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AN INVERSE PROBLEM WITH COMPOSITIONS OF BLASCHKE PRODUCTS

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Abstract: The zeros of compositions of Blaschke products with given parameters can be calculated by solving polynomial equations. In this paper we investigate the inverse problem, namely when we are given zeros of a composition of Blaschke products, can we find the parameters for the Blaschke products? The most simple case with forming the composition of two two-factor Blaschke products is considered. Along the way, reciprocal Blaschke functions are introduced. Interestingly many answers come in the form of Blaschke functions.

1. Introduction

In the recent decades Blaschke functions, Blaschke products and their applications have gained lots of interest. Both mathematical properties and practical applications are investigated by many researchers.

Blaschke functions and Blaschke products play an important role in many recent mathematical constructions related to rational bases, orthogonal and biorthogonal systems [2, 5, 6], multiresolution and wavelets [13, 14, 20], the Voice transform [15, 16, 17], FFT-like transforms [10, 19]. They provide an elegant way to describe and realize non-uniform dis-

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cretization [8]. Recent fields of application are found in system and control theory for the identification of systems [1, 18], as well as in signal processing for the analysis of ECG signals [3, 4]. Several publications shed light on their importance in various research and applied areas [7, 11, 12].

The motivation to this research is twofold. On one hand it is mathematical: Blaschke products are used to acquire complete orthogonal systems, but special constructions with compositions of Blaschke products (resulting again in Blaschke products) also allow to exercise fast transforms, see [2, 19]. On the other hand, our research for modeling ECG signals using rational systems with three poles, may benefit from solving the below problems, see [3, 4].

Let us now define the notions of Blaschke functions, Blaschke products and Blaschke compositions which we will use later on. We will also enumerate a few important properties of them.

The set of complex numbers will be denoted by $\mathbb{C}$, furthermore we will use the open unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ and the unit circle or torus $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$.

Definition 1 (Blaschke functions)**.** The functions with parameter $a \in \mathbb{D}$ defined as

$$B_a \colon \mathbb{C} \setminus \{ 1/\overline{a} \} \to \mathbb{C}, \qquad B_a(z) = \frac{z-a}{1-\overline{a}z}$$

are called *Blaschke functions.*

Sometimes a second parameter $\varepsilon \in \mathbb{T}$ is also introduced as a factor to the above defined $B_a(z)$ value, but we will not need the acquired *two-parameter Blaschke functions* in our current setting.

Blaschke functions have many interesting properties. They are both $\mathbb{D} \to \mathbb{D}$ and $\mathbb{T} \to \mathbb{T}$ bijections. The function B_a has a zero (of multiplicity one) at $z = a$, and a pole (of order one) at $z = 1/\overline{a}$ – which one gets from a by forming its inverse with respect to the unit circle. The inverse of B_a exists, it is B_{-a}. For the special case with $a = 0$ we have $B_0(z) = z$. Furthermore these functions act as isometric transformations in the Poincaré disk model of hyperbolic geometry. (The two-parameter Blaschke functions form an even more beautiful set, which is closed with respect to the composition of functions, thereby forming a group – this is not true for the one-parameter Blaschke functions – and by the means of them all of the isometries on the disk can be achieved.)

Definition 2 (Blaschke products)**.** The product of Blaschke functions is called a *Blaschke product,* i.e. with some $n \geq 0$ natural number and $a_1, \dots, a_n \in \mathbb{D}$

$$A_{a_1,\dots,a_n}(z) := \prod_{k=1}^{n} B_{a_k}(z) \qquad (z \in \mathbb{C}).$$

An empty product is naturally considered as the constant 1 function. Henceforth we will only consider two-factor Blaschke products, i.e. for given $a_1, a_2 \in \mathbb{D}$, the function

$$A_{a_1,a_2}(z) = B_{a_1}(z) \cdot B_{a_2}(z).$$

Among the properties of (general n-factor) Blaschke products one could find that they are *n-fold maps* on both $\mathbb{D}$ and $\mathbb{T}$: thus for all $w \in \mathbb{D}$, there exists n distinct $z_1, \dots, z_n \in \mathbb{D}$ such that they are all mapped to w. (And the same holds for $\mathbb{T}$ instead of $\mathbb{D}$.[1]) Specifically the functions of the form A_{a_1,a_2} will have two zeros inside $\mathbb{D}$, namely a_1 and a_2.

Definition 3 (Blaschke compositions)**.** We will call the composition of some Blaschke products a *Blaschke composition.*

We omit a more formal definition here, since in our current investigation we will only consider the composition of two two-factor Blaschke products, i.e. functions of the form $A_{a_3,a_4} \circ A_{a_1,a_2}$.

One might want to go deeper with composing more and more Blaschke compositions, but in general it is unnecessary, since we can find that the composition of Blaschke products will again result in a Blaschke product (times a factor in $\mathbb{T}$). Additionally the number of factors in the result will be the multiple of the factors of the terms of the composition, and the parameters (the zeros) of the composition can be calculated by finding the roots of appropriate polynomial equations, see [19].

Specifically the functions of the form $A_{a_3,a_4} \circ A_{a_1,a_2}$ will be four-factor Blaschke products, thus having 4 zeros in $\mathbb{D}$, and these zeros are the roots of the quadratic equations

$$B_{a_3}(z) \cdot B_{a_4}(z) = a_i \qquad (i = 1, 2).$$

We get two roots from both equations. We will call the 4 zeros of this simple composition b_1, b_2, b_3 and b_4.

In Fig. 1 we present a Blaschke function B_a with parameter $a = 0.4 + 0.3i$ and a Blaschke composition of the form $A_{a_3,a_4} \circ A_{a_1,a_2}$ with $a_1 = 0.1, a_2 = 0.2 + 0.2i, a_3 = -0.1 - 0.3i, a_4 = 0.3 - 0.1i$. Contour

[1] If $w = 0$ then we might have zeros of higher multiplicity if some parameters of the product are equal.

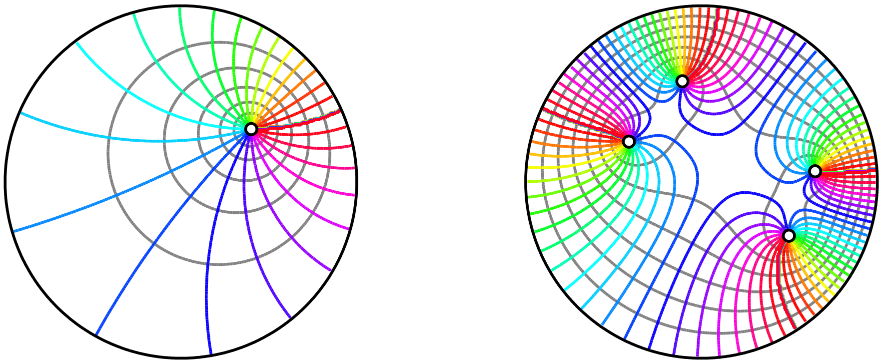

Figure 1. Left: Example of a Blaschke function. Right: Example of a Blaschke composition of the form $A_{a_3,a_4} \circ A_{a_1,a_2}$. The locations of the zeros are also marked (on both images).

lines are used to display the change in both magnitude and phase of the complex function values on the $\mathbb{D}$, see description of the `plotd` command in [10].

2. Formulating the problem

From the previous section we can conclude that for distinct $a_1, a_2, a_3, a_4 \in \mathbb{D}$ values, the map $z \mapsto A_{a_3,a_4}(A_{a_1,a_2}(z))$ has four zeros inside the unit disk, these can be simply found, let us call them b_1, b_2, b_3, b_4.

We will investigate the inverse problem. Given four values $b_1, b_2, b_3, b_4 \in \mathbb{D}$ can we find the values $a_1, a_2, a_3, a_4 \in \mathbb{D}$, for those the zeros of $A_{a_3,a_4} \circ A_{a_1,a_2}$ will be exactly the given numbers? Naturally, additional questions arise. Can we find a solution for any arbitrary set of b_i values? If not, then how can be a good set of zeros characterised? Is a solution unique or are there many solutions?

It turns out that we can fix only three of the desired zeros, and we have three valid choices for the fourth. But for each good setting we have infinitely many solutions, one for each complex number in the unit disk.

Let us now rewrite the problem in terms of formulae. Consider the set $\{b_1, b_2, b_3, b_4\} \subset \mathbb{D}$ prescribed. We are to find the values $\{a_1, a_2, a_3, a_4\} \subset$

$\subset \mathbb{D}$ such as

$$A_{a_3,a_4}(A_{a_1,a_2}(z)) = 0, \qquad z \in \{ b_1, b_2, b_3, b_4 \}$$

will hold. By the definition of A_{a_3,a_4} this is equivalent to the expression

$$B_{a_3}(A_{a_1,a_2}(z)) \cdot B_{a_4}(A_{a_1,a_2}(z)) = 0$$

at the four prescribed points. On one hand (because $B_a(z) = 0$ being equivalent to $z = a$) a_3 and a_4 should be set as

$$(1) \qquad a_3 := A_{a_1,a_2}(z), \qquad z \in \{ b_1, b_2 \},$$

$$(2) \qquad a_4 := A_{a_1,a_2}(z), \qquad z \in \{ b_3, b_4 \},$$

with the grouping of the b_i values now being arbitrary, let us set it this way. On the other hand – so that a_3 and a_4 will be well-defined – the A values should be equal, i.e.

$$(3) \qquad A_{a_1,a_2}(b_1) = A_{a_1,a_2}(b_2) \quad \text{and}$$

$$(4) \qquad A_{a_1,a_2}(b_3) = A_{a_1,a_2}(b_4)$$

are required. Therefore we must find a two-factor Blaschke product which maps two pairs of points to the same values. E.g. (3) is satisfied with the choice $a_1 := b_1, a_2 := b_2$ and (4) with $a_1 := b_3, a_2 := b_4$. Is there a choice of a_1, a_2 such that both are satisfied?

3. Solutions and experiments

In this section we will present our results with appropriate proofs and examples through numerical investigations to the problems at hand.

3.1. Reciprocal Blaschke functions

Let us start the analysis of the problem with examining only one of the equations in (3) and (4). Can we find the pair (a_1, a_2) such that (3) holds for given b_1 and b_2 inside $\mathbb{D}$? (After this is solved we should continue with comparing the solutions for the analogous problem in (4).)

Equation (3) is equivalent to

$$B_{a_1}(b_1) \cdot B_{a_2}(b_1) = B_{a_1}(b_2) \cdot B_{a_2}(b_2).$$

Without loss of generality we can assume that $a_1 \neq b_1$ and $a_2 \neq b_2$, since (4) would not be satisfied in general. So we can order the same functions on one side of the equation:

$$\frac{B_{a_2}(b_1)}{B_{a_2}(b_2)} = \frac{B_{a_1}(b_2)}{B_{a_1}(b_1)} = \left(\frac{B_{a_1}(b_1)}{B_{a_1}(b_2)}\right)^{-1}. \tag{5}$$

So we shall find two Blaschke functions with reciprocal function value quotients at the given points.

Definition 4 (Reciprocal Blaschke function)**.** For fixed $b_1, b_2, a_2 \in \mathbb{D}$ we will call the Blaschke function with parameter a_1 a *reciprocal Blaschke function* to B_{a_2} with respect to points b_1 and b_2, if the above (5) is satisfied.

Remark. We may consider the extended set of complex numbers, i.e. we may allow 0 or ∞ in the denominator as long as we agree on $1/\infty = 0$ and $1/0 = \infty$. This way B_{b_1} and B_{b_2} are reciprocal Blaschke functions with respect to points b_1 and b_2.

In the spirit of the above definition we shall find a pair of reciprocal Blaschke functions. It is easy to see that the quotient of a Blaschke functions values at two arbitrary points of $\mathbb{D}$ can be any $c \in \mathbb{C}$ complex number. (Because the function values can be arbitrary complex numbers in $\mathbb{D}$.) So for fixed $b_1, b_2, a_2 \in \mathbb{D}$ the left-hand side of (5) can be considered as a fixed number $c \in \mathbb{C}$. Now our search for a reciprocal Blaschke function (a good a_1) can be equivalently formulated as the task of characterizing the set

$$\left\{ a \in \mathbb{D} \, : \, \frac{B_a(b_1)}{B_a(b_2)} = c; b_1, b_2 \in \mathbb{D}, c \in \mathbb{C} \right\} \tag{6}$$

with the simplified notation a used instead of a_1, and c being set as

$$c := \left(\frac{B_{a_2}(b_1)}{B_{a_2}(b_2)}\right)^{-1}.$$

3.1.1. Numerical experiments

For the computer aided analysis of this problem we rewrite the condition in (6) as

$$\frac{b_1 - a}{1 - \overline{a}b_1} \cdot \frac{1 - \overline{a}b_2}{b_2 - a} = c,$$
$$(b_1 - a)(1 - \overline{a}b_2) = c(b_2 - a)(1 - \overline{a}b_1),$$
$$b_1 - a - \overline{a}b_1b_2 + a\overline{a}b_2 = cb_2 - ca - c\overline{a}b_1b_2 + ca\overline{a}b_1,$$

and finally order the terms of a as

$$(b_2 - cb_1)a\overline{a} - (1 - c)a - (b_1b_2 - cb_1b_2)\overline{a} + (b_1 - cb_2) = 0. \tag{7}$$

Equation (7) needs to be solved for a.

Defining the function $T\colon \mathbb{C} \to \mathbb{C}$ according to (7) as

$$T(z) := (b_2 - cb_1)z\overline{z} - (1 - c)z - (b_1b_2 - cb_1b_2)\overline{z} + (b_1 - cb_2),$$

we have plotted $|T(z)|$ for many $z \in \mathbb{C}$ values. E.g. Fig. 2 shows $|T(z)|$ with parameters $b_1 = -0.4 + 0.3i, b_2 = 0.2 + 0.2i$ and $c = 0.5$, the zero in $\mathbb{D}$ is also marked. These experiments suggest that there are always two solutions for (7), i.e. zeros of T, with one of them inside the unit circle, and one of them outside.[2]

With numerical optimization techniques we determined the approximate value of the zeros in $\mathbb{D}$ for fixed $b_1, b_2 \in \mathbb{D}$ and for each (sufficiently many) $a_2 \in \mathbb{D}$ values. (Recall that the choice of b_1, b_2 and a_2 determine the value $c \in \mathbb{C}$.) This $\mathbb{D} \to \mathbb{D}$ map ($a_2 \mapsto a_1$) is visualized as seen in Fig. 2 for the same b_1, b_2 values as above. Notice the strong similarity of this map to an ordinary Blaschke function. But if it is a Blaschke function, what is its parameter?

3.1.2. Analytical solution

Our next theorem summarizes the results related to finding reciprocal Blaschke functions.

Theorem 1 (Existence and uniqueness of reciprocal Blaschke functions)**.** *For arbitrary fixed $b_1, b_2, a_2 \in \mathbb{D}$ values there exists a unique $a_1 \in \mathbb{D}$ so that B_{a_1} is a reciprocal Blaschke function to B_{a_2} with respect to points b_1 and b_2. Furthermore*

$$a_1 = -B_{p(b_1,b_2)}(a_2)$$

with

$$p\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}, \qquad p(b_1, b_2) = \frac{(b_1 + b_2) - \overline{(b_1 + b_2)}b_1b_2}{1 - |b_1b_2|^2}.$$

[2]We have chosen the symbol T for this function, because of the strong resemblance of its contour plot to a *Turtle* facing towards the Reader.

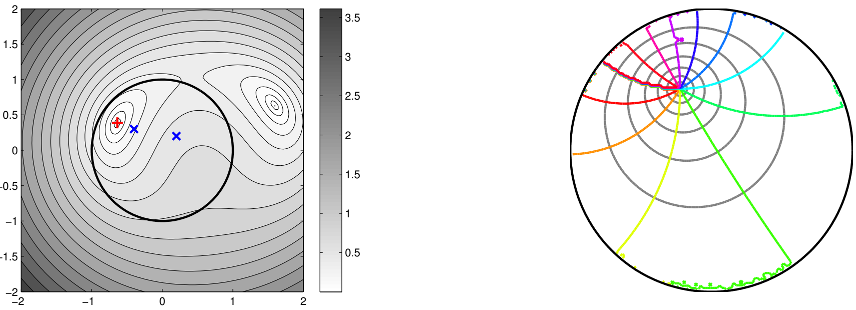

Figure 2. Left: The function $|T(z)|$. Right: The map $a_2 \mapsto a_1$.

Proof. We have three things to prove. **(I.)** p really maps to $\mathbb{D}$, so it is a valid parameter of a Blaschke function, **(II.)** the function B_{a_1} has the desired property of being a reciprocal Blaschke function to B_{a_2}, and finally **(III.)** discuss the uniqueness of the solution.

(I.) We shall prove that for any choice of $b_1, b_2 \in \mathbb{D}$, it follows that $p(b_1, b_2)$ is also in $\mathbb{D}$, i.e. $|p(b_1, b_2)| < 1$. The key to this proof lies in the exponential form of these values. Set $b_1 = r_1 e^{i\varphi_1}$ and $b_2 = r_2 e^{i\varphi_2}$ with $0 \le r_1, r_2 < 1$ and $\varphi_1, \varphi_2 \in \mathbb{R}$. This way $p(b_1, b_2)$ is

$$\frac{(r_1 e^{i\varphi_1} + r_2 e^{i\varphi_2}) - (r_1 e^{-i\varphi_1} + r_2 e^{-i\varphi_2}) r_1 r_2 e^{i(\varphi_1+\varphi_2)}}{1 - |r_1 r_2 e^{i(\varphi_1+\varphi_2)}|^2} =: \frac{N}{D}.$$

It is sufficient to show that

$$\left|\frac{N}{D}\right|^2 = \frac{N\overline{N}}{D\overline{D}} < 1.$$

We have

$$\begin{aligned}
N &= r_1 e^{i\varphi_1} + r_2 e^{i\varphi_2} - r_1^2 r_2 e^{i\varphi_2} - r_1 r_2^2 e^{i\varphi_1} = \\
&= r_1(1-r_2^2) e^{i\varphi_1} + r_2(1-r_1^2) e^{i\varphi_2}; \\
N\overline{N} &= [r_1(1-r_2^2)]^2 e^{i(\varphi_1-\varphi_1)} + [r_2(1-r_1^2)]^2 e^{i(\varphi_2-\varphi_2)} + \\
&\quad + [r_1 r_2 (1-r_1^2)(1-r_2^2)] \left(e^{i(\varphi_1-\varphi_2)} + e^{i(\varphi_2-\varphi_1)}\right); \\
D &= 1 - (r_1 r_2)^2.
\end{aligned}$$

Noticing that $e^0 = 1$ and $e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi$ with $\varphi := \varphi_1 - \varphi_2$ we get

$$\frac{N\overline{N}}{D\overline{D}} = \frac{[r_1(1-r_2^2)]^2 + [r_2(1-r_1^2)]^2 + [r_1 r_2 (1-r_1^2)(1-r_2^2)] \cos\varphi}{[1-(r_1 r_2)^2]^2}.$$

Set $F^2(\varphi)$ to the above value and note that $\cos\varphi \le 1$ and equality holds for e.g. $\varphi = 0$. Thus

$$\left|\frac{N}{D}\right|^2 =: F^2(\varphi) \le F^2(0).$$

Now we only need to show that

$$F(0) = \frac{r_1(1-r_2^2) + r_2(1-r_1^2)}{1-(r_1r_2)^2} < 1,$$

which is now easily achieved through

$$\begin{aligned} F(0) &= \frac{r_1 - r_1r_2^2 + r_2 - r_1^2r_2}{1-(r_1r_2)^2} = \frac{(r_1+r_2)(1-r_1r_2)}{(1+r_1r_2)(1-r_1r_2)} = \\ &= \frac{r_1+r_2}{1+r_1r_2} < 1 \quad \Longleftrightarrow \quad 0 < (1-r_1)(1-r_2), \end{aligned}$$

since we defined $r_1, r_2 < 1$.

So we can conclude that $p(b_1, b_2) \in \mathbb{D}$ for any $b_1, b_2 \in \mathbb{D}$, thus it can serve as a parameter for a Blaschke function.

(II.) We shall prove that B_{a_1} is a reciprocal Blaschke function to B_{a_2} with respect to points b_1, b_2, i.e.

$$\frac{B_{a_1}(b_1)}{B_{a_1}(b_2)} = \left(\frac{B_{a_2}(b_1)}{B_{a_2}(b_2)}\right)^{-1},$$

is satisfied, which is equivalent to

$$B_{a_1}(b_1) \cdot B_{a_2}(b_1) = B_{a_1}(b_2) \cdot B_{a_2}(b_2). \tag{8}$$

Notice that a_1 is itself given as a Blaschke function's value with the parameter being quite complicated. So instead of evaluating both sides, or showing their quotient being one, we will conclude this proof by showing that $B_{a_1}(b_1) \cdot B_{a_2}(b_1)$ is symmetric with respect to b_1 and b_2: so by evaluating either side of (8) we would get the same result.

To simplify the notation, we will use a instead of a_2 and p instead of $p(b_1, b_2)$.

$$\begin{aligned} B_a(b_1) \cdot B_{a_1}(b_1) &= B_a(b_1) \cdot B_{-B_p(a)}(b_1) = \\ &= B_a(b_1) \cdot \frac{b_1 + B_p(a)}{1 + \overline{B_p(a)}b_1} = B_a(b_1) \cdot \frac{b_1 + \frac{a-p}{1-\overline{p}a}}{1 + \frac{\overline{a}-\overline{p}}{1-p\overline{a}}} = \\ &= B_a(b_1) \cdot \frac{1-p\overline{a}}{1-\overline{p}a} \cdot \frac{b_1 - \overline{p}ab_1 + a - p}{1 - p\overline{a} + \overline{a}b_1 - \overline{p}b_1}. \end{aligned}$$

We shall denote $\frac{1-p\bar{a}}{1-\bar{p}a}$ by $\tau(a,b_1,b_2)$ and notice that $\tau\in\mathbb{T}$ and it is a symmetric function of b_1 and b_2, since the same is true for $p=p(b_1,b_2)$.

The calculation would go on with substituting the value of $p(b_1,b_2)$. The detailed presentation of the next steps shall be skipped, we would only give the following hints:

- It is advantageous to use the form $b_1b_2\overline{b_1b_2}$ in the denominator of $p(b_1,b_2)$ instead of $|b_1b_2|^2$.
- Later on one may notice that simplification is possible with the expression $(b_1^2\overline{b_1b_2}+1-b_1\overline{b_1}-b_1\overline{b_2})$.

Finally we arrive at

$$B_{a_2}(b_1)\cdot B_{a_1}(b_1)=-1\cdot\tau(a_2,b_1,b_2)\cdot B_{a_2}(b_1)\cdot B_{a_2}(b_2),$$

which is indeed a symmetric expression with respect to b_1 and b_2.

(III.) The uniqueness of the reciprocal Blaschke function B_{a_1} to B_{a_2} with respect to fixed points $b_1,b_2\in\mathbb{D}$ follows from the fact that a_1 is given as $-B_{p(b_1,b_2)}(a_2)$, and this map (a Blaschke function) is invertible on $\mathbb{D}$. This concludes the proof of Th. 1. $\Diamond$

3.2. Appropriate compositions and parameters

After solving the problem of reciprocal Blaschke functions, we now have infinitely many solutions for (3) and also for (4). Namely for any $a_2\in\mathbb{D}$ the value $a_1'=-B_{p(b_1,b_2)}(a_2)$ satisfies (3) and the value $a_1''==-B_{p(b_3,b_4)}(a_2)$ satisfies (4).

But unfortunately the attempts to find such an a_2 that the above $a_1'\in\mathbb{D}$ and $a_1''\in\mathbb{D}$ are equal, are generally doomed to fail. This problem stems from the following assertion.

The equation $B_{p_1}(z)=B_{p_2}(z)$ with $p_1,p_2\in\mathbb{D}$, $p_1\neq p_2$ has exactly two solutions $z_1,z_2\in\mathbb{C}$, furthermore $z_1,z_2\in\mathbb{T}$, and therefore $B_{p_i}(z_j)\in\mathbb{T}$ $(i,j=1,2)$.

So in general (for arbitrary $b_1,b_2,b_3,b_4\in\mathbb{D}$), the suitable a_2 and a_1 would both lie on $\mathbb{T}$, there is no solution in $\mathbb{D}$.

3.2.1. Analytical solution

So we must lower our expectations, and we will find that we may fix only three desired zeros on the unit disk, and we have three choices for the fourth. But we will find infinitely many sets of good parameters for the composed functions.

Definition 5 (Admissible sets of zeros)**.** We shall call the set $\{b_1, b_2, b_3, b_4\} \subset \mathbb{D}$ an *admissible set of zeros*, if there exists $a_1, a_2, a_3, a_4 \in \mathbb{D}$ parameters, such that

$$A_{a_3,a_4}(A_{a_1,a_2}(z)) = 0 \quad \Longleftrightarrow \quad z \in \{\, b_1, b_2, b_3, b_4 \,\}\,.$$

Our related observations are stated in the following theorem.

Theorem 2 (Parameters of a solution and admissible sets)**.** *For any fixed $b_1, b_2, b_3 \in \mathbb{D}$ values, the parameters*

- $a_1 \in \mathbb{D}$ *arbitrarily selected,*
- $a_2 = -B_{p(b_1,b_2)}(a_1) \in \mathbb{D}$,
- $a_3 = A_{a_1,a_2}(b_1) \in \mathbb{D}$, *and*
- $a_4 = A_{a_1,a_2}(b_3)$

satisfy

$$A_{a_3,a_4}(A_{a_1,a_2}(z)) = 0 \qquad (z \in \{\, b_1, b_2, b_3 \,\}). \tag{9}$$

Furthermore the equation

$$A_{a_1,a_2}(z) = a_4 \tag{10}$$

is independent (up to constant multiple) of the choice of a_1, it has two well-defined solutions in $\mathbb{D}$, namely b_3 and the fourth solution of (9) *that we may call b_4. Then $\{\, b_1, b_2, b_3, b_4 \,\}$ is an admissible set.*

Proof. Again we divide the proof into three parts. **(I.)** the given parameters actually satisfy (9), **(II.)** the equation (10) does not essentially depend on the initial choice of a_1, so b_4 is also well-defined and also satisfies (9), and finally **(III.)** the solutions of (10) lie in $\mathbb{D}$.

(I.) First we shall see that the values assigned to a_1, a_2, a_3, a_4 are well-defined parameters in $\mathbb{D}$ and with them (9) is satisfied. Actually this is a direct consequence of the analysis in Sec. 2 and Th. 1. Indeed:

- $a_1 \in \mathbb{D}$, because of its definition,
- a_2 is set according to Th. 1, such that $a_2 \in \mathbb{D}$ and B_{a_2} is a reciprocal Blaschke function to B_{a_1} with respect to points b_1 and b_2, and thus (5) and (3) are satisfied.

- This way a_3 is also well-defined, $a_3 \in \mathbb{D}$ since it is a product of two Blaschke functions values on the unit disk, furthermore by (1) it follows that b_1 and b_2 are roots of $B_{a_3}(A_{a_1,a_2}(z))$, thus also of (9).
- Also $a_4 \in \mathbb{D}$ holds, the argument is analogous to the case of a_3. The definition of a_4 also ensures that $B_{a_4}(A_{a_1,a_2}(b_3)) = 0$, thus b_3 also satisfies (9).

(II.) Let us now examine (10). By the definition of Blaschke products and Blaschke functions, (10) can be written as

$$\frac{z-a_1}{1-\overline{a_1}z} \cdot \frac{z-a_2}{1-\overline{a_2}z} = a_4,$$

which is equivalent to the quadratic equation

$$z^2 + \underbrace{\frac{a_4\overline{(a_1+a_2)} - (a_1+a_2)}{1-a_4\overline{a_1 a_2}}}_{C_1} z + \underbrace{\frac{a_1 a_2 - a_4}{1-a_4\overline{a_1 a_2}}}_{C_0} = 0.$$

It turns out that both coefficients, C_0 and C_1 are independent of the initial choice of a_1. We will skip the calculation, which consists of the substitution of the definition of a_2 and a_4 in the formulas of C_0 and C_1. Recall that $a_2 = -B_{p(b_1,b_2)}(a_1)$, and $a_4 = A_{a_1,a_2}(b_3)$. In the end, one finds that

$$C_0 = -b \cdot B_p(b), \qquad \text{and} \qquad C_1 = b \cdot \overline{p} \cdot B_p(b) - p,$$

with $b = b_3$ and $p = p(b_1, b_2)$; so both are independent of a_1.

However one might easily verify these formulas by a special case, say $a_1 = 0$. In this case we have

$$C_0 = \frac{-a_1 \cdot B_p(a_1) - B_{a_1}(b_3) \cdot B_{-B_p(a_1)}(b_3)}{1 + B_{a_1}(b_3) \cdot B_{-B_p(a_1)}(b_3) \cdot \overline{a_1} \cdot \overline{B_p(a_1)}},$$

and applying the identities $B_0(z) = z$ and $B_a(0) = -a$,

$$C_0 = \frac{0 \cdot p - b_3 \cdot B_p(b_3)}{1 + (-b_3) \cdot B_p(b_3) \cdot 0 \cdot (-\overline{p})} = -b_3 \cdot B_p(b_3).$$

C_1 can be verified in a similar manner.

Thus the two solutions of (10), b_3 and b_4, are independent of a_1, so b_4 is also well-defined, so $a_4 = A_{a_1,a_2}(z)$ also holds for b_4, and b_4 also satisfies (9).

(III.) It needs a few words to clarify that also $b_4 \in \mathbb{D}$. Assume indirectly that a solution z_0 to (10) has $|z_0| \geq 1$. But according to the properties of Blaschke functions, for the left-hand side $|B_{a_1}(z_0) \cdot B_{a_2}(z_0)| = |B_{a_1}(z_0)| \cdot |B_{a_2}(z_0)| \geq 1 \cdot 1 = 1$, while for the right-hand side $|a_4| < 1$. Contradiction!

This concludes the proof of Th. 2. $\Diamond$

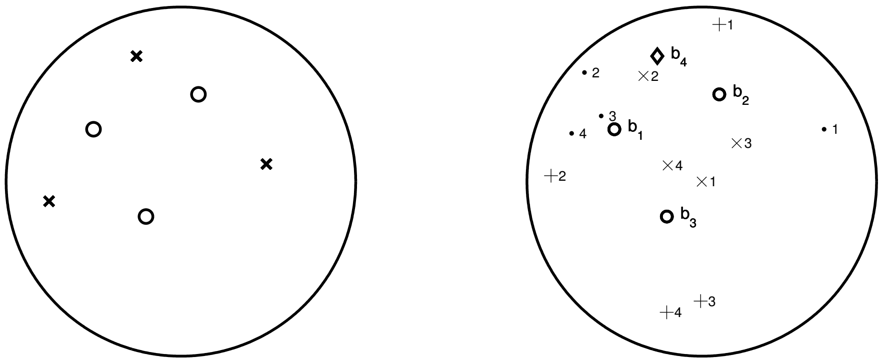

Figure 3. Admissible sets and solutions. Left: given b_1, b_2, b_3 values (marked with circles), and possible b_4 values (exes) to form an admissible set of four zeros. Right: 3 solutions for the same given b_1, b_2, b_3 values. The first solution is marked with exes, the second with dots and the third with plus signs.

3.2.2. Numerical experiments

Numerical calculations also confirm the independence of b_4 from the choice of a_1. For given b_1, b_2, b_3 values, b_4 is uniquely determined. It is also clear that if we switch b_1 and b_2, the resulting b_4 would be the same. However for further permutations of these three values we gain two more choices for b_4. Fig. 3 presents the three choices of b_4 for $b_1 = -0.5 + 0.3i, b_2 = 0.1 + 0.5i, b_3 = -0.2 - 0.2i$.

Furthermore we give three solutions for a_1, a_2, a_3, a_4 for the same values of b_1, b_2, b_3 as above with initial choices for a_1 being: 0, then $0.7 + 0.3i$ and $0.1 + 0.9i$. (See Fig. 3.)

4. Conclusions

In this paper we have investigated an inverse problem related to the most simple non-trivial Blaschke composition, namely the composition of two two-factor Blaschke products.

We found that not every set of four zeros can be achieved with this composition, but for any three points we have a choice of three for the fourth point to form an admissible set of zeros. And furthermore, to each admissible set we proved that there are infinitely many solutions for the parameters of the Blaschke products.

Along the way we introduced reciprocal Blaschke functions and proved their existence and uniqueness. Interestingly, again Blaschke functions play an important role in expressing the appropriate solutions.

The Matlab programs used in this research are available to download at `http://numanal.inf.elte.hu/~locsi/invblacomp/`.

5. Open questions

We mentioned that the equation $B_{p_1}(z) = B_{p_2}(z)$ with $p_1, p_2 \in \mathbb{D}$, $p_1 \neq p_2$ has exactly two solutions $z_1, z_2 \in \mathbb{T}$. The more detailed analysis and investigation of this assertion may lead to interesting results, e.g. with the argument functions (see [9]) of Blaschke functions: do they always have two intersections?

Furthermore when given three parameters $b_1, b_2, b_3 \in \mathbb{D}$, a choice for b_4, the fourth zero of our simple compositions at hand, looks like as if it has strong relation to hyperbolic transforms on the Poincaré disk model. It might coincide with the reflection of the hyperbolic triangle given by b_1, b_2, b_3 through the middle-point of one of its edges.

Of course also more complicated Blaschke compositions may be worth studying. E.g. how can one assure that prescribed zeros will arise for instance in an FFT-like construction? (See [10, 19].)

Similar compositions may be analyzed in the case of ordinary complex quadratic polynomials. The construction and results may be quite similar to the ones presented here.

Noticing that the condition in (6) can be also written as $B_a(b_1) = c \cdot B_a(b_2)$ and that $|B_a(b)|$ corresponds to the hyperbolic distance of the points a and b, by considering absolute values, we may find an analogue to the Apollonian circles in the Poincaré disk model. (The distance from b_1 is constant times the distance from b_2.)

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