products of strongly f-regular rings are strongly f-regular.

By the *order* of an element of a ring R, we mean the order of this element in the additive group R^+ of R. If p is a prime number, the subset $R_p = \{a \in R : \text{ the order of } a \text{ is a power of } p\}$ is an ideal of R, called the *p-component* of R. It is well known that every torsion ring is the (ring-theoretic) direct sum of its p-components. Hence, if R is a subdirectly irreducible torsion ring, then $R = R_p$ for some prime p. In what follows, $\operatorname{char}(R)$ denotes the characteristic of R, $\mathbb{Q}^+$ denotes the additive group of the field of rational numbers and, for any positive integer n, $\mathbb{Z}(n)$ denotes the cyclic group of order n. We now determine the additive groups of strongly f-regular rings which are either torsion or torsion-free.

Corollary 3 (see [9], Cor. 1)**.** *Let G be an abelian group. Then the following statements are equivalent:*

(i) *G is the additive group of a strongly f-regular ring which is either torsion or torsion-free;*

(ii) *$G \cong \bigoplus_{\alpha} \mathbb{Q}^{+}$ or $G \cong \oplus_{p} \bigoplus_{\alpha_p} \mathbb{Z}(p)$, where p is prime and α, α_p are cardinals.*

Proof. (i) implies (ii). Assume first that G is the additive group of a strongly f-regular torsion-free ring R. Then it can be shown, as in [9], that $G \cong \bigoplus_{\alpha} \mathbb{Q}^{+}$ for some cardinal α. Assume next that G is the additive group of a strongly f-regular torsion ring R. Then $R = \oplus_p R_p$, where p runs over all primes dividing the order of some element of R^{+}. Since R is reduced, each R_p^{+} is an elementary p-group. Hence $R_p^{+} \cong \bigoplus_{\alpha_p} \mathbb{Z}(p)$ for some cardinal α_p.

(ii) implies (i). If $G \cong \bigoplus_{\alpha} \mathbb{Q}^{+}$, then it is known that G is the additive group of a field and a field is obviously a strongly f-regular ring. On the other hand, it is also known that $\bigoplus_{\alpha} \mathbb{Z}(p)$ is the additive group of a field. Hence, if $G \cong \oplus_p \bigoplus_{\alpha_p} \mathbb{Z}(p)$, then G is the additive group of a direct sum of fields and so the result follows. ◊

A ring is said to be *classical* if it coincides with its classical ring of quotients. In what follows, let $Z(R)$ denote the centre of the ring R.

Proposition 4 (see [9], Cor. 2)**.** *Let R be a strongly f-regular ring. If $Z(R)$ contains a regular element (that is, a nonzero element that is not a zero divisor), then R has identity and $Z(R)$ is a classical ring.*

Proof. By ([10], Prop. 1.5), $Z(R)$ is strongly regular. Hence, for each $0 \neq a \in Z(R)$, there exists $b \in Z(R)$ such that $a = a^2 b$. If a is a regular element, then, as in ([3], Th. 1.3), it follows that ab is an identity element of R and so b is the inverse of a. ◊

Following Blair and Tsutsui [3], a ring R is said to be integral over $Z(R)$ if, for each element $a \in R$, there exists a monic polynomial $f(x)$ with coefficients in $Z(R)$ such that $f(a) = 0$.

Theorem 5. *If R is a right Goldie strongly f-regular ring with identity which is integral over $Z(R)$, then R is a finite direct sum of division rings.*

Proof. Let R be a right Goldie strongly f-regular ring with identity which is integral over $Z(R)$. Then, by Goldie's Theorem and the fact that R is reduced, the classical ring of quotients Q of R is a finite direct sum of division rings. As in ([3], Th. 3.1), it follows that every regular element in R is invertible and hence $R = Q$. Indeed, let c be a regular element in R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in Z(R)[x]$ be the minimal polynomial of c over $Z(R)$. Then, as shown in [3], $a_0 \neq 0$. Moreover, a_0 is a regular element in $Z(R)$. In fact, assuming the contrary, there exists $0 \neq d \in Z(R)$ such that $da_0 = 0$. Now $0 = d(c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0) = d(c^{n-1} + a_{n-1}c^{n-2} + \cdots + a_1)c$ and, since c is regular, $d(c^{n-1} + a_{n-1}c^{n-2} + \cdots + a_1) = 0$. Thus c is a root of the nonzero polynomial $dx^{n-1} + da_{n-1}x^{n-2} + \cdots + da_1 \in Z(R)[x]$; a contradiction. Therefore a_0 is invertible, by Prop. 4. Consequently, c is invertible. ◊

As usual, we say that a ring is a *P.I.-ring* if it satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible.

Proposition 6. *If R is a P.I.-ring, then the following conditions are equivalent:*

(i) *R is strongly f-regular;*

(ii) *R is strongly regular;*

(iii) *R is reduced and f-regular.*

Proof. (i) implies (ii). Let R be a strongly f-regular ring and $\overline{R}$ a prime factor ring of R. Then $\overline{R}$ is a domain and, by Prop. 4, $\overline{R}$ has identity. As in the proof of ([9], Prop. 1), it follows that $\overline{R}$ is a division ring. By ([13], Th. 2), R is strongly regular.

(ii) is equivalent to (iii). This equivalence follows from ([1], Th. 1).

It is clear that (ii) implies (i). ◊

Recall that a *(Kurosh–Amitsur) radical* γ is a class of rings which

(i) is closed under homomorphic images;

(ii) is closed under extensions (if I is an ideal of a ring R and I and R/I are in γ, then R is in γ);

(iii) has the inductive property (if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_\alpha \subseteq \dots$ is a chain of ideals of the ring R and each I_α is in γ, then $\cup I_\alpha$ is in γ).

For further details concerning radical theory of rings, we refer the reader to [7].

As is known [7], the class of all f-regular rings is a hereditary radical class, the largest subidempotent radical class. We shall now show that the class of all strongly f-regular rings is also a hereditary radical class. First, however, we show that the relation of being an ideal is transitive in the class of strongly f-regular rings.

Lemma 7. *Let R be a strongly f-regular ring. If I is an ideal of R and J is an ideal of I, then J is an ideal of R.*

Proof. By Andrunakievich's Lemma, $J_R^3 \subseteq J$, where J_R denotes the ideal of R generated by J. Since R is f-regular, $J_R^3 = J_R$ and the result follows. ◊

Theorem 8. *The class $\mathcal{F}_s$ of all strongly f-regular rings is a hereditary radical class.*

Proof. It is obvious that $\mathcal{F}_s$ is closed under homomorphic images.

To show that $\mathcal{F}_s$ has the inductive property, let $I_1 \subseteq I_2 \subseteq \cdots \subseteq \subseteq I_\alpha \subseteq \ldots$ be a chain of ideals of the ring R such that each I_α is in $\mathcal{F}_s$. If $a \in \cup I_\alpha$ then $a \in I_\alpha$ for some α and so, by Th. 2(ii), $a \in I_\alpha a^2 I_\alpha \subseteq \subseteq (\cup I_\alpha)\, a^2\, (\cup I_\alpha)$.

To prove that $\mathcal{F}_s$ is closed under extensions, let I be an ideal of R and suppose that both I and $\overline{R} = R/I$ are in $\mathcal{F}_s$. Take any $0 \neq a \in R$. If $a \in I$, then $a \in Ia^2I \subseteq Ra^2R$. On the other hand, if $a \notin I$, then we have $0 \neq \overline{a} = a + I \in \overline{R}$ and $\overline{a} \in \overline{R}\overline{a}^2\overline{R}$. Hence $\overline{a} = \sum_{i=1}^n \overline{u_i}\overline{a}^2\overline{v_i}$ for a certain positive integer n and $\overline{u_i}, \overline{v_i} \in \overline{R}$. This implies that $b = a - \sum\limits_{i=1}^{n} u_i a^2 v_i \in I$ and so $b \in Ib^2I \subseteq Ra^2R$. Consequently, $a \in Ra^2R$.

Finally, $\mathcal{F}_s$ is hereditary. Indeed, if $R \in \mathcal{F}_s$, I is a nonzero ideal of R and a is a nonzero element of I, then $a \in (a^2)_R$, where, by the previous lemma, $(a^2)_I = (a^2)_R$ and the theorem is proved. ◊

Let $\mathcal{R}$ denote the class of all subdirectly irreducible rings with heart having a nonzero nilpotent element and let U be the upper radical operator. Taking into account Th. 2, the following corollary is clear.

Corollary 9. $\mathcal{F}_s = \mathrm{U}\mathcal{R}$.

The supplementing radical of $\mathcal{F}_s$ is $U\mathcal{R}'$, where $\mathcal{R}'$ denotes the class of all subdirectly irreducible rings with reduced hearts ([7], Th. 3.9.5). We notice that $\mathcal{R}'$ coincides with the class of all subdirectly irreducible domains.

Denoting the f-regular radical of a ring R by $\mathcal{F}(R)$ and the full matrix ring of order n over a ring R by R_n, we notice that while $\mathcal{F}(R_n) = = (\mathcal{F}(R))_n$ for $n > 1$, as is well known, this does not hold for the strongly f-regular radical. Indeed, while $\mathcal{F}_s(R_n)$ is reduced, $(\mathcal{F}_s(R))_n$ contains nonzero nilpotent elements such as e_{21}, the respective matrix unit.

3. Proper strongly f-regular rings

In this section, we classify proper strongly f-regular rings and determine the structure of the additive groups of a subclass of these rings.

Theorem 10 (see [9], Th. 4)**.** *Let R be a ring. Then R is a proper strongly f-regular ring if and only if one the following holds:*

(i) *R is a strongly f-regular ring;*

(ii) *R is a simple ring with zero-divisors;*

(iii) *R is not a reduced ring and R is subdirectly irreducible with heart P such that R/P is strongly f-regular.*

Proof. Let R be a proper strongly f-regular ring and suppose that R is not reduced. If R is simple, then R satisfies (ii). Now assume that R is not simple. If R is subdirectly irreducible, then R satisfies (iii). If R is not subdirectly irreducible, then R is reduced and we have a contradiction. Indeed, let $a \in R$ such that $a^2 = 0$. Then, for any nonzero proper ideal I of R, R/I is reduced and so $(a + I)^2 = I$ implies that $a \in I$. Thus $a = 0$. The converse is clear. $\Diamond$

Arguing in a similar way to ([9] , Cor. 3), we have the following corollary. For ease of reading, we include the proof.

Corollary 11. *Let R be a proper strongly f-regular ring and let R^+ denote the additive group of R. If* $\operatorname{char}(R) \neq 0$ *or if R is torsion-free, then one of the following holds:*

(a) $R^+ \cong \bigoplus_\alpha \mathbb{Q}^+$ *for some cardinal α;*

(b) $R^+ \cong \bigoplus_{\alpha_1} \mathbb{Z}(p_1) \oplus \bigoplus_{\alpha_2} \mathbb{Z}(p_2) \oplus \cdots \oplus \bigoplus_{\alpha_k} \mathbb{Z}(p_k)$ *where k is a positive integer, the p_i are primes and the α_i are cardinals;*

(c) $R^+ \cong \bigoplus_\alpha \mathbb{Z}(p^2)$ *where p is a prime and α is a cardinal;*

(d) $R^+ \cong \bigoplus_\alpha \mathbb{Z}(p) \oplus \bigoplus_\beta \mathbb{Z}(p^2)$ *where p is a prime and α and β are cardinals.*

Proof. If R satisfies condition (i) of Th. 10, then either (a) or (b) holds, by Cor. 3. Next, suppose that R satisfies condition (ii) of Th. 10. Then $R^+ \cong \bigoplus_\alpha \mathbb{Z}(p)$ and hence (b) holds if $\mathrm{char}(R) = p$ and (a) holds if R is torsion-free. Suppose now that R satisfies (iii) of Th. 10. Then R is subdirectly irreducible with heart P. Moreover, every ideal of R properly containing P is idempotent. Assume that $\mathrm{char}(R) \neq 0$. Then $\mathrm{char}(R) = p^n$, where p is prime and n is a positive integer. If $pR = 0$, then $R^+ \cong \bigoplus_\alpha \mathbb{Z}(p)$. If, on the other hand, $pR \neq 0$, then $P = pR$ and $p^2R = 0$. Therefore, by ([6], Th. 17.2), R^+ satisfies (c) or (d). Next assume that R is torsion-free. If $P^2 = P$ then, since R is f-regular, $nR = n^2R$ for each positive integer n. Thus R^+ is a torsion-free divisible group and so R^+ satisfies (a), by ([6], Th. 23.1). If $P^2 = 0$, then $\mathrm{char}(R/P) = 0$. In fact, if $\mathrm{char}(R/P) = n \neq 0$, then $0 \neq nR \subseteq P$. Thus $n^2R = 0$; a contradiction. So R/P is a vector space over $\mathbb{Q}$, by Cor. 3, and hence the right R/P-module P is also a vector space over $\mathbb{Q}$. Then $R^+ \cong P^+ \oplus (R/P)^+$ and thus (a) holds. $\Diamond$

Example 12 (see [9], Example 2)**.** Let R be a strongly f-regular subdirectly irreducible ring with heart P. For example, R could be a simple domain. Then

$$S = \left\{ \begin{bmatrix} a & p \\ 0 & a \end{bmatrix} : a \in R,\ p \in P \right\}$$

with usual addition and multiplication of matrices is a proper f-regular subdirectly irreducible ring with heart

$$H = \left\{ \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} : p \in P \right\},$$

and $H^2 = 0$.

Acknowledgments. The author would like to thank the referee for his valuable suggestions and remarks which helped to improve this paper. This research was supported by FEDER and Portuguese funds through the Centre for Mathematics (University of Beira Interior) and the Portuguese Foundation for Science and Technology (FCT- Fundação para a Ciência e a Tecnologia), Project PEst-OE/MAT/UI0212/2012.

References

[1] ARMENDARIZ, E.P. and FISHER, J.W.: Regular P.I.-rings, *Proc. Amer. Math. Soc.* **39** (1973), 247–251.

[2] BLAIR, R.L.: A note on f-regularity in rings, *Proc. Amer. Math. Soc.* **6** (1955), 511–515.

[3] BLAIR, W.D. and TSUTSUI, H.: Fully prime rings, *Comm. in Algebra* **22** (1994), 5389–5400.

[4] COURTER, R.C.: Rings all of whose factor rings are semi-prime, *Canad. Math. Bull.* **12** (1969), 417–426.

[5] FILIPOWICZ, M. and PUCZYLOWSKI, E.R.: Left filial rings, *Algebra Colloq.* **11** (2004), 335–344.

[6] FUCHS, L.: *Infinite abelian groups*, vol. I, Academic Press, New York, 1970.

[7] GARDNER, B. J. and WIEGANDT, R.: Radical Theory of Rings, Marcel Dekker Inc., U.S.A., 2004.

[8] HEATHERLY, H. E. and TUCCI, R.P.: Right weakly regular rings: A Survey, *Ring and Module Theory (Trends in Mathematics)* (2010), 115–123.

[9] HIRANO, Y.: On rings all of whose factor rings are integral domains, *J. Austral. Math. Soc.* (Series A) **55** (1993), 325–333.

[10] JEON, Y. C., KIM, N. K. and LEE, Y.: On fully idempotent rings, *Bull. Korean Math. Soc.* **47** (2010), 715–726.

[11] KIM, N. K. and LEE, Y.: On rings whose prime ideals are completely prime, *J. Pure Appl. Algebra* **170** (2002), 255–265.

[12] MENDES. D. I. C.: A note on weak regularity in general rings, Contributions to Algebra and Geometry, DOI 10.1007/s13366-013-0162-6.

[13] SANDS, A. D.: Some subidempotent radicals, in: *Theory of Radicals* (Proc. Conf. Szekszárd, 1991), Colloq. Math. Soc. J. Bolyai, vol. 61, North-Holland, Amsterdam, 1993, pp. 239–248.

[14] SHIN, G.: Prime ideals and sheaf representation of a pseudo symmetric ring, *Trans. Amer. Math. Soc.* **184** (1973), 43–60.

[15] TSUTSUI, H.: Fully prime rings II, *Comm. in Algebra* **24** (1996), 2981–2989.