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ON FREDHOLM'S THEOREM OF THE ALTERNATIVE AND A COROLLARY OF ROHN'S RESIDUAL EXISTENCE THEOREM

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Abstract: By Fredholm's Theorem of the alternative, the system $\boldsymbol{Ax} = \boldsymbol{b}$ of linear equations has no solution if and only if $\boldsymbol{u}^T\boldsymbol{A} = \boldsymbol{o}^T$ and $\boldsymbol{u}^T\boldsymbol{b} \neq 0$ for some $\boldsymbol{u} \in \mathbb{R}^m$. Recently, Rohn proved as a corollary of the Residual Existence Theorem for linear equations [*Optim. Lett.* 4 (2010), 287–292] that the system $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if and only if the residual set $\{\, \boldsymbol{Ax} - \boldsymbol{b} : \boldsymbol{x} \in \mathbb{R}^n \,\}$ intersects all the orthants of $\mathbb{R}^m$. We study the relation between both the results in the more general setting of a vector space over a linearly ordered (possibly skew) field, obtain a new proof of the corollary, and give a generalisation of Fredholm's Theorem of the alternative.

1. Introduction

Jiří Rohn proved the Residual Existence Theorem for linear equations [7, Th. 2]. Then, as a corollary of it, he proved the next result [7, Th. 3]:

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Theorem 1 (Rohn's corollary in $\mathbb{R}^n$)**.** *Let a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$ be given. Then the system*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

is solvable if and only if the residual set

$$\{\, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} : \boldsymbol{x} \in \mathbb{R}^n \,\} \tag{2}$$

intersects all the orthants of $\mathbb{R}^m$.

Nevertheless, we also know Fredholm's Theorem of the alternative, cf. [5, Prop. at the end of Subsec. 9 (in § 2)].

Theorem 2 (Fredholm's Theorem in $\mathbb{R}^n$)**.** *Let a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$ be given. Then the system*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{3}$$

has no solution if and only if

$$\boldsymbol{u}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{o}^{T} \qquad \textit{and} \qquad \boldsymbol{u}^{\mathrm{T}}\boldsymbol{b} \neq 0 \tag{4}$$

for some $\boldsymbol{u} \in \mathbb{R}^m$.

Comparing the two results, Fredholm's Theorem 2 having been known, Th. 1 due to Rohn turns out to be interesting. It is natural to ask whether there is (and what is) the relation between both the results.

Both Th. 1 and Th. 2 are stated in the setting of the real finite-dimensional vector space $\mathbb{R}^n$. However, we know a generalisation of Fredholm's Theorem [1, Th. 3.2], see Th. 3 below. The generalisation is stated in a vector space of any dimension over any (possibly skew) field. An according generalisation of Rohn's Residual Existence Theorem [7, Th. 2], of which Th. 1 is a corollary, can be found in [2, Th. 5], see Th. 4 below. That is why, we shall study the relation between the results in the more general setting of a vector space over a linearly ordered (possibly skew) field, which we are now going to introduce.

2. Basic concepts, notation, and Fredholm's Theorem of the alternative

Let F be a field, which may be either commutative or skew. The latter case means the field is not commutative. In addition, let "$\leq$" be a binary relation on the field F such that, for all $\lambda, \mu \in F$, the next five statement hold true: First, we have $\lambda \geq 0$ or $\lambda \leq 0$. Second, if $\lambda \geq 0$ and $\lambda \leq 0$, then $\lambda = 0$. Third, if $\lambda \geq 0$ and $\mu \geq 0$, then $\lambda + \mu \geq 0$. Fourth, if $\lambda \geq 0$ and $\mu \geq 0$, then $\lambda \cdot \mu \geq 0$. Fifth, it holds $\lambda \leq \mu$ if and only if $\lambda - \mu \leq 0$. In the statements, we have used the usual convention that $\lambda \geq \mu$ means $\mu \leq \lambda$ for any $\lambda, \mu \in F$. Then F is a *linearly ordered (possibly skew) field*.

The field of the real numbers $\mathbb{R}$ or that of the rational numbers $\mathbb{Q}$ are examples of linearly ordered commutative fields. An example of a linearly ordered skew field was given as early as in 1901 by David Hilbert, see [3, Notes and comments to Ch. 1, p. 45, with Sec. 2.1 and Sec. 2.3, pp. 47–50 and 66] and [6, Ex. 1.7, p. 10, and above Prop. 18.7, p. 288].

The additive group of the field F can be seen as a vector space over the field F. Choose an element or vector $u \in F$. By ιu, i.e., by prepending the Greek letter iota, we shall denote the *right homothety of F given by the element u*. It is the mapping $\iota u \colon F \to F$ with $\iota u \colon b \mapsto \iota u(b) = b \cdot u$ for each $b \in F$.

Let m be a non-negative natural number. Consider the vector space F^m of column vectors. Choose a column vector $\boldsymbol{u} \in F^m$. Transposing it, we obtain the row $\boldsymbol{u}^T$. Then we can "multiply" it by the symbol "ι" from the right. Given another column vector $\boldsymbol{b} \in F^m$, we can multiply thus:

$$\begin{aligned} \iota \boldsymbol{u}^T \boldsymbol{b} = \iota \begin{pmatrix} u_1 & \dots & u_m \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \\ = \iota u_1 b_1 + \dots + \iota u_m b_m = \\ = b_1 \cdot u_1 + \dots + b_m \cdot u_m \,, \end{aligned}$$

where $u_1, \dots, u_m$ and $b_1, \dots, b_m$ are the components of the vectors $\boldsymbol{u}$ and $\boldsymbol{b}$, respectively.

Now, let W be a vector space over the field F. Let $\alpha \colon W \to F$ be a linear form. Note that, for any element or vector $u \in F$, the mapping $\iota u \colon F \to F$ with $\iota u \colon b \mapsto \iota u(b)$ for all $b \in F$ is linear. Hence, we can compose the form $\alpha \colon W \to F$ with $\iota u \colon F \to F$. We shall denote the composed mapping as $\iota u \alpha$. We have $\iota u \alpha(x) = \iota u\big(\alpha(x)\big) = \big(\alpha(x)\big) \cdot u$ for all $x \in W$.

More generally, let $A \colon W \to F^m$ be a linear mapping. To a point $x \in W$, the mapping assigns the column vector Ax of the m components

$\alpha_1(x), \dots, \alpha_m(x)$. Each $\alpha_i\colon W \to F$ is a linear form for $i = 1, \dots, m$. We note that, for a column vector $\boldsymbol{u} \in F^m$, the mapping $\iota\boldsymbol{u}^T\colon F^m \to F$ with $\iota\boldsymbol{u}^T\colon \boldsymbol{b} \mapsto \iota\boldsymbol{u}^T\boldsymbol{b}$ for all $\boldsymbol{b} \in F^m$ is linear. Then $\iota\boldsymbol{u}^T A$ is the composition of the mapping $A\colon W \to F^m$ with $\iota\boldsymbol{u}^T\colon F^m \to F$. We have $\iota\boldsymbol{u}^T Ax = = \iota u_1\alpha_1(x) + \dots + \iota u_m\alpha_m(x)$, i.e. the sum of the mappings $\iota u_i\alpha_i$ evaluated at x.

Now, having introduced the notation, we recall the following generalisation of Fredholm's Theorem 2, which can be found in [1, Th. 3.2]. Note that the field F need not be linearly ordered.

Theorem 3 (Fredholm's Theorem)**.** *Let $A\colon W \to F^m$ be a linear mapping where W is a vector space over a (possibly skew) field F and let $\boldsymbol{b} \in F^m$ be a vector. Then the system*

$$Ax = \boldsymbol{b} \tag{5}$$

has no solution if and only if

$$\iota\boldsymbol{u}^T A = o \qquad \text{and} \qquad \iota\boldsymbol{u}^T\boldsymbol{b} \neq 0 \tag{6}$$

for some $\boldsymbol{u} \in F^m$, where o is the zero linear form on W.

Fredholm's Theorem 2 is a special case of Fredholm's Theorem 3 when F is the field of the real numbers, $F = \mathbb{R}$, and W is finite-dimensional, $W = \mathbb{R}^n$.

3. Further concepts, notation, and Rohn's Residual Existence Theorem for linear equations

Let W be a vector space over a linearly ordered (possibly skew) field F. Given a set $X \subseteq W$, its convex hull and its (convex) conical hull can be defined in the usual way: The *convex hull* of X is the set $\operatorname{conv} X$ of all non-negative affine combinations of the points of the set X. The *conical hull* of X is the set $\operatorname{cone} X$ of all non-negative linear combinations of the points of X. The *Minkowski sum* of two sets $X, Y \subseteq W$ is the set $X + Y = \{\, x + y : x \in X,\ y \in Y \,\}$.

Let H be another vector space over the linearly ordered field F. The algebraic dual of the space H is the space $H^\#$ of all linear forms $\eta\colon H \to F$. Let H^* be any subspace of $H^\#$ such that, for any non-zero vector $h \in H$, there exists a linear form $\eta \in H^*$ with $\eta(h) \neq 0$. (For example, if H is a real Banach space, then H^* can be its topological dual, which is the space of all continuous linear functionals $\eta\colon H \to \mathbb{R}$.) Then (H, H^*) is a *dual pair of spaces.*

We say that two sets $R, S \subseteq H$ can be *strongly separated* by a hyperplane if there exists a linear form $\eta \in H^*$ and constants $\sigma, \tau \in F$ such that $\eta(r) \leq \sigma < \tau \leq \eta(s)$ for all $r \in R$ and $s \in S$. In particular, a set $R \subseteq H$ and a point $b \in H$ can be *strongly separated* by a hyperplane if there exists a linear form $\eta \in H^*$ and a constant $\tau \in F$ such that $\eta(r) \leq \tau < \eta(b)$ for all $r \in R$.

Having introduced the concepts, we recall the following generalisation of Rohn's Residual Existence Theorem [7, Th. 2], which can be found in [2, Th. 5].

Theorem 4 (Rohn's Residual Existence Theorem)**.** *Let W and (H, H^*) be a vector space and a dual pair of spaces, respectively, over a linearly ordered (possibly skew) field F. Given a linear mapping $A\colon W \to H$, a point $b \in H$, and finite subsets $X = \{x_1, \ldots, x_r\} \subseteq W$ and $Y = \{y_1, \ldots, y_s\} \subseteq$ $\subseteq W$, the linear equation*

$$Ax = b \tag{7}$$

has a solution in the set $\operatorname{conv} X + \operatorname{cone} Y$ *if and only if*

$$\eta(Ay_1), \ldots, \eta(Ay_s) \leq 0 \qquad \textit{implies} \qquad \max_{x_i \in X} \eta(Ax_i - b) \geq 0 \tag{8}$$

for all $\eta \in H^*$, which holds if and only if the set $A(\operatorname{conv} X + \operatorname{cone} Y)$ *and the point b cannot be strongly separated by a hyperplane.*

When F is the field of the real numbers, $F = \mathbb{R}$, both the spaces W and H are of finite dimension, $W = \mathbb{R}^n$ and $H = \mathbb{R}^m$, and the set Y is empty, $Y = \emptyset$, we obtain Rohn's result [7, Th. 2] as a special case of Th. 4.

The special choice $H = F^m$, whence we must have $H^* = H^{\#}$, will be significant in the following section.

4. The corollary of Rohn's Residual Existence Theorem for linear equations

Repeating Rohn's proof, we can establish a generalisation of his corollary [7, Th. 3] of the Residual Existence Theorem for linear equations in our setting.

Theorem 5 (Rohn's corollary)**.** *Let $A\colon W \to F^m$ be a linear mapping where W is a vector space over a linearly ordered (possibly skew) field F. Given a vector $\boldsymbol{b} \in F^m$, the system*

$$Ax = \boldsymbol{b} \tag{9}$$

is solvable if and only if the residual set

$$\{\, Ax - \boldsymbol{b} : x \in W \,\} \tag{10}$$

intersects all the orthants of the space F^m.

Proof. (Cf. [7, Th. 3].) The "only if" part is obvious. If $x \in W$ solves $Ax = \boldsymbol{b}$, then the origin $Ax - \boldsymbol{b} = \boldsymbol{o}$ lies in all the orthants of the space F^m.

It remains to prove the "if" part. Let $\mathcal{O}$ be the set of the 2^m orthants of the space F^m. We assume that, for each orthant $O \in \mathcal{O}$, there exists an $x_O \in W$ such that $Ax_O - \boldsymbol{b} \in O$. We shall apply Rohn's Residual Existence Theorem 4 with $H = F^m$ and $H^* = H^\#$, and with $X = \{x_O : O \in \mathcal{O}\}$ and $Y = \emptyset$.

Note that each $\boldsymbol{u} \in F^m$ induces a linear form $\eta\colon F^m \to F$ by $\eta(\boldsymbol{h}) = \iota\boldsymbol{u}^T\boldsymbol{h}$ for all $\boldsymbol{h} \in F^m$. Conversely, if $\eta\colon F^m \to F$ is a linear form, then it is induced by the vector $\boldsymbol{u} \in F^m$ with the components $u_i = \eta(\boldsymbol{e}_i)$ for $i = 1, \ldots, m$ where $\boldsymbol{e}_i$ are the canonical unit vectors of F^m.

Observe that, for each $\boldsymbol{u} \in F^m$, there exists an orthant $O \in \mathcal{O}$ with $\boldsymbol{u} \in O$. Moreover, the linear form $\eta\colon F^m \to F$ induced by such a vector $\boldsymbol{u}$ is non-negative on the O. We have $\eta(\boldsymbol{h}) = \iota\boldsymbol{u}^T\boldsymbol{h} \geq 0$ for all $\boldsymbol{h} \in O$. Hence, it follows that $\max_{x_O \in \mathcal{O}} \eta(Ax_O - \boldsymbol{b}) = \max_{x_O \in \mathcal{O}} \iota\boldsymbol{u}^T(Ax_O - \boldsymbol{b}) \geq 0$ for all $\eta \in (F^m)^\#$ or $\boldsymbol{u} \in F^m$, whence the system $Ax = \boldsymbol{b}$ has a solution (in $\operatorname{conv}\{\, x_O : O \in \mathcal{O} \,\}$) by Th. 4. $\Diamond$

Now, we should like to study the relation between Fredholm's Theorem 3 and Th. 5 (Rohn's corollary).

5. The relation between Fredholm's Theorem and the corollary of Rohn's Residual Existence Theorem

Let F be a (possibly skew) field; we shall assume in this section that the field is linearly ordered; and let W be a vector space over F.

Consider a linear mapping $A\colon W \to F^m$ and a vector $\boldsymbol{b} \in F^m$. Let us assume that the system of equations $Ax = \boldsymbol{b}$ has no solution. Then, by Fredholm's Theorem 3, there exists a $\boldsymbol{u} \in F^m$ such that $\iota\boldsymbol{u}^T A = o$ and $\iota\boldsymbol{u}^T\boldsymbol{b} \neq 0$. As the field F is linearly ordered, we may assume wlog that $\iota\boldsymbol{u}^T\boldsymbol{b} > 0$, considering $\boldsymbol{u} := -\boldsymbol{u}$ otherwise. Taking into consideration

- the proof of Th. 5 due to Rohn [7, Th. 3],
- the fact that Rohn's Residual Existence Theorem 4, which is used in the proof, is a separation theorem actually, see [2, Sec. 1],
- the fact that the vector $\boldsymbol{u}$ points inside a certain orthant of F^m, and
- the fact that the vector $\boldsymbol{u}$ is normal to the hyperplane $\{\, \boldsymbol{h} \in F^m : \iota\boldsymbol{u}^T\boldsymbol{h} = \tau \,\}$, where $\tau \in F$, such as $\tau = -\frac{1}{2}\iota\boldsymbol{u}^T\boldsymbol{b}$, is a constant,

we see that the hyperplane strongly separates the residual set $\{\, Ax - \boldsymbol{b} : x \in W \,\}$ and the orthant containing $\boldsymbol{u}$. Hence, the residual set does not intersect the orthant.

Note that we have just given another proof of the "if" part of Th. 5. Its "only if" part is obvious.

Now, the relation between Fredholm's Theorem 3 and Rohn's corollary, Th. 5, is apparent. Both the results say essentially the same. If the system $Ax = \boldsymbol{b}$ has no solution, then the residual set $\{\, Ax - \boldsymbol{b} : x \in W \,\}$ and the origin $\boldsymbol{o}$ of the space F^m can be strongly separated by a hyperplane. (Note that the residual set is a proper affine subspace of F^m. It is easy to see that the separating hyperplane is parallel with it. The hyperplane not passing through the origin $\boldsymbol{o}$, there exists an orthant of F^m which the hyperplane does not intersect. Hence, the hyperplane separates the residual set and that orthant as well.) Conversely, if the residual set and the origin – or even a whole orthant, which contains it – can be strongly separated by a hyperplane, then the system cannot have a solution. Finally, we observe that the residual set $\{\, Ax - \boldsymbol{b} : x \in W \,\}$ and the origin $\boldsymbol{o}$ can be strongly separated by a hyperplane if and only if the subspace $\{\, Ax : x \in W \,\}$ and the point $\boldsymbol{b}$ can be strongly separated by a hyperplane.

The last observation offers a new point of view on Fredholm's Theorem 3 [1, Th. 3.2], consequently on Gale's Theorem [1, Lemma 4.2], and other theorems of the alternative (Motzkin's Theorem [1, Th. 5.1], etc.), cf. [4, Th. 1].

The next theorem summarises the above considerations. It extends Rohn's corollary, Th. 5.

Theorem 6. *Let $A\colon W \to F^m$ be a linear mapping and let $\boldsymbol{b} \in F^m$ be a point where W is a vector space over a linearly ordered (possibly skew) field F. Then the next five statements are equivalent:*

1. *the system* (9) *has no solution,*
2. *the residual set* (10) *does not intersect all the orthants of the space F^m,*
3. *the residual set* (10) *and an orthant of F^m can be strongly separated by a hyperplane,*
4. *the residual set* (10) *and the origin $\boldsymbol{o}$ of F^m can be strongly separated by a hyperplane,*
5. *the range $\{ Ax : x \in W \}$ of the mapping A and the point $\boldsymbol{b}$ can be strongly separated by a hyperplane.*

6. A generalisation of Fredholm's Theorem and concluding remarks

Comparing Rohn's corollary, Th. 5, and Fredholm's Theorem 3, we can see they are similar to each other in a sense. We already know that the corollary can be proved by using Rohn's Residual Existence Theorem 4 or Fredholm's Theorem 3 – see the proof of Th. 5 or Sec. 5, respectively. Turning our attention back to Rohn's Residual Existence Theorem 4, the following generalisation of Fredholm's Theorem 3 is easy to prove. Recall that, when H is a vector space over any (possibly skew) field F, then $H^{\#}$ denotes its algebraic dual, i.e. the space of all linear forms $\eta\colon H \to F$.

Theorem 7 (Fredholm's Theorem)**.** *Let $A\colon W \to H$ be a linear mapping, where W and H are vector spaces over a (possibly skew) field F, and let $b \in H$ be a point. Then the linear equation*

$$Ax = b \tag{11}$$

has no solution if and only if

$$\eta A = o \qquad \text{and} \qquad \eta(b) \neq 0 \tag{12}$$

for some linear form $\eta \in H^{\#}$, where ηA denotes the composition of the mapping A with η and o is the zero linear form on W.

Proof. The "if" part is obvious. Should $Ax = b$ hold for some $x \in W$ and $\eta A = o$ with $\eta(b) \neq 0$ for some $\eta \in H^{\#}$, we would obtain $0 = \eta Ax = = \eta(b) \neq 0$, a contradiction.

We have to prove the "only if" part. Assume that $Ax = b$ has no solution. That is, the point b lies outside the linear subspace $L = = \{ Ax : x \in W \}$, the image or range of the mapping A. It follows hence that L is a proper subspace of the space H. Find a maximal linearly independent set $B \subseteq W \setminus L$. (If we know in advance that the dimension of the space H is finite, $H = F^m$, say, we can find the set B by induction. If the dimension of H is either known to be infinite or not known at all, we shall find the set by using the Axiom of Choice.) Now, each point $h \in H$ can be written in the form $h = h_L + \lambda_1 b_1 + \cdots + \lambda_n b_n$ for a unique $h_L \in L$, a unique natural number n, unique basis elements $b_1, \ldots, b_n \in B$, and unique non-zero scalars $\lambda_1, \ldots, \lambda_n \in F$. So, we can define a linear form $\eta\colon H \to F$ in the following way. We put $\eta(h_L) = 0$ for all $h_L \in L$ and $\eta(b') = 1$, say, for all $b' \in B$. Then, obviously, we have $\eta A = o$ and $\eta(b) \neq 0$, which means we are done. $\Diamond$

Assuming that the field F is linearly ordered and using the reasoning of Sec. 5, we obtain the following analogy of Th. 6.

Theorem 8. *Let W and H be vector spaces over a linearly ordered (possibly skew) field F. Let $A\colon W \to H$ be a linear mapping and let $b \in H$ be a point. Then the next three statements are equivalent:*

1. *the linear equation* (11) *has no solution,*
2. *the residual set $\{ Ax - b : x \in W \}$ and the origin 0 of the space H can be strongly separated by a hyperplane, i.e., there exists an $\eta \in H^{\#}$ such that $\eta(Ax - b) < -\frac{1}{2}\eta(b) < \eta(0)$ for all $x \in W$,*
3. *the range $\{ Ax : x \in W \}$ of the mapping A and the point b can be strongly separated by a hyperplane.*

On the one hand, Fredholm's Theorem 7 and its corollary, Th. 8, are quite general because we do not consider any topology on the vector spaces W and H and because the field F can be any one. On the other hand, unlike Rohn's Residual Existence Theorem 4, we have to work with the whole algebraic dual $H^{\#}$.

Therefore, it is a challenging task to prove an analogy of Fredholm's Theorem 7 according with Rohn's Residual Existence Theorem 4, i.e. an analogy such that the linear form η in (12) can be restricted to be from a subspace H^* of $H^\#$ which makes (H, H^*) be a dual pair of spaces. As noted in Sec. 5, statement 5 of Th. 6, provides a new point of view on Fredholm's Theorem, indicating a way to prove the sought-after analogy. Consequently, we may be able to obtain an according generalisation of Gale's Theorem of the alternative (see [1, Lemma 4.2]), which is fundamental in the proof of the Duality Theorem for linear programming (see [1, Th. 6.3]). Thus, we may expect that we could be able to consider more general problems of linear programming in infinite-dimensional spaces (in an algebraic setting, cf. [1]) and to establish the strong Duality Theorem for them.

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