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A NOTE ON GENERALIZED KENMOTSU MANIFOLDS

T. **Tshikuna-Matamba**

Département de Mathématiques, Institut Supérieur Pédagogique, B.P. 282-Kananga, République Démocratique du Congo

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Abstract: In this paper, we characterize a new class of almost contact metric manifolds in Kenmotsu geometry. One establishes the inclusions relations of the new class with the well-known ones.

1. Introduction

In [9], K. Kenmotsu studied a particular class of almost contact metric manifolds that is neither of Sasakian nor of cosymplectic type. This class is defined by the warped product of the real line with a Kähler manifold.

Following this formalism, many other classes were characterized in [15]. Several authors have produced many papers such as [2], [4], [5], [7], [8], [10], [11], [12], [14], [16] and [17], on what we can call *Kenmotsu Geometry.*

This note intends to characterize a new class obtained as the warped product of the real line with a Hermitian manifold.

The paper is organized as follows:

Sec. 2 is devoted to recall some fundamental notions of almost Hermitian and almost contact metric manifolds to be used in the sequel.

E-mail address: tshikmat@yahoo.fr

At Sec. 3, we characterize the new class and construct an example.

Sec. 4 is concerned with the inclusion relations between this new class and the known ones. In particular, we show that this class contains the class of Kenmotsu manifolds; it is included in the class of semi-Kenmotsu normal; in the class of G_1−semi-Kenmotsu and in the class of G_1-Kenmotsu manifolds.

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2. Almost Hermitian and almost contact metric manifolds

An almost Hermitian manifold is a Riemannian manifold, (M, g), furnished with a tensor field, J, of type $(1, 1)$ satisfying the following two conditions:

(i) $J^2D = -D$, and

(ii) $g(JD, JE) = g(D, E)$, for all $D, E \in \chi(M)$.

It is known that any almost Hermitian manifold, (M, g, J), is of even dimension, say $2m$, and possesses a fundamental 2−form, Ω, defined by $\Omega(D, E) = g(D, JE)$.

Noting by ∇ the Levi-Civita connection of M, we recall some remarkable identities

$(\nabla_D J)E = \nabla_D JE - J\nabla_D E$;

$(\nabla_D \Omega)(E, G) = g(G, (\nabla_D J)E) = -g((\nabla_D J)G, E)$.

Let $\{E_1, ..., E_m, JE_1, ..., JE_m\}$ be a local J-basis of an open subset of M, then the codifferential δ of Ω is defined by

$$\delta\Omega(D) = -\sum_{i=1}^{m} \left\{(\nabla_{E_i}\Omega)(E_i, D) + (\nabla_{JE_i}\Omega)(JE_i, D)\right\}.$$

From the classification of almost Hermitian structures, obtained by Gray and Hervella [6], we shall be interested with the following:

(a) *the Kähler manifold*, defined by $\nabla J = 0$;

(b) *Hermitian* or ($W_3 \oplus W_4$-manifold) if $N_J = 0$, where N_J denotes the Nijenhuis tensor of J or equivalently
$$(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0;$$

(c) G_1-*manifold* if $(\nabla_D\Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E) = 0$;

(d) a G_2-*manifold* or ($W_2 \oplus W_3 \oplus W_4$-manifold) if
$\mathcal{G}\{(\nabla_D\Omega)(E,G) - (\nabla_{JD}\Omega)(JE,G)\} = 0$ or
$\mathcal{G}\{g(N_J(D,E),JG)\} = 0$, where $\mathcal{G}$ denotes the cyclic sum over D, E and G.

An almost contact structure on a differentiable manifold, M, is a triple (φ, ξ, η) where:

(i) ξ is a characteristic vector field,

(ii) η is a differential 1-form such that $\eta(\xi) = 1$, and

(iii) φ is a tensor field of type $(1,1)$ satisfying

$$\varphi^2 D = -D + \eta(D)\xi, \quad \text{for all } D \in \chi(M).$$

If in addition, M admits a Riemannian metric g such that

$$g(\varphi D, \varphi E) = g(D,E) - \eta(D)\eta(E),$$

then g is called a compatible metric. In this case, $(M, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. These manifolds are of odd dimension $2m+1$.

As in the case of almost Hermitian manifolds, the fundamental 2-form, ϕ, of an almost contact metric manifold is defined by $\phi(D,E) = = g(D, \varphi E)$.

Among some remarkable identities we have:

(1) $(\nabla_D\eta)E = g(E, \nabla_D\xi)$;

(2) $2d\eta(D,E) = (\nabla_D\eta)E - (\nabla_E\eta)D$.

$$3d\Omega(D,E,G) = \mathcal{G}\{(\nabla_D\Omega)(E,G)\}.$$

Let $\{E_1, ..., E_m, \varphi E_1, ..., \varphi E_m, \xi\}$ be a local φ-basis of an open subset of M, then the codifferential δ is given by

$$\delta\phi(D) = -\sum_{i=1}^{m}\{(\nabla_{E_i}\phi)(E_i, D) + (\nabla_{\varphi E_i}\phi)(\varphi E_i, D)\} - (\nabla_\xi\phi)(\xi, D);$$

$$\delta\eta = -\sum_{i=1}^{m}\{(\nabla_{E_i}\eta)E_i + (\nabla_{\varphi E_i}\eta)\varphi E_i\}.$$

In [13], S. Sasaki and Y. Hatakeyama have defined two tensors fields $N^{(1)}$ and $N^{(2)}$ of type $(0,2)$ by setting

(a) $N^{(1)}(D,E) = N_\varphi(D,E) + 2d\eta(D,E)\xi$,

(b) $N^{(2)}(D,E) = (\mathcal{L}_{\varphi D}\eta)E - (\mathcal{L}_{\varphi E}\eta)D$

where N_φ is the Nijenhuis tensor of φ while $\mathcal{L}$ denotes the Lie derivative.

Recall that, in [14], the tensor N_φ of a nearly Kenmotsu manifold is obtained by

$$N_\varphi(D,E) = ((\nabla_{\varphi D}\varphi)E - (\nabla_{\varphi E}\varphi)D)) - \varphi(\nabla_D\varphi)E - (\nabla_E\varphi)D)).$$

If $N^{(1)} = 0$, the manifold is said to be normal and in this case $N^{(2)} = 0$.

Let us recall the well known structures in the topic of Kenmotu manifolds. An almost contact metric manifold is said to be:

(1) *almost Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E,G)\right\};$

(2) *Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E,G)\right\}$, $d\eta = 0$ and $N^{(1)} = 0$;

(3) G_1-*Kenmotsu* if $(\nabla_D\phi)(D,E) - (\nabla_{\varphi D}\phi)(\varphi D,E) - \eta(D)\phi(E,D) = 0 = d\eta$;

(4) G_1-*semi-Kenmotsu* if it is G_1-Kenmotsu and $\delta\phi = 0$;

(5) G_2-*Kenmotsu* if $\mathcal{G}\left\{(\nabla_D\phi)(E,G) - (\nabla_{\varphi D}\phi)(\varphi E,G) - \eta(D)\phi(E,G)\right\} = 0 = d\eta$;

(6) G_2-*semi-Kenmotsu* if it is G_2-Kenmotsu and $\delta\phi = 0$;

(7) *nearly Kenmotsu* if $(\nabla_D\varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$;

(8) *semi-Kenmotsu normal* if $(\nabla_D\phi)(E,G) - (\nabla_{\varphi D}\phi)(\varphi E,G) = \eta(E)\phi(G,D)$, $\delta\phi = 0$ and $d\eta = 0$;

(9) *quasi-Kenmotsu* if $(\nabla_D\phi)(E,G) + (\nabla_{\varphi D}\phi)(\varphi E,G) = \eta(E)\phi(G,D) + 2\eta(G)\phi(D,E)$ and $d\eta = 0$;

(10) *almost trans-Kenmotsu* if $\mathcal{G}\left\{(\nabla_D\phi)(E,G) - \frac{1}{m}\phi(D,E)\delta\phi(\varphi G) - 2\eta(D)\phi(E,G)\right\} = 0$ and $d\eta = 0$.

3. Characterization of the new class

Let (M', g', J') be an almost Hermitian manifold and $M = \mathbb{R}\times_f M'$ the warped product [1] where f is the warping function defined by $f(t) = ce^t$ with $c \succ 0$. The vector fields of $\chi(M)$ are $(t\frac{d}{dt}, D')$ where $t \in \mathbb{R}$ and $D' \in \chi(M')$.

On this warped product, the following formulae hold.

$$(\nabla_D\varphi)E = f^2(t)(\nabla'_{D'}J')E' + g(\varphi D,E)\xi - \eta(E)\varphi D. \tag{3.1}$$

$$(\nabla_D\phi)(E,G) = f^2(t)(\nabla'_{D'}\Omega')(E',G') + \eta(G)\phi(D,E) + \eta(E)\phi(G,D). \tag{3.2}$$

$$\delta\phi = f^2(t)\delta\Omega'. \tag{3.3}$$

This warped product is characterized by the terms such as $\eta(.)\phi(.,.)$ which appear in (3.2). Hereafter, we admit that any almost contact metric manifold furnished with this property is a warped product.

Definition 3.1. Let $\bar{M} = \mathbb{R} \times_f M'$ be the warped product of the real line with an almost Hermitian manifold M'. Then $\bar{M}$ is called a *generalized Kenmotsu manifold* if M' is Hermitian.

Theorem 3.1. *Let* $(M^{2m+1}, g, \varphi, \xi, \eta)$ *be an almost contact metric manifold. Then* M *is a generalized Kenmotsu manifold if and only if*

$$(\nabla_D \phi)(E, G) - (\nabla_{\varphi D} \phi)(\varphi E, G) = \eta(E)\phi(G, D)$$

and $d\eta = 0$.

Proof. Suppose that M is a generalized Kenmotsu manifold. Thus, M is the warped product of a real line with a Hermitian manifold. Since $\eta \circ \varphi = 0$, then (3.2) leads to

$$(\nabla_{\varphi D} \phi)(\varphi E, G) = f^2(t)(\nabla'_{J'D'} \Omega')(J'E', G') + \eta(G)\phi(D, E). \tag{3.4}$$

On the other hand, on a Hermitian manifold, we have

$$(\nabla'_{D'} \Omega')(E', G') - (\nabla'_{J'D'} \Omega')(J'E', G') = 0.$$

Subtracting (3.4) from (3.2) we get the required relation.

Conversely, assume that

$$(\nabla_D \phi)(E, G) - (\nabla_{\varphi D} \phi)(\varphi E, G) = \eta(E)\phi(G, D) \tag{3.5}$$

This equation shows that we are in the case of a warped product of a real line with an almost Hermitian manifold. We have to show that we have a warped product of a real line with a Hermitian manifold. In fact, equation (3.2) leads to

$$(\nabla_{\varphi D} \phi)(\varphi E, G) = f^2(t)(\nabla'_{J'D'} \Omega')(J'E', G') + \eta(G)\phi(\varphi D, \varphi E) \tag{3.6}$$

because $\eta(\varphi E) = 0$; since $\phi(\varphi D, \varphi E) = \phi(D, E)$, then (3.6) gives rise

$$(\nabla_{\varphi D} \phi)(\varphi E, G) = f^2(t)(\nabla'_{J'D'} \Omega')(J'E', G') + \eta(G)\phi(D, E). \tag{3.7}$$

Calculating (3.2)–(3.7) and using (3.5) we get

$$(\nabla'_{D'} \Omega')(E', G') - (\nabla'_{J'D'} \Omega')(J'E', G') = 0,$$

which is the defining relation of a Hermitian structure. $\Diamond$

Example. It is known that any odd dimensional sphere is furnished with the Sasakian structure. But any Sasakian manifold is normal. Corollary of Proposition 3.5 of Capursi [3, p. 78], states that the direct product of two odd dimensional spheres is Hermitian. Thus, the warped product $\bar{M} = \mathbb{R} \times_f (S^{2p+1} \times S^{2p'+1})$ is a generalized Kenmotsu manifold.

4. Inclusion relations

Now, let us turn to the inclusions relations with the well known structures.

Proposition 4.1. *A generalized Kenmotsu manifold is G_1-Kenmotsu.*

Proof. Putting $E = D$ in the defining relation of a generalized Kenmotsu manifold, one obtains

$$(\nabla_D \phi)(D, G) - (\nabla_{\varphi D} \phi)(\varphi D, G) = \eta(D)\phi(G, D),$$

which is the defining relation of G_1-Kenmotsu. ♢

Proposition 4.2. *A generalized Kenmotsu manifold is G_2-Kenmotsu.*

Proof. Consider the defining relation of a generalized Kenmotsu manifold; if we consider a cyclic sum over D, E and G, we get

$$\mathcal{G}\left\{(\nabla_D \phi)(E, G) - (\nabla_{\varphi D} \phi)(\varphi E, G) - \eta(E)\phi(G, D)\right\} = 0,$$

which defines a G_2-Kenmotsu structure. ♢

It is clear that Prop. 4.1 and Prop. 4.2 show that a generalized Kenmotsu manifold is both G_1 and G_2-Kenmotsu. This is true because Hermitian manifold is the intersection of a G_1 and a G_2-manifold.

Proposition 4.3. *A generalized Kenmotsu manifold is semi-Kenmotsu normal.*

Proof. It suffices to add $\delta\phi = 0$ to the defining relation of a generalized Kenmotsu structure to obtain the defining relation of semi-Kenmotsu normal. ♢

Proposition 4.4. *A Kenmotsu manifold is a generalized Kenmotsu.*

Proof. If M is Kenmotsu, then (3.2) gives

$$(\nabla_D \phi)(E, G) = \eta(G)\phi(D, E) + \eta(E)\phi(G, D), \tag{4.1}$$

which implies $(\nabla_{\varphi D} \phi)(\varphi E, G) = \eta(G)\phi(\varphi D, \varphi E) + \eta(\varphi E)\phi(G, \varphi D)$ and this can be transformed in

$$(\nabla_{\varphi D} \phi)(\varphi E, G) = \eta(G)\phi(D, E). \tag{4.2}$$

Subtracting (4.2) from (4.1) we get

$$(\nabla_D \phi)(E, G) - (\nabla_{\varphi D} \phi)(\varphi E, G) = \eta(E)\phi(G, D)$$

which defines a generalized Kenmotsu structure. ♢

Propositions 4.1, 4.2, 4.3 and 4.4 lead to the following inclusions relations.

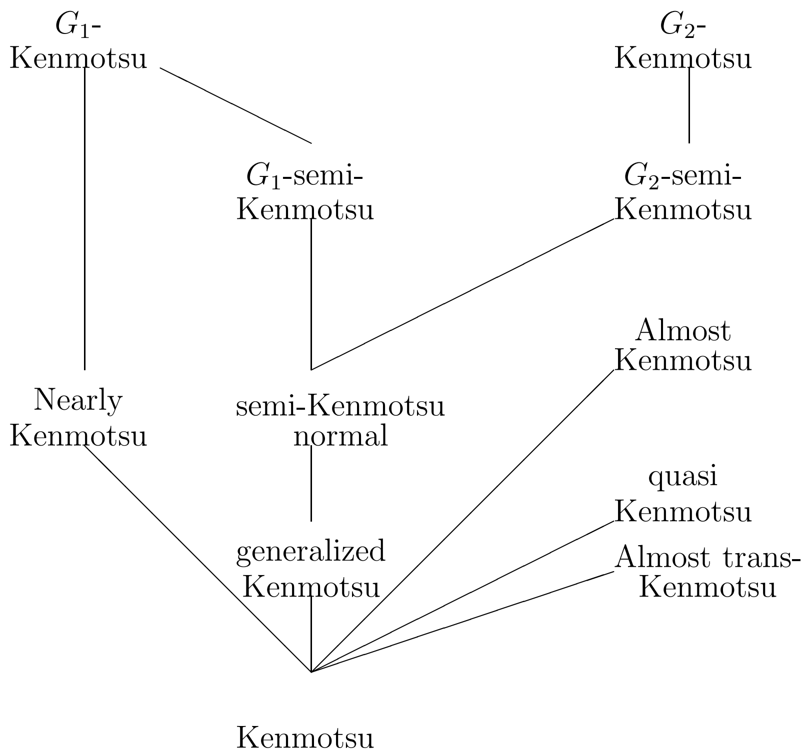

Figure 1. Diagram of the strict inclusions

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