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ON SINGULAR ALGEBRAS WITH COMPOSITION

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Abstract: Generalizing concepts from semigroups, we initiate a study of singular algebras. We determine their endomorphisms and investigate their algebras of selfmaps with the operation of composition adjoined.

I. Introduction

We recall that a groupoid (called "magma" by Bourbaki) is an algebra $\langle A, +\rangle$ with a single binary equation, $+$, on A. If the operation is defined by $x + y = x$ ($x + y = y$) for $x, y \in A$ then $\langle A, +\rangle$ is called a left (right) singular groupoid. In [3], Clifford and Preston call those groupoids (which are semigroups) left (right) zero semigroups and state ([3] p. 4) "In spite of their triviality these semigroups arise naturally in a number of investigations". As an illustration in support of this statement we recall that every rectangular band is a product of left singular semigroups and right singular semigroups.

Singular semigroups also arise naturally when considering mutations of monoids. In fact, let $\langle M, +, 0\rangle$ be a commutative monoid and let ϕ, ψ be commuting idempotent, 0-preserving endomorphisms of $\langle M, +, 0\rangle$. Define a new operation, $\oplus$, on M by $x \oplus y = \phi x + \psi y$, $x, y \in M$. The

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groupoid $\langle M, \oplus\rangle$ is a semigroup, called the (ϕ, ψ)-mutation of $\langle M, +, 0\rangle$. One is interested in which properties of the given monoid remain invariant under mutations (see [7], [9]). When $\phi = id_M$ and $\psi = 0_M$ or $\phi = 0_M$ and $\psi = id_M$, we get singular semigroups. When $\phi = 0_M = \psi$, we get a constant semigroup, i.e., $x \oplus y = 0$, $x, y \in M$.

In 1962, Tamura [14], announced that a groupoid $\langle A, +\rangle$ has the property that every selfmap on A is an endomorphism of $\langle A, +\rangle$ if and only if $\langle A, +\rangle$ is singular, and published a proof in 1965 [15]. In 2000, Chris Devillier [4], extended this work by classifying groupoids in which every selfmap is either an endomorphism or a translation.

The purpose of this investigation is to generalize and extend the above results on singular semigroups to appropriate algebras. We first recall some basic concepts and definitions from universal algebras. By "algebra", we mean a set A with a collection, F, of operations on A, denoted by $\langle A, F\rangle$. Further, there exists a function $\tau : F \to \mathbb{N} = \{0, 1, 2, \dots\}$ called the *type of* A or arity function of the operation ω. If $\tau(\omega) = n$ then ω is a function, $\omega : A^n \to A$, and we say ω is an n-ary operation. We refer the reader to the books of Burris and Sankappanavar [2], or Romanowska and Smith [12], for further notions of universal algebra.

We indicate two classes of possible operations on a set A. The *projections* on $A, \pi_i^n : A^n \to A$, $(a_1, \dots, a_n) \to a_i, n \geq 1, \iota \in \{1, \dots, n\}$, and for an arbitrary but fixed $e \in A$, the *constants* $e^n : A^n \to A$, $(a_1, \dots, a_n) \to$ $\to e, n \geq 1$. For an algebra $\langle A, F\rangle$ we denote the projections on A contained in F by P_A and we let $e_A = \{e^n \in F\}$. In this paper we restrict our attention to algebras $\langle A, F\rangle$ in which $F = P_A \cup e_A$, that is, an operation is either a projection or a constant. If $e_A = \varnothing$, Pöschel and Reichel [11], call these algebras, $\langle A, P_A\rangle$, projection algebras. However, in the present literature, particularly in that of computer science, the phrase "projection algebra" refers to a completely different kind of algebra (see [6]). Thus following the terminology from semigroups we call algebras $\langle A, F\rangle$ with $F = P_A$, *singular algebras*, and in the case $F = P_A \cup e_A, e_A \neq \varnothing$, we say $\langle A, F\rangle$ is an *e-singular algebra*. In the sequel when we write $F = P_A \cup e_A$ we are taking $e_A \neq \varnothing$.

In the sequel we also restrict our attention to algebras $\langle A, F\rangle$ of finite type, that is $\{\tau(\omega) | \omega \in F\}$ is a finite set. For future use we let $\mu = \max\{\tau(\omega) | \omega \in F\}$ for an algebra $\langle A, F\rangle$.

For an arbitrary algebra, $\langle A, F\rangle$, we let $M(A)$ denote the collection of selfmaps on A. As is well-known, $M(A)$ can be considered as an algebra with the same set F of operations as A in the following manner: for $\omega \in F$ with $\tau(\omega) = n$, and $(f_1, \dots, f_n) \in (M(A))^n$ we define the function $\omega(f_1, \dots, f_n) : A \to A$ by $\omega(f_1, \dots, f_n)(x) = \omega(f_1(x), \dots, f_n(x)), x \in A$. We then have the algebra $\langle M(A), F \rangle$. We note that the composition, $\circ$, of functions is a binary operation on $M(A)$ and we adjoin this to F obtaining the enrichment $\langle M(A), F \cup \{\circ\} \rangle$. One then investigates the structure of the function algebra $\langle M(A), F \cup \{\circ\} \rangle$. This is part of the program suggested by Guenter Pilz ([10] p. 42).

We close this section with a short summary of the paper. In the next section we generalize the result of Tamura mentioned above to singular and e-singular algebras. In Sec. III we turn to the algebras of selfmaps of singular and e-singular algebras with the operation of composition adjoined. We focus on endomorphisms and subalgebras.

II. Endomorphisms of singular and *e*-singular algebras

We begin this section with the following two straightforward observations:

(II.A) The lattice of subalgebras of a singular algebra $\langle A, P_A \rangle$ is isomorphic to the Boolean algebra of subsets of A.

(II.B) The lattice of subalgebras of an e-singular algebra, $\langle A, P_A \cup e_A \rangle$ is isomorphic to the Boolean algebra of subsets of A which contain e_A.

Neither the converse of (II.A) nor (II.B) is true. (See Sasaki [13], and the references given there.)

Let $\langle A, F \rangle$ be an arbitrary algebra and let $\omega \in F, \tau(\omega) = n$. Then, for $f \in M(A)$, $f\omega = \omega f^n$ if for $x_1, \dots, x_n$ in A, $f(\omega(x_1, \dots, x_n)) = \omega(f(x_1), \dots, f(x_n))$. When $f \in M(A)$ and $f\omega = \omega f^{\tau(\omega)}$, for all $\omega \in F$, we say f *is an endomorphism of* A. The collection of endomorphisms of A is denoted by $\mathrm{End}(A)$, so we have $\mathrm{End}(A) = \{f \in M(A) | f\omega = \omega f^{\tau(\omega)}$, for each $\omega \in F\}$.

Discussions with Erhard Aichinger in Linz led to the next theorem. The author wishes to thank Professor Aichinger for his assistance.

Theorem II.1. i] *Let* $\langle A, F \rangle$ *be an algebra with* $|A| \geq \mu$. *Then* $\mathrm{End}(A) = M(A)$ *if and only if* $\langle A, F \rangle$ *is a singular algebra.*

ii] *Let* $\langle A, f\rangle$ *be an algebra with* $|A| \geq \mu + 1$. *Then* $\mathrm{End}(A) = M_{\{e\}}(A) := \{f \in M(A) | f(e) = e\}$ *if and only if* $F = P_A \cup e_A$.

Proof. i] Suppose $\langle A, F\rangle$ is a singular algebra. We take $f \in M(A)$ and show $f \in \mathrm{End}(A)$. To this end, let $\omega \in F$, $\tau(\omega) = n$, say $\omega = \pi_i^n$, and take $(a_1, \ldots, a_n) \in A^n$. Then $f\omega(a_1, \ldots, a_n) = f(a_\iota) = \pi_i^n(f(a_1), \ldots, f(a_n))$, so $f \in \mathrm{End}(A)$.

For the converse we let $\omega \in F$ be arbitrary and show ω is a projection. Suppose $\tau(\omega) = n$ and choose n distinct elements, $a_1, \ldots, a_n$ from A. This is possible since $|A| \geq \mu$. We let $\omega(a_1, \ldots, a_n) = a$. If $a \notin \{a_1, \ldots, a_n\}$ then there exist $f, g \in M(A) = \mathrm{End}(A)$ such that $f(a_\iota) = g(a_\iota)$, $\iota = 1, 2, \ldots, n$ but $f(a) \neq g(a)$. But this gives $f(a) = f(\omega(a_1, \ldots, a_n)) = \omega(f(a_1), \ldots, f(a_n)) = \omega(g(a_1), \ldots, g(a_n)) = g(a)$, a contradiction thus $a \in \{a_1, \ldots, a_n\}$ say $a = a_j$. Now let $x_1, \ldots, x_n$ be arbitrary in A. There exists $h \in \mathrm{End}(A)$ such that $h(a_i) = x_i$, $\iota = 1, 2, \ldots, n$ so $x_j = h(a_j) = h\omega(a_1, \ldots, a_n) = \omega(h(a_1), \ldots, h(a_n)) = \omega(x_1, \ldots, x_n)$ hence ω is a projection and $\langle A, F\rangle$ is a singular algebra.

ii] If $\langle A, F\rangle$ is an e-singular algebra then $F = P_A \cup e_A$ and $e_a \neq \varnothing$. We take $e^n \in F$, $f \in \mathrm{End} A$ and $(a_1, \ldots, a_n) \in A^n$. We get

$$e = e^n(f(a_1), \ldots, f(a_n)) = f(e^n(a_1, \ldots, a_n)) = f(e)$$

so $\mathrm{End}(A) \subseteq M_{\{e\}}(A)$.

Now let $\mathrm{End}(A) = M_{\{e\}}(A)$ and let $\omega \in F$ with $\tau(\omega) = n$. Since $|A| \geq \mu + 1 > n$, there exist n distinct elements $a_1, \ldots, a_n \in A - \{e\}$ and, if $\omega(a_1, \ldots, a_n) = a$ with $a \notin \{a_1, \ldots, a_n, e\}$ as above we obtain a contradiction thus $a \in \{a_1, \ldots, a_n\}$ so $a = e$ or $a = e_j$ for some $j \in \{1, \ldots, n\}$. In the first case $\omega = e^n$ and in the second case ω is a projection. Hence $\langle A, F\rangle$ is e-singular. $\Diamond$

Without some cardinality condition, the above result need not be true.

Example II.2. For i], let $a = \{a, b\}$ and let ω be the ternary operation defined as follows: for any triple $(x, y, z) \in A^3$, at least two components are the same element of A. We define $\omega(x, y, z)$ to be that element. Let $f \in M(A)$ and $(x, y, z) \in A^3$ with, say, $x = y$. Then $f(\omega(x, y, z) = f(x)$ and $\omega f^3(x, y, z) = \omega(f(x), f(y), f(z)) = f(x)$. Hence $M(A) = \mathrm{End}(A)$ but ω is not a projection.

For ii], the group $\langle \mathbb{Z}_2 = \{0, 1\}, F = \{+\}\rangle$ has $\mathrm{End}(\mathbb{Z}_2) = M_{\{0\}}(\mathbb{Z}_2)$, the zero preserving functions on $\mathbb{Z}_2$, but the binary operation, $+$, is neither a projection nor a constant.

We recall that an arbitrary algebra $\langle A, F\rangle$ is called *entropic* or *medial* if any two $\sigma, \omega \in F$ commute. That is, if $\tau(\omega) = m$ and $\tau(\sigma) = n$, and $(x_1, \dots, x_m), \dots, (x_{n_1}, \dots, x_{n_m})$ are in A^m then we have the *entropic identity.*

[E]: $$\sigma(\omega(x_{11}, \dots, x_{1m}), \omega(x_{21}, \dots, x_{2m}), \dots, \omega(x_{m1}, \dots, x_{nm}))$$
$$= \omega(\sigma(x_{11}, \dots, x_{n1}), \sigma(x_{12}, \dots, x_{n2}), \dots, \sigma(x_{1m}, \dots, x_{nm}))$$

is satisfied in $\langle A, F\rangle$. (See [12], p. 235–237 for additional details.)

Calculations show that any $e^n \in e_A$ and $\pi_j^n \in P_A$ commute as well as $e^n, e^m \in e_A$ and $\pi_j^n, \pi_i^m \in P_A$. Thus, from Prop. 5.1 of [12] we obtain the next result.

Theorem II.3. *If $\langle A, F\rangle$ is a singular or e-singular algebra, then $\langle \mathrm{End}(A), F\rangle$ is a subalgebra of $\langle M(A), F\rangle$.*

Note that the entropic property $[E]$ guarantees that for each $\omega \in F$, say $\tau(\omega) = n$ and for $f_1, \dots, f_n \in \mathrm{End}(A)$, $\omega(f_1, \dots, f_n) \in \mathrm{End}(A)$. We note further that each projection π_ι^n satisfies the idempotent identity $\pi_\iota^n(a_1, \dots, a_n) = a_i$, i.e. a singular algebra is idempotent.

Corollary II. 4. *A singular algebra $\langle A, F\rangle$ is an idempotent, entropic algebra, hence, by definition, a mode. (See [12].)*

Corollary II.5. i] *If $\langle A, F\rangle$ is a singular algebra then the lattice of subalgebras of $\langle M(A), F\rangle$ is the Boolean algebra of subsets of $M(A)$, each subset having the set F of operations.*

ii] *If $\langle A, F = P_A \cup e_A\rangle$ is the Boolean algebra of subsets of $M_{\{e\}}(A)$, each subset having the set F of operations.*

Proof. ii] One observes that if $\langle W, F\rangle$ is a subalgebra of $\langle M(A), F\rangle$ then for each $g \in W, g(e) = e$. $\Diamond$

III. Singular and e-singular algebras with composition

For any algebra, $\langle A, F\rangle$, composition of functions is a (binary) operation on the sets $M(A)$ and $\mathrm{End}(A)$. We denote composition by "$\circ$", so for $f, g \in M(A)$, $x \in A$, $\circ(f, g)(x) = f(g(x))$. (As usual, we often use $f \circ g$ or just fg to denote $\circ(f, g)$.) We let $F^\circ := F \cup \{\circ\}$, hence $\langle M(A), F^\circ\rangle$ is an *enrichment* of $\langle M(A), F\rangle$. (See [12], p. 18.) If $\langle A, F\rangle$ satisfies the entropic property, $[E]$, then $\langle \mathrm{End}(A), F^\circ\rangle$ is a subalgebra of $\langle M(A), F^\circ\rangle$. In these cases we say $\langle M(A), F^\circ\rangle$ and $\langle \mathrm{End}(A), F^\circ\rangle$ are *function algebras with composition.*

For $\omega \in F, \tau(\omega) = n$ and $f_1, \ldots, f_n \in M(A)$ we have
$$\circ(\omega(f_1, \ldots, f_n), g) = \omega(f_1, \ldots, f_n) \circ g,$$
so for $a \in A$,
$$\begin{aligned}(\omega(f_1, ..., f_n) \circ g)(a) &= \omega(f_1, ..., f_n)(g(a)) \\ &= \omega(f, g(a), ..., f_n g(a)) = \omega(f_1 g, \ldots, f_n g)(a).\end{aligned}$$
We say g is right composition distributive over F or composition is right distributive over F. Further, we say $g \in M(A)$ is *composition distributive over* F, if for arbitrary $\omega \in F, \tau(\omega) = n$ and arbitrary $f_1, \ldots, f_n$ in $M(A)$ we have $\circ(g, \omega(f_1, \ldots, f_n)) = \omega(gf_1, \ldots, gf_n)$. The collection of composition distributive elements in $M(A)$ is denoted by $\mathrm{Dist}(M(A))$. The following results follow directly from the definitions and previous statements.

Proposition III.1. *For any algebras* $\langle A, F\rangle, \mathrm{Dist}(M(A)) = \mathrm{End}(A)$.

For any algebra, $\langle A, F\rangle$, we say a subalgebra $\langle W, F^\circ\rangle$ of $\langle M(A), F^\circ\rangle$ is composition distributive if every element $h \in W$ is composition distributive over F.

Corollary III.2. *Let* $\langle A, F\rangle$ *be an algebra with* $|A| \geq \mu$. *The following are equivalent:*

i) $\langle A, F\rangle$ *is singular;*

ii) $\langle M(A), F^\circ\rangle = \langle \mathrm{End}(A), F^\circ\rangle$*;*

iii) *Every subalgebra of* $\langle M(A), F^\circ\rangle$ *is composition distributive.*

Corollary III.3. *If* $\langle A, F\rangle$ *satisfies the entropic property* $[E]$*, then* $\langle \mathrm{End}(A), F^\circ\rangle$ *is the unique maximal composition distributive function algebra in* $\langle M(A), F^\circ\rangle$ *or* $\langle \mathrm{End}(A), F^\circ\rangle = \langle M(A), F^\circ\rangle$. *(We use the word "maximal" to mean proper.)*

Proof. Since $\langle A, F\rangle$ satisfies the entropic property, $\langle \mathrm{End}(A), f^\circ\rangle$ is a composition distributive algebra. From Prop. III.1, $\mathrm{Dist}(M(A)) =$ $= \mathrm{End}(A)$, so for any composition distributive algebra, $W = \langle \mathrm{End}(A), f^\circ\rangle$ or $W = \langle M(A), F^\circ\rangle$. $\Diamond$

We remark that in the above discussion the composition distributive functions are relative to all functions in $M(A)$. In contrast, consider the case (say) where $\langle A, F\rangle$ is an infinite abelian group, then $\langle \mathrm{End}(A), F^\circ\rangle$ is the unique maximal composition distributive algebra (ring) in the near-ring $\langle M(A), F^\circ\rangle$ but $\langle \mathrm{End}(A), F^\circ\rangle$ need not be a maximal ring in $\langle M(A), F^\circ\rangle$. (See [5].)

Using Th. II.1 we know when $\langle \mathrm{End}(A), F^\circ\rangle = \langle M(A), F^\circ\rangle$.

Corollary III.4. i] *Let $\langle A, F\rangle$ be an algebra with $|A| \geq \mu$. Then $\langle \mathrm{End}(A), F^{\circ}\rangle = \langle M(A), F^{\circ}\rangle$ if and only if $\langle A, F\rangle$ is singular.*

ii] *If $\langle A, F\rangle$ is an algebra with $|A| \geq \mu + 1$ then*

$$\langle \mathrm{End}(A), F^{\circ}\rangle = \left\langle M_{\{e\}}(A), F^{\circ}\right\rangle$$

if and only if $F = P_A \cup e_A$.

Let $\langle A, F\rangle$ be a singular algebra. From the above, $\langle \mathrm{End}(A), F^{\circ}\rangle$ is, in general, not a maximal composition distributive algebra in $\langle M(A), F^{\circ}\rangle$, however every subalgebra of $\langle M(A), F^{\circ}\rangle$ is composition distributive. We next identify the maximal composition distributive subalgebras of $\langle M(A), F^{\circ}\rangle$ for a finite singular algebra $\langle A, F\rangle$. From Cor. II.5, every subset of H of $M(A)$ determines a subalgebra $\langle H, F\rangle$ of $\langle M(A), F\rangle$. Thus to characterize the maximal subalgebras of $\langle M(A), F^{\circ}\rangle$ one needs only to describe the maximal subsemigroups $\langle H, \circ\rangle$ of $\langle M(A), \circ\rangle$. For finite A this was done by Bayramov [1], as follows. Let $|A| = m$ and for $\iota \leq r \leq m$ let $K(m, r) := \{f \in M(A) || Imf| \leq r\}$. Further let $\langle \mathrm{Perm}(A), \circ\rangle$ denote the group of permutations of A.

With the notation above, Bayramov [1], determines that $\langle H, \circ\rangle$ is a maximal subsemigroup of $\langle M(A), \circ\rangle$ if and only if $H = K(m, m-2) \cup$ $\cup \mathrm{Perm}(A)$ or $H = K(m, m-1) \cup G$ where $\langle G, \circ\rangle$ is a maximal subgroup of $\langle \mathrm{Perm}(A), \circ\rangle$. The maximal subgroups of $\langle \mathrm{Perm}(A), \circ\rangle$ for $|A| < \infty$ have been classified [8]. For further information on the lattice of subsemigroups of $\langle M(A), \circ\rangle$ for $|A| < \infty$, see [16] and the references given there. We can state a description of the maximal subgroups of $\langle M(A), F\rangle$ for finite A.

Theorem III.5. *Let $\langle A, F\rangle$ be a singular algebra with $\mu \leq |A| =$ $= m < \infty$, then $\langle H, F^{\circ}\rangle$ is a maximal subalgebra of $\langle M(A), F^{\circ}\rangle$ if and only if $H = K(m, m-2) \cup \mathrm{Perm}(A)$ or $H = K(m, m-1) \cup G$ where $\langle G, \circ\rangle$ is a maximal subgroup of $\langle \mathrm{Perm}(A), \circ\rangle$.*

We turn our attention now to e-singular algebras. Contrary to the case for singular algebras, in the following example we show that not every subalgebra of $\langle M(A), F^{\circ}\rangle$ is composition distributive for an e-singular algebra $\langle A, F\rangle$.

Example III.6. Let $A = \{e, a, b\}$ and let F consist of the single binary operation $\cdot(x, y) = x \cdot y = e$, for $x, y \in A$. Then $\langle A, F\rangle$ is an e-singular algebra. For $x \in A$, let k_x denote the constant function on A with value x. For $T = \{k_e, k_a, k_b\}$, $\langle T, F^{\circ}\rangle$ is a subalgebra of $\langle M(A), F^{\circ}\rangle$ but $\langle T, F^{\circ}\rangle \subsetneqq \left\langle M_{\{e\}}(A), F^{\circ}\right\rangle = \langle \mathrm{End}(A), F^{\circ}\rangle$.

In Th. III.5 we found the maximal composition distribution algebras in $\langle M(A), F^\circ\rangle$ for a singular algebra $\langle A, F\rangle, \mu \leq |A| < \infty$. To complete our program we next turn to the analogous problem for e-singular algebras, $\langle A, F = P_A \cup e_A\rangle$. As above, using Cor. II.5, we see that we need to characterize subsemigroups $\langle Y, F^\circ\rangle$ of $\left\langle M_{\{e\}}(A), F\right\rangle$ since for any such semigroup, $\langle Y, F^\circ\rangle$ is a subalgebra of $\langle \mathrm{End}(A), F^\circ\rangle$. We take $\langle A, F = P_A \cup e_A\rangle$ as an e-singular algebra with $\mu + 1 \leq |A| = m < \infty$. By definition, $e_A \neq \varnothing$, so we have a distinguished element e. We fix some notation:

i) $A^- := A - \{e\}$,

ii) $P := \mathrm{Perm}(A^-) := \{f \in \mathrm{End}(A) | f \text{ is a bijection of } A^-\}$,

iii) $H_1 := \{f \in \mathrm{End}(A) | f(A^-) \subseteqq A^-\}$,

iv) $\widehat{H}_1 := \{f \in H_1 |\ |f(A^-)| < m - 2\}$,

v) $H_2 := \{f \in \mathrm{End}(A) | f(A^-) \nsubseteqq A^-\}$,

vi) $\widehat{H}_2 := \{f \in H_2 |\ |f(A^-)| \leq m - 2\}$.

We state some straightforward observations.

Lemma III.7. *Using the above notation,*

a) $H_1 \cup P = \mathrm{End}(A^-)$;

b) $\langle H_1 \cup P, \circ\rangle, \langle H_2 \cup P, \circ\rangle, \left\langle H_1 \cup P \cup \widehat{H}_2, \circ\right\rangle$ and $\left\langle \widehat{H}_1 \cup P \cup H_2, \circ\right\rangle$ are subsemigroups of $\langle \mathrm{End}(A), \circ\rangle$;

c) $H_1 \cup H_2 \cup P = \mathrm{End}(A)$.

Lemma III.8. *Using the notation above,* $\left\langle H_1 \cup P \cup \widehat{H}_2, \circ\right\rangle$ *and* $\left\langle \widehat{H}_1 \cup P \cup H_2, \circ\right\rangle$ *are maximal subsemigroups of* $\langle \mathrm{End}(A), \circ\rangle$.

Proof. We consider $\left\langle H_1 \cup P \cup \widehat{H}_2, \circ\right\rangle$, the other case being similar. Let $A^- = \{a_1, \ldots, a_m\}$ and let $g \in H_2 \setminus \widehat{H}_2$, so $g(a_\iota) = e$ for some $a_\iota \in A^-$. Since $g \notin \widehat{H}_2$, $|g(A^-)| = m - 1$, hence g is a one-one function on A^-. We show that any $k \in H_2 \setminus \widehat{H}_2$ is in the subsemigroup of $\langle \mathrm{End}(A), \circ\rangle$ generated by $H_1 \cup P \cup \widehat{H}_2 \cup \{g\}$, hence the result. Suppose $k(a_j) = e$ and let $\sigma \in P$ be the transposition interchanging a_ι and a_j. We have

$$g\sigma(a_\iota) = \begin{cases} g(a_\ell), & l \neq i, j, \\ e, & l = j, \\ g(a_j), & l = i, \end{cases} \tag{1}$$

and we note $|g\sigma(A^-)| = m-1$. Thus there exists $\rho \in P$ with

$$\rho g(a_e) = \begin{cases} k(a_\ell), & l \neq i, j, \\ k(a_\iota), & l = \iota, \end{cases} \tag{2}$$

and $\rho(a_j) = a$ where $A^- \setminus k(A^-) = \{a\}$. From this, $k = \rho g \sigma$ is in $\left\langle H_1 \cup P \cup \widehat{H}_2, \circ \right\rangle$. $\Diamond$

Next we note that, since $\langle H_1 \cup P \cup H_2, \circ\rangle = \langle \mathrm{End}(A), \circ\rangle$, for any maximal subgroup $\langle G, \circ\rangle$ of $\langle P, \circ\rangle$, $\langle H_1 \cup H_2 \cup G, \circ\rangle$ is a maximal subsemigroup of $\langle \mathrm{End}(A), \circ\rangle$. We show that we have now identified all maximal subsemigroups of $\langle \mathrm{End}(A), \circ\rangle$.

To this end let $\langle M, \circ\rangle$ be a maximal subsemigroup of $\langle \mathrm{End}(A), \circ\rangle$ where again, $\langle A, F = P_A \cup e_A\rangle$ is a finite e-algebra with $\mu + 1 \leq |A| = m < \infty$. We have $M = M \cap \mathrm{End}(A) = M \cap (H_1 \cup H_2 \cup P) = (M \cap H_1) \cup (M \cap H_2) \cup (M \cap P)$, a disjoint union. Note $M \cap P \neq \phi$ for otherwise $\langle M, \circ\rangle \subsetneqq \langle M \cup \{id_A\}, \circ\rangle$ and thus $\langle M \cup \{id\}, \circ\rangle = \langle \mathrm{End}(A), \circ\rangle$, a contradiction. Suppose first $M \cap P \subsetneqq P$. then $M \subseteq H_1 \cup H_2 \cup (M \cap P)$. If $\langle M \cap P, \circ\rangle$ is not a maximal subgroup of $\langle P, \circ\rangle$, we get a contradiction, so in this case we have $M = H_1 \cup H_2 \cup G$, $\langle G, \circ\rangle$ a maximal subgroup of $\langle P, \circ\rangle$.

The case, $M \cap P = P$ remains. We have $M = (M \cap H_1) \cup (M \cap H_2) \cup P$. If $H_2 \cap M \subseteq \widehat{H}_2$, then $M = (M \cap H_1) \cup (M \cap H_2) \cup P \subseteq H_1 \cup \widehat{H}_2 \cup P$. From Lemma III.8, $\left\langle H_1 \cup \widehat{H}_2 \cup P, \circ \right\rangle$ is a maximal subsemigroup of $\left\langle \mathrm{End}(A), \circ \right\rangle$, so $\langle M, \circ\rangle = \left\langle H_1 \cup \widehat{H}_2 \cup P, \circ \right\rangle$. Similarly, if $H_1 \cap M \subseteq \widehat{H}_1$ we get $\langle M, \circ\rangle = \left\langle \widehat{H}_1 \cup H_2 \cup P, \circ \right\rangle$. Thus, it remains to consider the situation where we have $M \cap H_1 \nsubseteqq \widehat{H}_1$ and $M \cap H_2 \nsubseteqq \widehat{H}_2$. Since $H_2 \cap M \nsubseteqq \widehat{H}_2$, there exists $g \in (M \cap H_2) \setminus \widehat{H}_2$. As in Lemma III.8, we find that every $f \in H_2$ with $|f(A^-)| = m - 1$ is in $M \cap H_2$ (since $P \subseteq M$) and then one finds $H_2 \subseteq M$. But then $M = (H_1 \cap M) \cup H_2 \cup P$. Using $M \cap H_1 \nsubseteqq \widehat{H}_1$ we have $H_1 \subseteq M$ which now gives $M = H_1 \cup H_2 \cup P$, a contradiction. We have the following characterization result.

Theorem III.9. *Let $\langle A, F = P_A \cup e_A\rangle$ be an e-singular algebra with $\mu + 1 \leq |A| = m < \infty$. Using the above notation we have that $\langle M, F^\circ\rangle$ is a maximal subalgebra of $\langle \mathrm{End}(A), \circ\rangle$ if and only if $M = H_1 \cup H_2 \cup G$ where $\langle G, \circ\rangle$ is a maximal subgroup of $\langle P, \circ\rangle$ or $M = H_1 \cup P \cup \widehat{H}_2$ or $M = \widehat{H}_1 \cup P \cup H_2$.*

The infinite case remains open.

Problem. What are the analogues for Th. III.5 and Th. III.9 when A is infinite?

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