Mathematica Pannonica
24/1 (2013), 15–31

SOME CONFORMALLY INVARIANT TENSORS ON ANTI-KÄHLER MANIFOLDS AND THEIR GEOMETRICAL PROPERTIES

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Received: December 2011

MSC 2010: 53 A 30, 53 B 15, 53 C 15, 53 C 56

Keywords: Anti-Kähler space, anti-Kähler condition, conformally related metrics, metric connection, J-connection, generator, curvature tensor.

Abstract: We consider a class of conformal transformation on anti-Kähler manifolds and find even five invariants for such a transformation. Three of them are curvature-like tensors. Besides, we find that one of these curvature-like conformal invariants is also an invariant of two special connections on such a kind of spaces.

1. Facts about anti-Kähler manifolds

An anti-Kähler manifold $(\mathcal{M}, g, J)$ is a differentiable manifold $\mathcal{M}$, $\dim \mathcal{M} = 2n$, endowed with parallel complex structure J and anti-Hermitian metrics g. With respect to the local coordinates, these condi-

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tions are

$$J^i_a J^a_j = -\delta^i_j, \text{ (a)}; \quad g_{ab} J^a_i J^b_j = -g_{ij}, \text{ (b)}; \quad \nabla_k J^i_j = 0, \text{ (c)}; \tag{1.1}$$

where ∇ is Levi-Civita connection operator with respect to metric g. We note that $J^a_a = 0$.

The anti-Kähler manifolds were first investigated by A. P. Norden [3] (in the case $\dim M = 4$). He named them B-manifolds to distinguish them from the Kähler spaces (A-manifolds). Low-dimensional spaces were also investigated in some contemporary papers, for example, [8]. Anti-Kähler manifolds were very well presented in the paper [1], with all of its geometric characteristics. Also, even now, the Yano's monography [9] can be useful.

The condition (1.1)(b) implies

$$F_{ij} = g_{aj} J^a_i = g_{ia} J^a_j = F_{ji}. \tag{1.2}$$

Let R_{ijkh} be a component of the Riemannian curvature tensor of Levi-Civita connection. Then the first and the second Ricci tensors are

$$\rho_{jh} = R_{ijhk} g^{ik}, \quad \widetilde{\rho}_{jh} = R_{ajhk} J^a_i g^{ik}, \tag{1.3}$$

and the first and the second scalar curvatures are

$$\varkappa = \rho_{jh} g^{jh}, \quad \widetilde{\varkappa} = \widetilde{\rho}_{jh} g^{jh}. \tag{1.4}$$

The second Ricci tensor like the first one, is symmetric. Namely

$$\widetilde{\rho}_{jh} = R_{ajhk} J^a_i g^{ik} = R_{khja} J^k_i g^{ia} = \widetilde{\rho}_{hj}.$$

The condition (1.1)(c), in view of the Ricci identity implies

$$R_{ijab} J^a_h J^b_k = -R_{ijhk}. \tag{1.5}$$

It is known that the condition

$$R_{ijab} J^a_h J^b_k = R_{ijhk}$$

characterizes the class of Kähler manifolds among the Hermitian manifolds. Thus, we say that condition (1.5) is the *condition of anti-Kähler type.*

The condition (1.5) implies

$$R_{abcd} J^a_i J^b_j J^c_h J^d_k = R_{ijhk}$$

and ([4], [5])

$$\rho_{aj}J_i^a = \widetilde{\rho}_{ji}, \qquad \rho_{ij} = -\widetilde{\rho}_{aj}J_j^a,$$
$$\rho_{ab}J_i^a J_j^b = -\rho_{ij}, \qquad \widetilde{\rho}_{ab}J_i^a J_j^b = -\widetilde{\rho}_{ij}. \tag{1.6}$$

Finally, in view of (1.5), (1.6) and (1.7), we have the following identity:

$$\begin{aligned} R_{ijhk} = \frac{1}{8}\big[&R_{ijhk} + R_{abcd}J_i^a J_j^b J_h^c J_k^d - \\ &-R_{abhk}J_i^a J_j^b - R_{ajbk}J_i^a J_h^b - R_{ajhb}J_i^a J_k^b - \\ &-R_{iabk}J_j^a J_h^b - R_{iahb}J_j^a J_h^b - R_{ijab}J_h^a J_k^b\big]. \end{aligned} \tag{1.7}$$

2. Conformal change of metric

Let us consider Riemannian manifolds ($\overline{\mathcal{M}}$,g) and ($\mathcal{M}, g$) and a diffeomorphism $\varphi : \mathcal{M} \to \overline{\mathcal{M}}$ such that $\overline{g}_{ij} = e^{2f} g_{ij}$, where f is a scalar function. Then we say that $\varphi : \mathcal{M} \to \overline{\mathcal{M}}$ is a conformal mapping. Since φ is a diffeomorphism, we can suppose that locally it maps points onto points with the same local coordinates, that is, locally, we can presume that $\mathcal{M} = \overline{\mathcal{M}}$ ([1]). We shall consider the conformal mapping of the anti-Kähler manifolds ($\mathcal{M}$,$\overline{g}, J$) and ($\mathcal{M}, g, J$). Thus $\dim \mathcal{M} = 2n$ and $\overline{F}_{ij} = e^{2f} F_{ij}$. From now on, all geometric objects in ($\mathcal{M}$,$\overline{g}, J$) will be denoted by analogous letters as in ($\mathcal{M}, g, J$), but with "bar".

It is well known that the Riemannian curvature tensors, $\overline{R}_{ijhk}$ and R_{ijhk}, are related as follows (see, for example, [2]):

$$e^{-2f}\overline{R}_{ijhk} = R_{ijhk} + g_{ik}\sigma_{jh} + g_{jh}\sigma_{ik} - g_{ih}\sigma_{jk} - g_{jk}\sigma_{ih}, \tag{2.1}$$

where

$$\sigma_{jh} = \nabla_j \sigma_h - \sigma_j \sigma_h + \frac{1}{2} g_{jh} \triangle_1 \sigma, \qquad \sigma_h = \frac{\partial f}{\partial x^h}, \qquad \triangle_1 \sigma = \sigma_a \sigma_b g^{ab}.$$

We note that $\sigma_{ij} = \sigma_{ji}$.

If we contract (2.1) with respect to $g^{ik} = e^{2f}\overline{g}^{ik}$, we obtain

$$\overline{\rho}_{jh} = \rho_{jh} + 2(n-1)\sigma_{jh} + g_{jh}\sigma_{ab}g^{ab}. \tag{2.2}$$

Now, contracting (2.2) with respect to g^{jh}, we find

$$\sigma_{ab}g^{ab} = \frac{e^{2f}\overline{\varkappa} - \varkappa}{2(2n-1)}, \tag{2.3}$$

such that (2.2) becomes

$$\sigma_{jh} = \frac{1}{2(n-1)}\left[\overline{\rho}_{jh} - \frac{\overline{\varkappa}}{2(2n-1)}\overline{g}_{jh}\right] - \frac{1}{2(n-1)}\left[\rho_{jh} - \frac{\varkappa}{2(2n-1)}g_{jh}\right].$$

Thus, and in view of (1.8),

$$\sigma_{ab}J^a_j J^b_h = -\frac{1}{2(n-1)}\left[\overline{\rho}_{jh} - \frac{\overline{\varkappa}}{2(2n-1)}\overline{g}_{jh}\right] + \frac{1}{2(n-1)}\left[\rho_{jh} - \frac{\varkappa}{2(2n-1)}g_{jh}\right].$$

Therefore

$$\sigma_{jh} - \sigma_{ab}J^a_j J^b_h = \frac{1}{(n-1)}\left[\overline{\rho}_{jh} - \frac{\overline{\varkappa}}{2(2n-1)}\overline{g}_{jh}\right] - \frac{1}{(n-1)}\left[\rho_{jh} - \frac{\varkappa}{2(2n-1)}g_{jh}\right] \tag{2.4}$$

and

$$\sigma_{ja}J^a_h + \sigma_{ah}J^a_j = \frac{1}{n-1}\left[\overline{\widetilde{\rho}}_{jh} - \frac{\overline{\varkappa}}{2(2n-1)}\overline{F}_{jh}\right] - \frac{1}{n-1}\left[\widetilde{\rho}_{jh} - \frac{\varkappa}{2(2n-1)}F_{jh}\right]. \tag{2.5}$$

On the other hand, the relation (2.1) yields

$$e^{-2f}\overline{R}_{ajhk}J^a_i = R_{ajhk}J^a_i + F_{ik}\sigma_{jh} + g_{jh}\sigma_{ak}J^a_i - F_{ih}\sigma_{jk} - g_{jk}\sigma_{ah}J^a_i,$$

from which, contracting with respect to g^{ik}, we obtain

$$\overline{\widetilde{\rho}}_{jh} = \widetilde{\rho}_{jh} + g_{jh}(\sigma_{ak}J^a_i)g^{ik} - (\sigma_{ja}J^a_h + \sigma_{ah}J^a_j).$$

Contracting this relation with respect to g^{jh}, we get

$$\sigma_{ak}F^{ak} = \frac{e^{2f}\overline{\widetilde{\varkappa}} - \widetilde{\varkappa}}{2(n-1)}, \tag{2.6}$$

such that the preceding relation becomes

$$\sigma_{ja}J^a_h + \sigma_{ah}J^a_j = -\overline{\widetilde{\rho}}_{jh} + \frac{\overline{\widetilde{\varkappa}}}{2(n-1)}\overline{g}_{jh} + \widetilde{\rho}_{jh} - \frac{\widetilde{\varkappa}}{2(n-1)}g_{jh}, \tag{2.7}$$

wherefrom, using (1.8), we get

$$\sigma_{jh} - \sigma_{ab}J^a_j J^b_h = -\left[\overline{\rho}_{jh} + \frac{\overline{\widetilde{\varkappa}}}{2(n-1)}\overline{F}_{jh}\right] + \left[\rho_{jh} + \frac{\widetilde{\varkappa}}{2(n-1)}F_{jh}\right]. \tag{2.8}$$

Comparing (2.4) and (2.8), we find

$$2n\overline{\rho}_{jh} - \frac{\overline{\varkappa}}{2n-1}\overline{g}_{jh} + \overline{\widetilde{\varkappa}}F_{jh} = 2n\rho_{jh} - \frac{\varkappa}{2n-1}g_{jh} + \widetilde{\varkappa}F_{jh}. \tag{2.9}$$

Thus, we can state

Theorem 1. *For an anti-Kähler manifold, the tensor*

$$2n\rho_{jh} - \frac{\varkappa}{2n-1}g_{jh} + \widetilde{\varkappa}F_{jh}$$

is conformally invariant.

3. Conformally invariant curvature tensors

For the manifold ($\mathcal{M}$,$\overline{g}$, J) the relation (1.9) is

$$\begin{aligned} e^{-2f}\overline{R}_{ijhk} = \frac{1}{8}\big[&\overline{R}_{ijhk} + \overline{R}_{abcd}J_i^aJ_j^bJ_h^cJ_k^d - \overline{R}_{abhk}J_i^aJ_j^b - \overline{R}_{ajbk}J_i^aJ_h^b - \\ &-\overline{R}_{ajhb}J_i^aJ_k^b - \overline{R}_{iabk}J_j^aJ_h^b - \overline{R}_{iahb}J_j^aJ_k^b - \overline{R}_{ijab}J_h^aJ_k^b\big], \end{aligned}$$

wherefrom, substituting (2.1), we find

$$\begin{aligned} e^{-2f}\overline{R}_{ijhk} = R_{ijhk} + \frac{1}{4}\big\{&g_{ik}(\sigma_{jh} - \sigma_{ab}J_j^aJ_h^b) + g_{jh}(\sigma_{ik} - \sigma_{ab}J_i^aJ_k^b) - \\ &- g_{ih}(\sigma_{jk} - \sigma_{ab}J_j^aJ_k^b) - g_{jk}(\sigma_{ih} - \sigma_{ab}J_i^aJ_h^b) - \\ &- F_{ik}(\sigma_{ah}J_j^a + \sigma_{ja}J_h^a) - F_{jh}(\sigma_{ak}J_i^a + \sigma_{ia}J_k^a) + \\ &+ F_{ih}(\sigma_{ak}J_j^a + \sigma_{ja}J_k^a) + F_{jk}(\sigma_{ah}J_i^a + \sigma_{ia}J_h^a)\big\}. \end{aligned} \tag{3.1}$$

Contracting (3.1) with respect to $g^{ik} = e^{2f}\overline{g}^{ik}$, we obtain

$$\overline{\rho}_{jh} = \rho_{jh} + \frac{1}{2}\big[(n-2)(\sigma_{jh} - \sigma_{ab}J_j^aJ_h^b) + (\sigma_{ab}g^{ab})g_{jh} - (\sigma_{ab}F^{ab})F_{jh}\big]. \tag{3.2}$$

Contracting (3.2) with respect to g^{jh}, we obtain

$$\sigma_{ab}g^{ab} = \frac{e^{2f}\overline{\varkappa} - \varkappa}{2(n-1)}, \tag{3.3}$$

while after contracting it with respect to $F^{jh} = e^{2f}\overline{F}^{jh}$, we find

$$\sigma_{ab}F^{ab} = \frac{e^{2f}\overline{\widetilde{\varkappa}} - \widetilde{\varkappa}}{2(n-1)}.$$

Thus, relation (3.2) becomes

$$(3.4)\ \sigma_{jh} - \sigma_{ab} J_j^a J_h^b = \frac{2}{n-2}\left[\overline{\rho}_{jh} - \frac{\overline{\varkappa}}{4(n-1)}\overline{g}_{jh} + \frac{\overline{\widetilde{\varkappa}}}{4(n-1)}\overline{F}_{jh}\right] - \\ -\frac{2}{n-2}\left[\rho_{jh} - \frac{\varkappa}{4(n-1)}g_{jh} + \frac{\widetilde{\varkappa}}{4(n-1)}F_{jh}\right],$$

wherefrom there follows

$$(3.5)\ \sigma_{ja} J_h^a + \sigma_{ah} J_j^a = \frac{2}{n-2}\left[\overline{\widetilde{\rho}}_{jh} - \frac{\overline{\widetilde{\varkappa}}}{4(n-1)}\overline{g}_{jh} - \frac{\overline{\varkappa}}{4(n-1)}\overline{F}_{jh}\right] - \\ -\frac{2}{n-2}\left[\rho_{jh} - \frac{\widetilde{\varkappa}}{4(n-1)}g_{jh} - \frac{\varkappa}{4(n-1)}F_{jh}\right].$$

Now, substituting (3.4) and (3.5) into (3.1), we have

$$e^{-2f}\Big\{\overline{R}_{ijhk} - \frac{1}{2(n-2)}(\overline{g}_{ik}\overline{\rho}_{jh} + \overline{g}_{jh}\overline{\rho}_{ik} - \overline{g}_{ih}\overline{\rho}_{jk} - \overline{g}_{jk}\overline{\rho}_{ih} - \\ -\overline{F}_{ik}\overline{\widetilde{\rho}}_{jh} - \overline{F}_{jh}\overline{\widetilde{\rho}}_{ik} + \overline{F}_{ih}\overline{\widetilde{\rho}}_{jk} + \overline{F}_{jk}\overline{\widetilde{\rho}}_{ih}) + \\ +\frac{\overline{\varkappa}}{4(n-1)(n-2)}(\overline{g}_{ik}\overline{g}_{jh} - \overline{g}_{ih}\overline{g}_{jk} - \overline{F}_{ik}\overline{F}_{jh} + \overline{F}_{ih}\overline{F}_{jk}) - \\ -\frac{\overline{\widetilde{\varkappa}}}{4(n-1)(n-2)}(\overline{g}_{ik}\overline{F}_{jh} + \overline{g}_{jh}\overline{F}_{ik} - \overline{g}_{ih}\overline{F}_{jk} - \overline{g}_{jk}\overline{F}_{ih})\Big\} = \\ = R_{ijkh} - \frac{1}{2(n-2)}(g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih} - \\ -F_{ik}\widetilde{\rho}_{jh} - F_{jh}\widetilde{\rho}_{ik} + F_{ih}\widetilde{\rho}_{jk} + F_{jk}\widetilde{\rho}_{ih}) + \\ +\frac{\varkappa}{4(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk}) - \\ -\frac{\widetilde{\varkappa}}{4(n-1)(n-2)}(g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih}).$$

In other words, the tensor

$$(3.6)\quad \overset{1}{B}_{ijhk} = R_{ijkh} - \frac{1}{2(n-2)}(g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih} - \\ -F_{ik}\widetilde{\rho}_{jh} - F_{jh}\widetilde{\rho}_{ik} + F_{ih}\widetilde{\rho}_{jk} + F_{jk}\widetilde{\rho}_{ih}) + \\ +\frac{\varkappa}{4(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk}) -$$

$$-\frac{\widetilde{\varkappa}}{4(n-1)(n-2)}(g_{ik}F_{jh}+g_{jh}F_{ik}-g_{ih}F_{jk}-g_{jk}F_{ih})$$

satisfies the condition

$$e^{-2f}\ \overset{1}{\overline{B}}=\overset{1}{B}_{ijhk}\ . \tag{3.7}$$

The tensor (3.6) is obtained in [4] and [5] using the pseudoconformal correspondence

$$\overline{g}_{ij}=\alpha g_{ij}+\beta F_{ij},$$

where α and β are scalar functions.

On the other hand, substituting (2.4) and (2.5) into (3.1), we obtain

$$\begin{aligned}
&e^{-2f}\Big\{\overline{R}_{ijhk}-\frac{1}{4(n-1)}(\overline{g}_{ik}\overline{\rho}_{jh}+\overline{g}_{jh}\overline{\rho}_{ik}-\overline{g}_{ih}\overline{\rho}_{jk}-\overline{g}_{jk}\overline{\rho}_{ih}-\\
&\qquad-\overline{F}_{ik}\overline{\widetilde{\rho}}_{jh}-\overline{F}_{jh}\overline{\widetilde{\rho}}_{ik}+\overline{F}_{ih}\overline{\widetilde{\rho}}_{jk}+\overline{F}_{jk}\overline{\widetilde{\rho}}_{ih})+\\
&\qquad+\frac{\varkappa}{4(n-1)(2n-1)}(g_{ik}g_{jh}-g_{ih}g_{jk}-F_{ik}F_{jh}+F_{ih}F_{jk})\Big\}=\\
&=R_{ijhk}-\frac{1}{4(n-1)}(g_{ik}\rho_{jh}+g_{jh}\rho_{ik}-g_{ih}\rho_{jk}-g_{jk}\rho_{ih}-\\
&\qquad-F_{ik}\widetilde{\rho}_{jh}-F_{jh}\widetilde{\rho}_{ik}+F_{ih}\widetilde{\rho}_{jk}+F_{jk}\widetilde{\rho}_{ih})+\\
&\qquad+\frac{\varkappa}{4(n-1)(2n-1)}(g_{ik}g_{jh}-g_{ih}g_{jk}-F_{ik}F_{jh}+F_{ih}F_{jk}).
\end{aligned}$$

This means that the tensor

$$\begin{aligned}
\overset{2}{B}_{ijhk}=R_{ijhk}&-\frac{1}{4(n-1)}(g_{ik}\rho_{jh}+g_{jh}\rho_{ik}-g_{ih}\rho_{jk}-g_{jk}\rho_{ih}-\\
&-F_{ik}\widetilde{\rho}_{jh}-F_{jh}\widetilde{\rho}_{ik}+F_{ih}\widetilde{\rho}_{jk}+F_{jk}\widetilde{\rho}_{ih})+\\
&+\frac{\varkappa}{4(n-1)(2n-1)}(g_{ik}g_{jh}-g_{ih}g_{jk}-F_{ik}F_{jh}+F_{ih}F_{jk})
\end{aligned} \tag{3.8}$$

satisfies the condition

$$e^{-2f}\ \overset{2}{\overline{B}}_{ijhk}=\overset{2}{B}_{ijhk}\ .$$

But, if instead of (2.4) and (2.5), we use (2.7) and (2.8), we obtain

$$e^{-2f}\ \overset{3}{\overline{B}}_{ijhk}=\overset{3}{B}_{ijhk}$$

where

$$\overset{3}{B}_{ijhk} = R_{ijhk} + \frac{1}{4}\big(g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih} - \\ -F_{ik}\widetilde{\rho}_{jh} - F_{jh}\widetilde{\rho}_{ik} + F_{ih}\widetilde{\rho}_{jk} + F_{jk}\widetilde{\rho}_{ih}\big) + \\ +\frac{\widetilde{\varkappa}}{4(n-1)}(g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih}). \tag{3.9}$$

If we put

$$T_{ijhk} = g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih} - \\ -F_{ik}\widetilde{\rho}_{jh} - F_{jh}\widetilde{\rho}_{ik} + F_{ih}\widetilde{\rho}_{jk} + F_{jk}\widetilde{\rho}_{ih},$$

$$\overset{1}{\pi}_{ijhk} = g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk},$$

$$\overset{2}{\pi}_{ijhk} = g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih},$$

we have

$$\overset{1}{B} = R - \frac{1}{2(n-2)}T + \frac{\varkappa}{4(n-1)(n-2)}\overset{1}{\pi} - \frac{\widetilde{\varkappa}}{4(n-1)(n-2)}\overset{2}{\pi},$$

$$\overset{2}{B} = R - \frac{1}{4(n-1)}T + \frac{\varkappa}{4(n-1)(2n-1)}\overset{1}{\pi},$$

$$\overset{3}{B} = R + \frac{1}{4}T + \frac{\widetilde{\varkappa}}{4(n-1)}\overset{2}{\pi}.$$

Thus, we have

$$\overset{1}{B} = \frac{1}{n-2}[(n-1)\overset{2}{B} - \overset{3}{B}] + \frac{n}{4(n-1)(n-2)(2n-1)}\varkappa\overset{1}{\pi}. \tag{3.10}$$

We note that the relations (2.3) and (3.3) yield

$$\frac{e^{2f}\overline{\varkappa} - \varkappa}{2(2n-1)} = \frac{e^{2f}\overline{\varkappa} - \varkappa}{2(n-1)},$$

wherefrom

$$e^{2f}\overline{\varkappa} = \varkappa,$$

and therefore

$$e^{-2f}\overline{\varkappa}\overset{1}{\overline{\pi}} = \varkappa\overset{1}{\pi}.$$

Thus, we can state

Theorem 2. *For an anti-Kähler manifold, each of tensors* (3.6), (3.8), (3.9) *and* $\varkappa\overset{1}{\pi}$ *is algebraic curvature tensor and satisfies*

(a) *the anti-Kähler condition of type* (1.5), *and*

(b) *conformal condition of type* (3.7). *These tensors are mutually related such that condition* (3.10) *holds.*

4. The class of proper follower connections and its invariant

On an anti-Kähler space, we shall consider a class of connections given by their coefficients

$$\Gamma^i_{jh} = \{^{\,i}_{jh}\} - p_j\delta^i_h + p^i g_{jh} + q_j J^i_h - q^i F_{jh}, \tag{4.1}$$

where vector (p_i) is its generator and vector (q_i) is generator's image by the structure, i. e. $q_i = p_a J^a_i$. We shall call such kind of connection a **proper follower connection**, because it is a metric J-connection, but, of course, non-symmetric.

The components of Riemannian curvature tensor of proper follower connection are given by

$$\begin{aligned} M_{ijhk} = R_{ijhk} - g_{ih}p_{kj} + g_{ik}p_{hj} - g_{jk}p_{hi} + g_{jh}p_{ki} + \\ + F_{ih}q_{kj} - F_{ik}q_{hj} + F_{jk}q_{hi} - F_{jh}q_{ki}, \end{aligned} \tag{4.2}$$

where R_{ijhk} is a component of curvature tensor of Levi-Civita connection and

$$p_{kj} = \nabla_k p_j + p_k p_j - q_k q_j + \frac{1}{2}p_s q^s F_{kj} - \frac{1}{2}p_s p^s g_{kj}, \tag{4.3}$$

$$q_{kj} = \nabla_k q_j + p_k q_j + q_k p_j - \frac{1}{2}p_s p^s F_{kj} - \frac{1}{2}p_s q^s g_{kj}. \tag{4.4}$$

First, we can notice that $q_{kj} = p_{ka}J^a_j$. Second, we can notice that the component of the tensor (4.2) is skew-symmetric in first two indices. Now we want it to be invariant under changing places of first and second pair of indices, i.e. we want it to be an algebraic curvature tensor. Then we obtain that there holds

$$\begin{aligned} g_{ih}(p_{jk} - p_{kj}) - g_{ik}(p_{jh} - p_{hj}) + g_{kj}(p_{ih} - p_{hi}) - g_{jh}(p_{ik} - p_{ki}) + \\ + F_{ih}(q_{kj} - q_{jk}) - F_{ik}(q_{hj} - q_{jh}) + F_{jk}(q_{hi} - q_{ih}) - F_{jh}(q_{ki} - q_{ik}) = 0. \end{aligned} \tag{4.5}$$

After transvection by g^{ik}, we obtain

$$(2n-3)(p_{kj} - p_{jk}) - J^a_j J^b_k(p_{ba} - p_{ab}) = 0. \tag{4.6}$$

Applying (4.6) once again, we obtain that there holds

$$p_{jk} - p_{kj} = \frac{1}{(2n-3)^2}(p_{jk} - p_{kj}). \tag{4.7}$$

If we exclude the cases of low dimensions ($2n = 2, 2n = 4$ [8]), we obtain that the tensor p_{jk} is symmetric:

$$p_{jk} = p_{kj}.$$

Then, using (4.5), we can easily obtain, after transvection by F^{ih}, that the tensor q_{jk} is also symmetric:

$$q_{jk} = q_{kj}.$$

We are going to calculate a curvature-like invariant of such a connection, like it has been done in [6], [7].

As both p_{kj} and q_{kj} are consisting of five addends and the sum of last four of them is symmetric in both cases, then there holds

$$\nabla_k p_j = \nabla_j p_k; \quad \nabla_k q_j = \nabla_j q_k, \tag{4.8}$$

what means that both the generator and its image by the structure are gradients. We can also notice that Riemannian curvature tensor of proper follower connection satisfies the first Bianchi identity

$$M_{ijhk} + M_{ihkj} + M_{ikjh} = 0$$

and, moreover, it satisfies the anti-Kähler condition (1.5).

Now we shall transvect (4.2) by g^{ik} and obtain

$$\mu_{jh} = \rho_{jh} + 2(n-2)p_{hj} + g_{jh}p^s_s - F_{jh}q^s_s,$$

where μ_{jh} is a component of the Ricci tensor of the proper follower connection. Then, after another transvection of upper equality by g^{jh}, we obtain

$$p^s_s = \frac{\mu - \varkappa}{4(n-1)}, \tag{4.9}$$

where μ is scalar curvature of the connection (4.1).

If we transvect the upper equality by F^{jk}, we obtain that there holds

$$q^s_s = \frac{\widetilde{\mu} - \widetilde{\varkappa}}{4(n-1)}, \tag{4.10}$$

where $\widetilde{\mu}$ stands for $\mu_{aj}F^{aj}$. Now we shall obtain that there holds

$$p_{hj} = \frac{1}{2(n-2)}\left[\mu_{hj} - \rho_{hj} - \frac{\mu - \varkappa}{4(n-1)}g_{hj} + \frac{\widetilde{\mu} - \widetilde{\varkappa}}{4(n-1)}F_{hj}\right]. \tag{4.11}$$

If we transvect (4.10) by F^{ik}, we obtain

$$\widetilde{\mu}_{jh} = \widetilde{\rho}_{jh} + 2(n-2)q_{jh} + g_{jh}q^s_s - F_{jh}p^s_s, \tag{4.12}$$

where $\widetilde{\mu}_{jh} = M_{ijhk}J^i_a g^{ak} = M_{ijhk}F^{ik}$ and $\widetilde{\rho}_{jh}$ is given by (1.3). Then, we obtain

$$q_{hj} = \frac{1}{2(n-2)}\left[\widetilde{\mu}_{hj} - \widetilde{\rho}_{hj} - \frac{\widetilde{\mu} - \widetilde{\varkappa}}{4(n-1)}g_{hj} - \frac{\mu - \varkappa}{4(n-1)}F_{hj}\right]. \tag{4.13}$$

But, using the relationship between p_{hj} and q_{hj}, we can also obtain that there holds

$$q_{hj} = p_{ha}F^a_j = \tag{4.14}$$

$$= \frac{1}{2(n-2)}\left[\mu_{ha}J^a_j - \rho_{ha}J^a_j - \frac{\mu - \varkappa}{4(n-1)}F_{hj} - \frac{\widetilde{\mu} - \widetilde{\varkappa}}{4(n-1)}g_{hj}\right]. \tag{4.15}$$

Using (1.6), we have that the first member in parentheses of (4.13) is equal to the first member in parentheses of (4.14). So, we shall use (4.13) rather the (4.14). From (4.12) and (4.13), we obtain that there holds

$$\begin{aligned} R_{ijkh} &- \frac{1}{2(n-2)}(g_{ik}\rho_{jh} + g_{jk}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih} - \\ &- F_{ik}\widetilde{\rho}_{jh} - F_{jh}\widetilde{\rho}_{ik} + F_{ih}\widetilde{\rho}_{jk} + F_{jk}\widetilde{\rho}_{ih}) + \\ &+ \frac{\varkappa}{4(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk}) - \\ &- \frac{\widetilde{\varkappa}}{4(n-1)(n-2)}(g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih}) = \\ = M_{ijkh} &- \frac{1}{2(n-2)}(g_{ik}\mu_{jh} + g_{jk}\mu_{ik} - g_{ih}\mu_{jk} - g_{jk}\mu_{ih} - \\ &- F_{ik}\widetilde{\mu}_{jh} - F_{jh}\widetilde{\mu}_{ik} + F_{ih}\widetilde{\mu}_{jk} + F_{jk}\widetilde{\mu}_{ih}) + \\ &+ \frac{\mu}{4(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk}) - \end{aligned} \tag{4.16}$$

$$- \frac{\widetilde{\mu}}{4(n-1)(n-2)}(g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih}).$$

So, we have proved that there holds:

Theorem 3. *On an anti-Kähler manifold* $(\mathcal{M}, g, J), \dim \mathcal{M} \neq 2, 4$, *if the generator of a proper follower connection (4.1) is a gradient, then the tensor on the right-hand side of* (4.15) *is independent on the choice of generator and is equal to conformal curvature invariant* $\overset{1}{B}$ *given by* (3.6).

5. The class of antiholomorphically projective connections and its invariant

On the same anti-Kähler space, we shall consider another class of connections, given by its coefficients

$$\Gamma^i_{jh} = \{^{\,i}_{jh}\} + p_j\delta^i_h + p^i g_{jh} - q_j J^i_h - q^i F_{jh}, \tag{5.1}$$

$(q_j = p_a J^a_j)$ which we call an antiholomorphically projective connection by the reason of evident similarity to holomorphically projective connection. Its coefficients look like coefficients of holomorphically projective connection ([9]), but with opposite signs on the structure image side. An antiholomorphically projective connection is a J-connection, nonsymmetric, but not a metric one.

We shall calculate the components of curvature tensor of such a connection. After lowering its upper index, we obtain that there holds

$$\begin{aligned} N_{ijhk} = R_{ijhk} + g_{ih}p_{kj} - g_{ik}p_{hj} - g_{jk}\overline{p}_{hi} + g_{jh}\overline{p}_{ki} - \\ - F_{ih}q_{kj} + F_{ik}q_{hj} + F_{jk}\overline{q}_{hi} - F_{jh}\overline{q}_{ki}. \end{aligned} \tag{5.2}$$

The meaning of upper abbreviations is

$$\begin{aligned} p_{kj} &= \nabla_k p_j - p_k p_j + q_k q_j + \frac{1}{2}p_s q^s F_{kj} - \frac{1}{2}p_s p^s g_{kj}; \\ \overline{p}_{kj} &= \nabla_k p_j + p_k p_j - q_k q_j - \frac{1}{2}p_s q^s F_{kj} + \frac{1}{2}p_s p^s g_{kj}; \\ q_{kj} &= \nabla_k q_j - p_j q_k - q_j p_k - \frac{1}{2}p_s q^s g_{kj} - \frac{1}{2}p_s p^s F_{kj}; \\ \overline{q}_{kj} &= \nabla_k q_j + p_j q_k + q_j p_k + \frac{1}{2}p_s q^s g_{kj} + \frac{1}{2}p_s p^s F_{kj}. \end{aligned} \tag{5.3}$$

There also hold the following relationships:

$$p_{ka}J^a_j = q_{kj}; \quad \overline{p}_{ka}J^a_j = \overline{q}_{kj}; \tag{5.4}$$

$$p_{kj} = \nabla_k p_j + S_{kj}; \quad \overline{p}_{kj} = \nabla_k p_j - S_{kj}; \tag{5.5}$$

and, consequently,

$$q_{kj} = \nabla_k q_j + S_{ka}J^a_j; \quad \overline{q}_{kj} = \nabla_k q_j - S_{ka}J^a_j. \tag{5.6}$$

Also, it is obvious from (5.3) that both tensors S_{kj} and $S_{ka}J^a_j$ are symmetric.

If we want the curvature tensor (5.2) to be an algebraic curvature tensor, i.e. to be skew-symmetric in first two indices, the necessary condition shall be satisfied

$$\begin{aligned} 0 = g_{ih}(p_{kj} + \overline{p}_{kj}) - g_{ik}(p_{hj} + \overline{p}_{hj}) + g_{jh}(p_{ki} + \overline{p}_{ki}) - g_{jk}(p_{hi} + \overline{p}_{hi}) + \\ + F_{ik}(q_{hj} + \overline{q}_{hj}) - F_{ih}(q_{kj} + \overline{q}_{kj}) + F_{jk}(q_{hi} + \overline{q}_{hi}) - F_{jh}(q_{ki} + \overline{q}_{ki}) \end{aligned}$$

and, taking into account (5.3):

$$\begin{aligned} 0 = g_{ih}\nabla_k p_j - g_{ik}\nabla_h p_j + g_{jh}\nabla_k p_i - g_{jk}\nabla_h p_i + \\ + F_{ik}\nabla_h q_j - F_{ih}\nabla_k q_j + F_{jk}\nabla_h q_i - F_{jh}\nabla_k q_i. \end{aligned} \tag{5.7}$$

We shall suppose that the curvature tensor of antiholomorphically projective connection is skew-symmetric in first two indices and that, besides, its generator (p_i) is a gradient. Then

$$\nabla_k p_j = \nabla_j p_k. \tag{5.8}$$

If we transvect (5.7) with F^{ih}, we obtain

$$(2n+1)\nabla_k q_j - \nabla_j q_k = g_{jk}F^{ih}\nabla_h p_i + F_{jk}\nabla_s p^s. \tag{5.9}$$

As the right-hand side of (5.9) is symmetric, its left-hand side will also be symmetric and, consequently

$$\nabla_k q_j = \nabla_j q_k \tag{5.10}$$

and the generator's image by the structure will also be a gradient.

From (5.9), we can see that

$$\nabla_k q_j = \alpha F_{kj} + \beta g_{kj} \tag{5.11}$$

and, as a consequence,

$$\nabla_k p_i = \alpha g_{ki} - \beta F_{ki}. \tag{5.12}$$

For the tensor S_{kj} ((5.3), (5.5)) there holds

$$S_s^s = -(n+2)p_s p^s; \quad S_{kj}F^{kj} = -(n+2)p_s q^s. \tag{5.13}$$

We want the tensor (5.2) to be invariant under changing places of first and second pair of indices. Then, using (5.5) and (5.6), we obtain that there holds

$$g_{jh}(p_{ki} + \overline{p}_{ki}) - g_{ik}(p_{hj} + \overline{p}_{hj}) = F_{jh}(q_{ki} + \overline{q}_{ki}) - F_{ik}(q_{hj} + \overline{q}_{hj})$$

or

$$g_{jh}\nabla_k p_i - g_{ik}\nabla_h p_j = F_{jh}\nabla_k q_i - F_{ik}\nabla_h q_j.$$

After transvection of the last equality by g^{jk}, we obtain

$$\alpha = \frac{1}{2n}\nabla_s p^s, \quad \beta = \frac{1}{2n}\nabla_s q^s, \tag{5.14}$$

for scalar functions appearing in (5.11) and (5.12).

Now we are going to construct a curvature-like invariant of that class of antiholomorphically projective connections. Transvecting (5.2) by g^{ik}, we obtain

$$\nu_{jh} = \rho_{jh} + 2(1-n)p_{hj} - 2\overline{p}_{hj} + g_{jh}\overline{p}_s^s - F_{jk}\overline{q}_s^s, \tag{5.15}$$

where ν_{jh} denotes the component of the Ricci tensor of antiholomorphically projecive connection. After transvection of (5.15) by g^{jk} and using (5.4), (5.5) and (5.13), we obtain

$$p_s p^s = \frac{\nu - \varkappa}{4(n-1)(n+2)}, \tag{5.16}$$

where ν stands for the scalar curvature of (5.1).

In fully analogous way, we can find, transvecting (5.2) by F^{ik} first and then by g^{jh} that there holds

$$p_s q^s = \frac{\widetilde{\nu} - \widetilde{\varkappa}}{4(n-1)(n+2)}, \tag{5.17}$$

where $\widetilde{\nu} = N_{ijhk}F^{ik}g^{jh}$. If we transvect (5.2) by $F^{ik}F^{jh}$, we obtain that

$$p_s p^s = \frac{\widetilde{\widetilde{\varkappa}} - \widetilde{\widetilde{\nu}}}{4(n-1)(n+2)}, \tag{5.18}$$

where $\widetilde{\widetilde{\varkappa}} = \widetilde{\rho}_{hj}F^{hj}$ and $\widetilde{\widetilde{\nu}} = N_{ijhk}F^{ik}F^{jh}$. From (5.16) and (5.18), we obtain that there holds

Lemma 1. *On an anti-Kähler space, for the scalar curvature ν and scalar function $\widetilde{\widetilde{\nu}} = N_{ijhk}F^{ik}F^{jh}$ of a class of antiholomorphically projective connections with gradient generator and curvature tensor which is invariant under changing places of the first and second pair of indices, there holds*

$$\nu + \widetilde{\widetilde{\nu}} = \varkappa + \widetilde{\widetilde{\varkappa}} \tag{5.19}$$

where $\varkappa$ and $\widetilde{\widetilde{\varkappa}}$ are the same quantities depending on Levi-Civita connection.

We can also notice that there hold

$$\begin{aligned} p^s_s &= 2n\alpha - (n+2)p_s p^s;\ \overline{p}^s_s = 2n\alpha + (n+2)p_s p^s, \\ q^s_s &= 2n\beta - (n+2)p_s q^s;\ \overline{q}^s_s = 2n\beta + (n+2)p_s q^s. \end{aligned} \tag{5.20}$$

From (5.3), (5.11) and (5.15), we obtain

$$\begin{aligned} \nu_{jh} = \rho_{jh} &+ 2(1-n)(\alpha g_{jh} - \beta F_{jh} + S_{jh}) - 2(\alpha g_{jh} - \beta F_{jh} + S_{jh}) + \\ &+ g_{jh}(2n\alpha - S^s_s) - F_{jk}(2n\beta - S_{ab}F^{ab}) = \\ = \rho_{jh} &+ 2(2-n)S_{jh} + (n+2)p_s p^s g_{jh} - (n+2)p_s q^s F_{jh}. \end{aligned}$$

From the upper equality, we obtain that there holds

$$S_{jh} = \frac{\rho_{jh} - \nu_{jh}}{2(n-2)} + \frac{1}{8(n-1)(n-2)}\big[(\nu - \varkappa)g_{jk} - (\widetilde{\nu} - \widetilde{\varkappa})F_{jh}\big], \tag{5.21}$$

using (5.16) and (5.17). We shall also need the tensor

$$\begin{aligned} S_{ha}J^a_j = \frac{\rho_{ha} - \nu_{ha}}{2(n-2)}J^a_j &+ \\ + \frac{1}{8(n-4)(n-2)}&[(\nu - \varkappa)F_{hj} + (\widetilde{\nu} - \widetilde{\varkappa})g_{hj}]. \end{aligned} \tag{5.22}$$

We can conclude from (5.22), as $\rho_{ha}J_j^a$ is symmetric, that $\nu_{ha}J_j^a$ is also symmetric Besides, it is easy to verify that this class of antiholomorphically projective connections satisfies the first Bianchi identity, anti-Kähler condition (1.5) and also conditions which are analogous to (1.7) and (1.8).

Substituting (5.11) and (5.12) into (5.2), we obtain that there holds

$$(5.23)\qquad \begin{aligned} N_{ijhk} = R_{ijhk} + g_{ih}S_{jk} - g_{ik}S_{jh} + g_{jk}S_{hi} - g_{jk}S_{ki} + \\ +F_{ik}S_{ha}J_j^a - F_{ih}S_{ka}J_j^a + F_{jh}S_{ka}J_i^a - F_{jk}S_{ha}J_i^a. \end{aligned}$$

Substituting (5.21) and (5.22) into (5.23), we obtain that there holds

$$(5.24)\qquad \begin{aligned} & R_{ijkh} - \frac{1}{2(n-2)}(g_{ik}\rho_{jh} + g_{jk}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih} \\ & -F_{ik}\widetilde{\rho}_{jh} - F_{jh}\widetilde{\rho}_{ik} + F_{ih}\widetilde{\rho}_{jk} + F_{jk}\widetilde{\rho}_{ih}) + \\ & +\frac{\varkappa}{4(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk}) \\ & -\frac{\widetilde{\varkappa}}{4(n-1)(n-2)}(g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih}) \\ & = N_{ijkh} - \frac{1}{2(n-2)}(g_{ik}\nu_{jh} + g_{jk}\nu_{ik} - g_{ih}\nu_{jk} - g_{jk}\nu_{ih} \\ & -F_{ik}\widetilde{\nu}_{jh} - F_{jh}\widetilde{\nu}_{ik} + F_{ih}\widetilde{\nu}_{jk} + F_{jk}\widetilde{\nu}_{ih}) \\ & +\frac{\nu}{4(n-1)(n-2)}(g_{ik}g_{jh} - g_{ih}g_{jk} - F_{ik}F_{jh} + F_{ih}F_{jk} \\ & -\frac{\widetilde{\nu}}{4(n-1)(n-2)}(g_{ik}F_{jh} + g_{jh}F_{ik} - g_{ih}F_{jk} - g_{jk}F_{ih}). \end{aligned}$$

So, we have proved that there holds

Theorem 4. *On an anti-Kähler space, if the generator of an antiholomorphically projective connection is a gradient and if its curvature tensor is an algebraic curvature tensor, then the tensor on the right-hand side of* (5.24) *is independent on the choice of generator. Moreover, it is equal to the tensor* (3.6)*, which is a conformal invariant of considered anti-Kähler space and also to the invariant of considered class of proper follower connections* (4.1)*.*

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