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KILLING AN END POINT WITH AN OPEN MAPPING REVISITED

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Abstract: This paper deals with the question if the end points of a smooth dendroid are mapped into the end points under an open mapping posed by J. J. Charatonik and W. J. Charatonik. We follow up on the previous negative answers by providing another rather simple example where this is not the case.

1. Introduction

In this paper, all the considered spaces are supposed to be metric and all the mappings are continuous.

Definition 1 (Interior at a point mapping). Let $f : X \to Y$ be a mapping. We say that f is *interior at point* $p \in X$ if for each open

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neighbourhood U of p in X, the point $f(p)$ is an interior point of the set $f(U)$ in Y.

Definition 2 (Open mapping)**.** A mapping $f : X \to Y$ is referred to as *open* provided that for each open subset U of X, its image $f(U)$ is an open subset of Y.

Obviously, a mapping is open if and only if it is interior at each point of its domain.

A *continuum* is a nonempty compact connected metric space. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. The symbol $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{N}$ stands for the set of all naturals.

A continuum X is *arcwise connected* provided each two points of X are contained in some arc contained in X.

A connected topological space S is said to be *unicoherent* provided that whenever A,B are closed connected subsets of S such that $S = A \cup B$, then $A \cap B$ is connected. A connected topological space is said to be *hereditarily unicoherent* provided that each of its closed connected subsets is unicoherent. A *dendroid* is an arcwise connected hereditarily unicoherent continuum.

A dendroid X is said to be *smooth* provided that there exists a point $v \in X$ such that for each point $x \in X$ and each sequence of points x_n tending to x, the sequence of arcs connecting v to x_n in X tends to the arc connecting v to x.

By a *simple m-od* with the *vertex p*, we mean the union of m arcs every pair of which have p as the only common point. A *simple triod* is a simple 3-od.

Let X be a continuum and p a point in X. Then p is said to be a *point of order at least m (in the classical sense*[1]) if p is the (central) vertex of a m-od contained in X. We say that p is a *point of order m* (in the classical sense) provided that m is the minimum cardinality for which the above condition is satisfied (see [1]).

A point of order 1 (in the classical sense) is called an *end point* of a continuum. A point of order 2 (in the classical sense) is called an *ordinary point* of a continuum. A *branch point* of a continuum is the

[1] Another definition of an order of a point was given by Whyburn in [7]. There are fundamental differences between these two notions. In this paper, we always refer to the order in the classical sense.

vertex of a simple triod lying in that continuum.

Notice that there are many continua without any arc and that the notion of the order of the point in the classical sense does not make sense for them.

2. The construction and its simplification

J. J. Charatonik and W. J. Charatonik in 1997 (see [2, Question 2.3, p. 3730], [3, Question 3.3, p. 103]) asked if the end points of a smooth dendroid are mapped into the end points under an open mapping. The first negative answer was provided by L. G. Oversteegen already in 1980 (cf. [5, Example 3.2, p. 118]). The author presents a very smart construction of a mapping that is in addition monotone (i.e. the pre-images of points are connected). However, this construction is very complicated and hence we do not describe it any further.

Another, and indeed much simpler, example of the case when the end points of a smooth dendroid are not mapped into the end points under and open mapping was published by Pyrih in 2011 [6]. We follow up on this paper and provide a construction of another example.

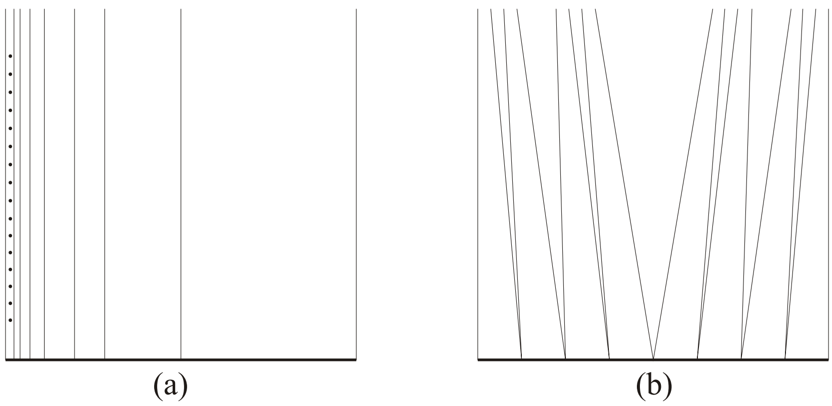

Figure 1: The harmonic comb and Cantor function comb

The following recalls the construction from [6, p. 2]. A *harmonic comb* (shown in Fig. 1a) is the union of countably many vertical line segments called *teeth* connected to a horizontal *body*. The most right and left teeth are called *first tooth* and the *last tooth* respectively. Fig. 1b depicts the Cantor function comb, in which tooth are the linear segments connecting $(x, 1)$ and $(\varphi(x), 0)$ for each $x \in C$, where φ is the Cantor function and C is the Cantor ternary set in $[0, 1]$ (see [6, sec. 2] for further details).

Consider $\mathbb{R}^3$ with points described by coordinates (x, y, z). For all of the following, we define projection

$$\pi : \mathbb{R}^3 \to \mathbb{R}^3$$
$$\pi((x, y, z)) = (x, 0, 0)$$

of each point on its x-coordinate.

Similarly to [6], let us create a continuum $X \subset \mathbb{R}^3$ by "gluing" two copies of the harmonic comb together with their bodies (Fig. 2a). These bodies together with the first teeth form a simple triod $T \subset X$.

We replace in X the segment cd by the singleton e and for each $y \in (1/2, 1)$ such that $(1, y, 0) \in X$ we replace in X the minimal arc in X joining points $(1, y, 0)$ and $(-1, y, 0)$ by the set

$$\left\{(x, y\exp(-x^2/(1-y)), x(1-y)) \in \mathbb{R}^3,\ x \in [-1, 1]\right\}$$

and obtain a continuum $\widetilde{X}$. See Fig. 2.

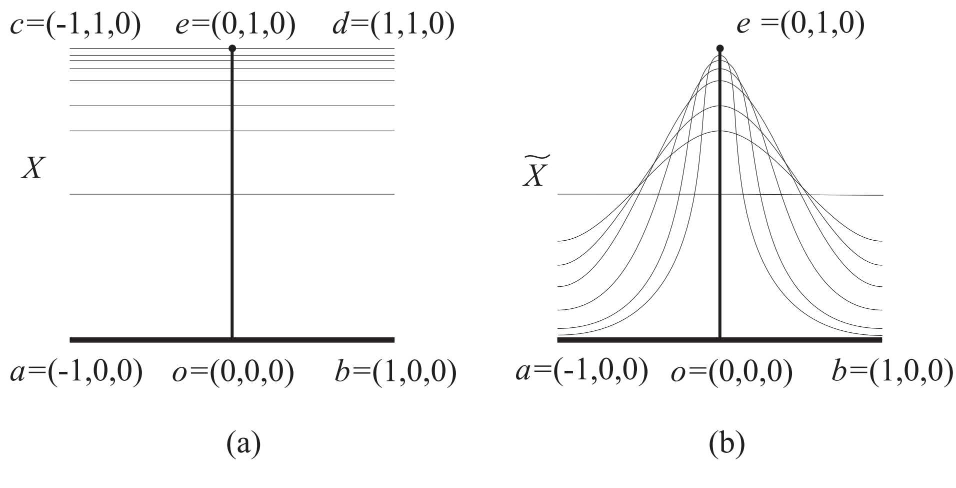

Figure 2: Illustration of the continuum $\tilde{X}$

The projection $\pi : \tilde{X} \to \pi(\tilde{X})$ is interior at e (Fig. 2b), but the interiority is not clear in all the ordinary points of the arc (o, e).

This construction is in [6] further enhanced to a full interiority of π using the (double) Cantor function comb. We suggest that this construction may be too complicated afterall, and provide a new example of the negative answer to question by J. J. Charatonik and W. J. Charatonik.

Surprisingly, the continuum $\tilde{X}$ alone is enough to find the desired example. However, we have to find a suitable mapping as the projection π may not work. To simplify the description of the mapping we use a continuum topologically equivalent to $\tilde{X}$ defined this way.

Consider the following construction of a continuum Y formed by the union

$$T \cup \bigcup_{n=4}^{\infty} \bigcup_{m=1}^{n^2} L_n^m ,$$

where T is as above and each L_n^m (for $n \in \mathbb{N}$, $1 \leq m \leq n^2$) is a piecewise linear curve going through the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}$ that are as follows:

$$\begin{aligned}
\mathfrak{a} &= \left(-1, \tfrac{m}{n^5}, \tfrac{\mathfrak{a}_x}{\lambda}\right), & \mathfrak{b} &= \left(-\tfrac{1}{n} - \tfrac{1}{n^3}, \mathfrak{a}_y, \tfrac{\mathfrak{b}_x}{\lambda}\right), \\
\mathfrak{c} &= \left(-\tfrac{1}{n}, \tfrac{m}{n^5} + \tfrac{m-1}{n^2}, \tfrac{\mathfrak{c}_x}{\lambda}\right), & \mathfrak{d} &= \left(-\tfrac{1}{n^2}, \mathfrak{c}_y, \tfrac{\mathfrak{d}_x}{\lambda}\right), \\
\mathfrak{e} &= \left(0, 1 - \tfrac{1}{n^3} + \tfrac{m}{n^7}, \tfrac{\mathfrak{e}_x}{\lambda}\right), & \mathfrak{f} &= \left(-\mathfrak{d}_x, \mathfrak{c}_y, \tfrac{\mathfrak{f}_x}{\lambda}\right), \\
\mathfrak{g} &= \left(-\mathfrak{c}_x, \mathfrak{c}_y, \tfrac{\mathfrak{g}_x}{\lambda}\right), & \mathfrak{h} &= \left(-\mathfrak{b}_x, \mathfrak{a}_y, \tfrac{\mathfrak{h}_x}{\lambda}\right), \\
\mathfrak{i} &= \left(1, \mathfrak{a}_y, \tfrac{\mathfrak{i}_x}{\lambda}\right),
\end{aligned}$$

with $\lambda = n^3$ and $(\mathfrak{p}_x, \mathfrak{p}_y, \mathfrak{p}_z)$ being the coordinates of a point $\mathfrak{p}$ (cf. Fig. 3).

The projection $\pi : Y \to \pi(Y)$ (defined as above) now is the desired open mapping killing an endpoint.

Indeed, each neighborhood of any point p of the segment oe contains horizontal segments $\mathfrak{c}\mathfrak{d}$, which are projected onto the segment $(-1/n, -1/n^2)$ and also contains horizontal segments $\mathfrak{f}\mathfrak{g}$, which are projected onto the segment $(1/n^2, 1/n)$ for all n sufficiently large (this is true due to the ladders contained in the corresponding L_n^m in the construction). The union of these segments form a neighborhood of o. This proves the interiority of the projection π at the point p.

Notice that this continuum Y is still topologically the same as the continuum $\tilde{X}$.

Let us now simplify this construction even more. Consider continuum Z constructed similarly as continuum Y with the difference that

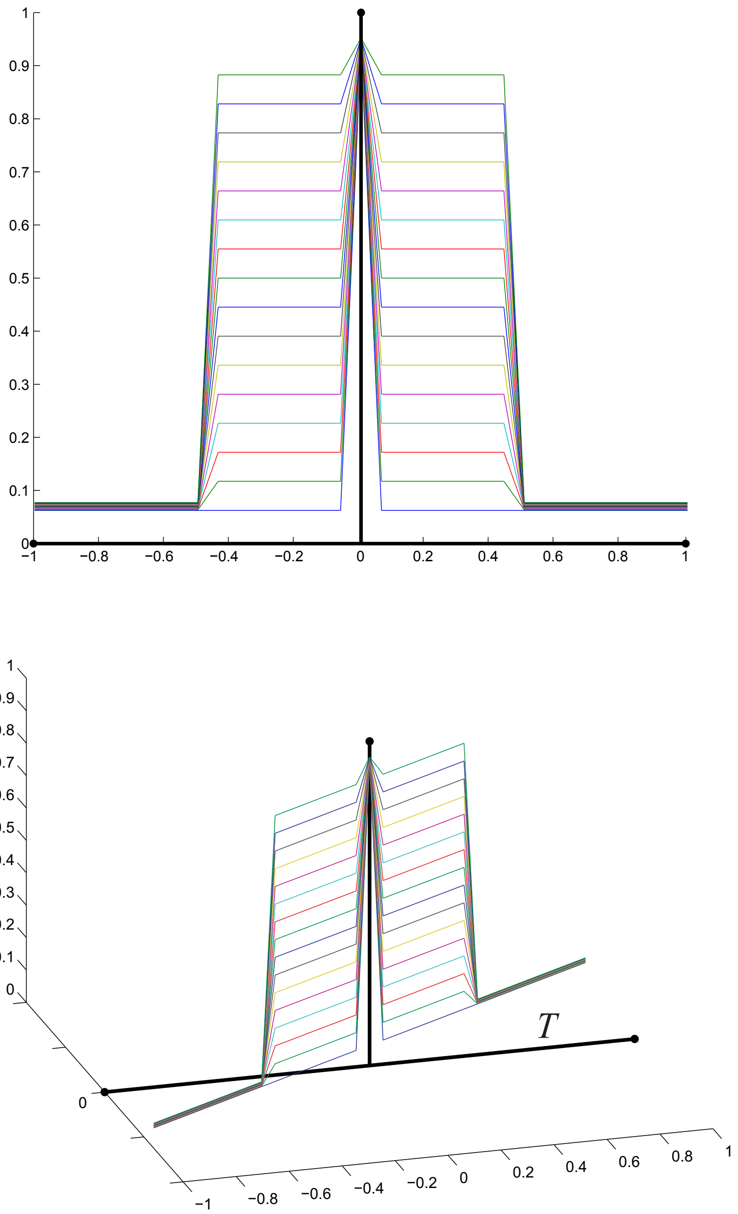

Figure 3: Curves L_4^m for $1 \leq m \leq 16$ used for the construction of Y

the piecewise linear curves L_n^m go through the points $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}$ (with the coordinates given above) only (Fig. 4). This is fully sufficient for $\pi : Z \to \pi(Z)$ to be an open mapping from Z to an arc killing the end point.

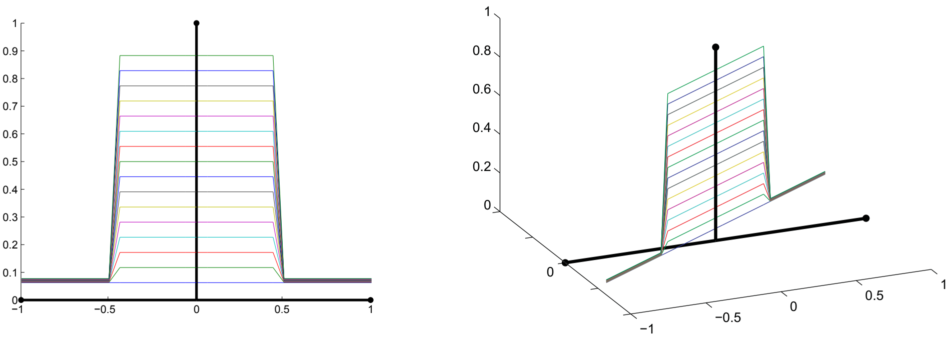

Figure 4: Curves L_4^m for $1 \leq m \leq 16$ used for the construction of Z

Obviously, all the dendroids in the examples above are smooth.

3. Conclusions

We gave two new examples of an open mapping from a smooth dendroid erasing an end point, which greatly simplify the constructions known so far. Furthermore, the concepts introduced within the examples are applicable in general and can lead to construction of the new ones. For instance, the continuum constructed in the similar fashion from the closure of the set

$$\left\{\left(x, e^{-nx^2}|\sin n|, \frac{x}{n}\right) \in \mathbb{R}^3 \,|\, x \in (-1,1),\, n \in \mathbb{N}\right\}$$

with the projection π works as well. However, we suggest that the constructions presented in this paper are the simplest possible.

In [6, sec. 7], the second author posed a question which our paper does not provide any answer for, and which still remains open:

Question 1. Is there a *planar* smooth dendroid and an open mapping not preserving the end points (in the classical sense)?

Errata

to Pavel Pyrih: KILLING AN END POINT WITH AN OPEN MAPPING, *Mathematica Pannonica* **22**/1 (2011), 1-7.

The construction of $\widetilde{X}$ and $\widetilde{Y}$ on pages 3–4 contains errors in the description. We restate the construction here in a correct way.

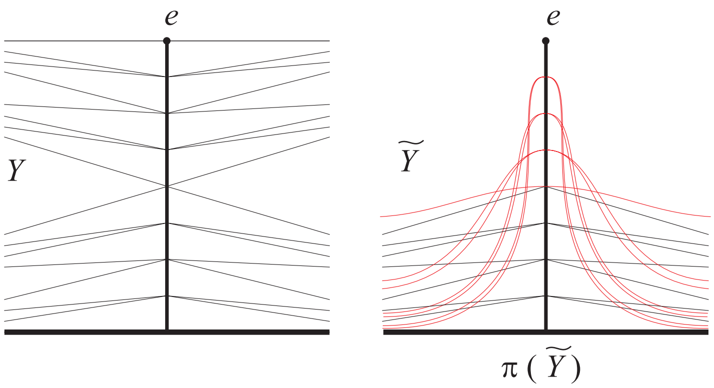

Figure 5: Combing a double Cantor function comb

We replace in X the segment cd by the singleton e and for each $y \in (1/2, 1)$ such that $(1, y, 0) \in X$ we replace in X the minimal arc in X joining points $(1, y, 0)$ and $(-1, y, 0)$ by the set

$$\{(x, y \exp(-x^2/(1-y)), x(1-y)) \in \mathbb{R}^3\,,\ x \in [-1, 1]\}$$

and obtain the continuum $\widetilde{X}$. See Fig. 2.

We denote by φ the Cantor function $\varphi : C \to [0, 1]$ from the Cantor ternary set onto the unit interval. Now we can similarly replace in Y the segment cd by the singleton e and for each $y \in (1/2, 1)$ such that $(1, y, 0) \in Y$ we replace in Y the minimal arc in Y joining points $(1, y, 0)$ and $(-1, y, 0)$ by the set

$$\{(x, \varphi(y) \exp(-x^2/(1-y)), x(1-y)) \in \mathbb{R}^3\,,\ x \in [-1, 1]\}$$

and obtain the continuum $\widetilde{Y}$. See Fig. 5.

Now the projection $\pi(x, y, z) = (x, 0, 0)$ is an open mapping on $\widetilde{Y}$ because the interiority of the projection is fixed at each point of $\widetilde{Y}$ due to the replaced arcs

$$\big\{(x, \varphi(y) \exp(-x^2/(1-y)), x(1-y)) \in \mathbb{R}^3,\ x \in [-1, 1]\big\}.$$

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