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CHEBYSHEV AND JENSEN INEQUALITIES FOR CHOQUET INTEGRAL

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Abstract: We supply Chebyshev and Jensen type inequalities for Choquet integral. Moreover, in the setting of normalized linear functionals on a real vector space of measurable functions including all bounded ones, we state the equivalence of these inequalities and the positivity condition of functionals.

1. Introduction and preliminaries

Some well-known integral inequalities (stated for the Lebesgue integral), such as Chebyshev inequality and Jensen inequality, play important roles not only from a theoretical point of view but also in applications.

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For this reason, in the last years, several authors have studied the validity of these inequalities, in the setting of non-additive set functions, for the Choquet integral (e.g. see [7], [8], [12]), for the Sugeno integral (e.g. see [2], [3], [6]) and for semi-normed fuzzy integrals (e.g. see [1], [9]).

In this paper, with reference to Choquet integral, we complete our results (obtained in [7]) on the validity of Chebyshev inequality and state a general form of Jensen inequality. Moreover, we show that, in the setting of normalized linear functionals on a real vector space of measurable functions including all bounded ones, these inequalities and the positivity condition of functionals are equivalent.

Now, we are going to recall briefly some notion, notation and results useful in the sequel. Given a measurable space $(\Omega, \mathcal{F})$, we shall denote by ω (with or without indices) any element of Ω and by F (with or without indices) any element of $\mathcal{F}$. Any set function $\mu : \mathcal{F} \to [0, +\infty[$ is called a *(real) monotone set function* if the following properties are satisfied:

(a) $\mu(\emptyset) = 0$;

(b) $\mu(F_1) \leq \mu(F_2)$, whenever $F_1 \subset F_2$ (monotonicity).

We call μ a *monotone probability*, if $\|\mu\| = \mu(\Omega) = 1$ and *additive*, if $\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$, whenever $F_1 \cap F_2 = \emptyset$. Finally, for any μ, we will consider the corresponding *conjugate* monotone set function on $\mathcal{F}$ defined as: $\overline{\mu}(F) = \|\mu\| - \mu(F^c)$.

Henceforth, X, Y always denote real-valued functions on Ω which are $\mathcal{F}$-Borel measurable and I_F denotes the indicator function of F. Moreover, given X, we put $\{X > t\} = \{\omega : X(\omega) > t\}$ for any $t \geq 0$. Finally, we recall that X, Y are said to be *comonotonic* if $X(\omega_1) > X(\omega_2)$ and $Y(\omega_1) < Y(\omega_2)$ is impossible for any ω_1, ω_2 (i.e. $(X(\omega_1) - X(\omega_2))\,(Y(\omega_1) - Y(\omega_2)) \geq 0$ for any ω_1, ω_2).

Now, given the monotone set function space $(\Omega, \mathcal{F}, \mu)$, the *Choquet integral* of X w.r.t. μ is defined as:

$$\Im_\mu(X) = \int_{-\infty}^{0} [\mu(\{X > t\}) - \|\mu\|\,]\,dt + \int_{0}^{+\infty} \mu(\{X > t\})\,d\,t,$$

whenever at least one of the involved integrals is finite (this integral is usually denoted, in the related literature, by $\mathrm{C}\int_\Omega X\,d\mu$). Note that $\Im_\mu(X) \in \overline{\mathbb{R}} = [-\infty, +\infty]$ and that $\Im_\mu(X)$ exists for any $X \geq 0$. Now, we recall some properties of this integral (see [5]):

- $\Im_\mu(I_F) = \mu(F)$;

- $\Im_\mu(\alpha X) = \alpha \Im_\mu(X)$ for any real $\alpha \geq 0$ (positive homogeneity);
- $\Im_\mu(-X) = -\Im_{\overline{\mu}}(X)$ (asymmetry);
- $\Im_\mu(X + \alpha) = \Im_\mu(X) + \alpha\|\mu\|$ for any real α (translatability);
- $\Im_\mu(X) \leq \Im_\mu(Y)$, whenever $X \leq Y$ (monotonicity).

Moreover, by translatability (put $X \equiv 0$), we have $\Im_\mu(\alpha I_\Omega) = \alpha\|\mu\|$ for any real α.

Note that, if μ is additive, $\mu = \overline{\mu}$ so that $\Im_\mu$ is homogeneous and $\Im_\mu(X) = \Im_\mu(X^+) - \Im_\mu(X^-)$, on noting that

$$\Im_\mu(X^-) = -\int_{-\infty}^{0} [\mu(\{X > t\}) - \|\mu\|\,]\, dt$$

(where X^+, X^- are, respectively, the positive and the negative part of X).

Finally, we call X *Choquet integrable (w.r.t. μ)*, whenever $\Im_\mu(X)$ exists and is finite; plainly, any bounded function X is Choquet integrable.

2. Main results

The following lemma paves the way for verifying that the comonotonicity condition is "nearly always" sufficient for the validity of Chebyshev inequality in the setting of monotone set functions.

Lemma 2.1. *Let μ be additive. Then, the set of Choquet integrable functions is a real vector space including all bounded $\mathcal{F}$-Borel measurable functions and $\Im_\mu$ is a real linear functional on it. Moreover, if $\Im_\mu(X)$ is not finite and $\Im_\mu(Y)$ is finite, then there exists $\Im_\mu(X+Y)$ and $\Im_\mu(X + Y) = = \Im_\mu(X)$.*

Proof. The first statement of the thesis immediately follows from Cor. 6.5 and Prop. 9.4 in [5]. Let $\Im_\mu(X)$ be not finite and $\Im_\mu(Y)$ finite. Now, recall that $(X+Y)^- \leq X^- + Y^-$, $(X+Y)^+ \leq X^+ + Y^+$ and $(X+Y)^+ + X^- + Y^- = (X+Y)^- + X^+ + Y^+$. First assume $\Im_\mu(X) = +\infty$. Then, $\Im_\mu(X^-)$ is finite and hence, by monotonicity, $\Im_\mu((X+Y)^-)$ is also finite, so that $\Im_\mu(X+Y)$ exists. Therefore, by Cor. 6.5 in [5], we have

$$\Im_\mu((X+Y)^+) + \Im_\mu(X^-) + \Im_\mu(Y^-) = \Im_\mu((X+Y)^-) + \Im_\mu(X^+) + \Im_\mu(Y^+)$$

and hence $\Im_\mu((X+Y)^+) = +\infty$, recalling that $\Im_\mu(X^+) = +\infty$ and $\Im_\mu(X^-)$, $\Im_\mu(Y^-)$ are finite.

Finally, if $\Im_\mu(X) = -\infty$, by similar arguments we get the thesis. $\Diamond$

The next theorem supplies some sufficient conditions assuring the validity of a Chebyshev type inequality in the setting of monotone set functions.

Theorem 2.2. *Let X, Y be comonotonic such that $\Im_\mu(X), \Im_\mu(Y)$ and $\Im_\mu(XY)$ exist. Assume one of the following conditions being valid:*

(i) $X, Y \geq 0$;

(ii) *Let μ be additive and assume one of the following conditions being valid:*

(ii1) X, Y *are Choquet integrable;*

(ii2) $\Im_\mu(X) = +\infty$ *and there is* ω_0 *such that* $Y(\omega_0) > 0$;

(ii3) $\Im_\mu(X) = -\infty$ *and there is* ω_0 *such that* $Y(\omega_0) < 0$.

Then, the Chebyshev type inequality $\|\mu\| \, \Im_\mu(XY) \geq \Im_\mu(X)\Im_\mu(Y)$ *holds.*

Proof. When (i) or, for μ additive, (ii1) holds, the thesis immediately follows from Th. 2.2 and Th. 2.5 in [7].

Now, let μ be additive. First assume (ii2). Let $\Im_\mu(Y)$ be finite. By comonotonicity we have $0 \leq [X - X(\omega_0)][Y - Y(\omega_0)] = XY - X(\omega_0)Y - Y(\omega_0)X + X(\omega_0)Y(\omega_0)$ and hence

$$XY \geq X(\omega_0)Y + Y(\omega_0)X - X(\omega_0)Y(\omega_0).$$

Note that, by positive homogeneity, $\Im_\mu(Y(\omega_0)X) = Y(\omega_0)\Im_\mu(X) = +\infty$ and, by Lemma 2.1, $\Im_\mu(X(\omega_0)Y - X(\omega_0)Y(\omega_0))$ is finite. Therefore, by Lemma 2.1 and monotonicity, we get

$$\Im_\mu(XY) \geq \Im_\mu(Y(\omega_0)X + [X(\omega_0)Y - X(\omega_0)Y(\omega_0)]) = \Im_\mu(Y(\omega_0)X) = +\infty.$$

Now, let $\Im_\mu(Y) = +\infty$ (if $\Im_\mu(Y) = -\infty$, the thesis is trivial). Then, $\Im_\mu(X^+) = \Im_\mu(Y^+) = +\infty$. On noting that $(XY)^+ \geq X^+Y^+$, by monotonicity and (i), we get

$$\Im_\mu((XY)^+) \geq \Im_\mu(X^+Y^+) \geq \Im_\mu(X^+)\Im_\mu(Y^+) = +\infty$$

so that $\Im_\mu(XY) = +\infty$ ($\Im_\mu(XY)$ exists!).

Finally, assume (ii3). The thesis follows from (ii2) and homogeneity, on noting that $-X, -Y$ are comonotonic. $\Diamond$

Remark 2.3. (i) Note that under assumption of (i) or, for μ additive, (ii1) in the previous theorem, the hypothesis of the existence of $\Im_\mu(XY)$ can be dropped (see proof of Th. 2.5 in [7]).

(ii) As the next example shows, in the previous theorem the existence of ω_0 assumed in (ii2) and (ii3) can not be dropped for its validity. Let Ω be the set of positive natural numbers and μ' the additive probability on the finite-cofinite field on Ω defined as: $\mu'(A) = 0$, if A is finite, and $\mu'(A) = 1$, if A is cofinite. Now, let μ be an additive extension of μ' on $\mathcal{F} = 2^\Omega$ (see Th. 3.4.4 in [4]) and consider the following two comonotonic functions on Ω defined as: $X(\omega) = \omega^2$ and $Y(\omega) = -\frac{1}{\omega}$. On noting that, for any real $t \geq 0$, the sets $\{X > t\}$, $\{(XY)^- > t\}$ are cofinite and $\{Y^- > t\}$ is finite, we easily get that $\Im_\mu(X) = +\infty$, $\Im_\mu(XY) = -\infty$ and $\Im_\mu(Y) = 0$. Consequently, adopting the usual rule $+\infty \cdot 0 = 0$ of arithmetic in the extended real line, we have $\|\mu\| \, \Im_\mu(XY) < 0 = \Im_\mu(X)\Im_\mu(Y)$.

Now, we are going to study the validity of the well-known Jensen inequality in the setting of monotone probabilities. We recall that any real convex function g, defined on a real interval I (bounded or not), is monotone or there is $a \in I$ such that g is decreasing to the left of a and increasing to the right (so that g admits limits at the endpoints of I); moreover, for any interior points x_1, x_2 of I with $x_1 < x_2$, we have:

$$g'_+(x_1) \leq \frac{g(x_2) - g(x_1)}{x_2 - x_1} \leq g'_+(x_2),$$

where g'_+ is the right-hand derivative of g which is finite on the interior of I and supplies a subgradient of g at any interior point. Finally, for any interior point x_0, we consider the difference quotient w.r.t. x_0 defined on I as:

$$R^{(g)}_{x_0}(x) = \begin{cases} \frac{g(x)-g(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ \frac{g'_+(x_0)+g'_-(x_0)}{2} & \text{if } x = x_0 \end{cases},$$

where g'_- is the left-hand derivative of g. Note that this function is increasing for any interior point x_0.

Now we show that Jensen inequality does not hold, in general, in the setting of monotone probabilities for the most used non additive integrals.

Example 2.4. Let $\Omega =]0, 2[$, $\mathcal{F} = 2^\Omega$ and $S =]0, \frac{3}{2}]$. Consider then the (non additive) monotone probability μ_S defined as: $\mu_S(A) = 1$, if $A \supset S$, and $\mu_S(A) = 0$, otherwise (introduced in [10], p.91). Plainly, we get $\Im_{\mu_S}(X) = \inf X(S)$ for any X. Now, consider the function on Ω defined as: $X(\omega) = 2 - \omega$ and the convex function g on Ω defined as: $g(\omega) = (\omega - 1)^2$. Consequently, we get $g \circ X = g$, $\Im_{\mu_S}(X) = \frac{1}{2}$, $\Im_{\mu_S}(g \circ X) = 0$ and $g(\Im_{\mu_S}(X)) = \frac{1}{4}$ and hence $g(\Im_{\mu_S}(X)) > \Im_{\mu_S}(g \circ X)$.

On noting that Choquet, Sugeno and Shilkret integrals of X and $g \circ X$ w.r.t. μ_S coincide, this example shows that Jensen inequality does not hold, in general, for all these integrals (for definition and some properties of these integrals see [13] and [11]).

The following theorem supplies a generalization of Jensen inequality for Choquet integral which becomes the classical one in the setting of additive probabilities.

Theorem 2.5. *Given $I =]i_0, i_1[\subset \mathbb{R}$ (bounded or not), let $X : \Omega \to I$ and $g : I \to \mathbb{R}$ such that $\Im_\mu(X)$ and $\Im_\mu(g \circ X)$ exist. If g is a convex function and μ is a monotone probability, then we have:*

$$g(\Im_\mu(X)) \leq \max(\Im_\mu(g \circ X), \Im_{\overline{\mu}}(g \circ X)),$$

where $g(\Im_\mu(X)) = g(i_1^-)$, if $\Im_\mu(X) = i_1$, and $g(\Im_\mu(X)) = g(i_0^+)$, if $\Im_\mu(X) = i_0$.

More precisely: if $\Im_\mu(X) \in I$, then $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$, if $g'_+(\Im_\mu(X)) \geq 0$, and $g(\Im_\mu(X)) \leq \Im_{\overline{\mu}}(g \circ X)$, if $g'_+(\Im_\mu(X)) < 0$; if $\Im_\mu(X) = i_1$, then $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$.

Finally, if μ is additive, then we get the usual Jensen inequality $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$.

Proof. Let $x_0 = \Im_\mu(X)$ and note that, by monotonicity, $x_0 \in [i_0, i_1] \subset \overline{\mathbb{R}}$. Since the desired inequality is obvious if $g(x_0) = -\infty$, assume $g(x_0) > -\infty$. Now, the proof is carried out in the following steps.

1°. Let $x_0 \in I$. Since g is convex, we have $g(x) \geq g(x_0) + g'_+(x_0)(x - x_0)$ for any $x \in I$ and hence

$$g(X(\omega)) \geq g(x_0) + g'_+(x_0)(X(\omega) - x_0) \tag{2.1}$$

for any ω. Now, assume $g'_+(x_0) \geq 0$. Then, by monotonicity, positive homogeneity and translatability, we have

$$\Im_\mu(g \circ X) \geq g(x_0) + g'_+(x_0)(\Im_\mu(X) - x_0) = g(x_0) = g(\Im_\mu(X)).$$

On the other hand, if $g'_+(x_0) < 0$, from (1) we get

$$(-g)(X(\omega)) \leq -g(x_0) + (-g'_+(x_0))(X(\omega) - x_0)$$

for any ω and then $\Im_\mu(-(g \circ X)) \leq -g(x_0)$. Hence, by asymmetry, $-\Im_{\overline{\mu}}(g \circ X) = \Im_\mu(-(g \circ X)) \leq -g(x_0)$, i.e. $g(\Im_\mu(X)) \leq \Im_{\overline{\mu}}(g \circ X)$.

2°. Let $x_0 = i_0$. If g is increasing, then $g(x_0) \in \mathbb{R}$ (recall $g(x_0) > -\infty$) and $g(X(\omega)) \geq g(x_0)$ for any ω, so that, by monotonicity, we get $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$. If g is not increasing, then there is $a \in I$ such that $g'_+(a) < 0$. Consider a sequence $(x_n)_{n \geq 1}$ in I such that $a > x_n \downarrow x_0$. Consequently, given n, we have $g(X(\omega)) \geq g(x_n) + g'_+(x_n)(X(\omega) - x_n)$ for any ω and $g'_+(x_n) < 0$; therefore, by monotonicity, positive homogeneity and translatability, we have

$$\Im_\mu(-(g\circ X)) \le -g(x_n) + (-g'_+(x_n))(\Im_\mu(X) - x_n) \le -g(x_n),$$

recalling that $\Im_\mu(X) = i_0 < x_n$ for all n. Hence, by asymmetry, we obtain $\Im_{\overline{\mu}}(g\circ X) \ge g(x_n)$ for all n. Now, by carrying out the passage to the limit, we get $\Im_{\overline{\mu}}(g\circ X) \ge g(x_0^+) = g(i_0^+)$ and hence, by definition, $g(\Im_\mu(X)) \le \Im_{\overline{\mu}}(g\circ X)$.

3°. Let $x_0 = i_1$. If g is decreasing, then $g(X(\omega)) \ge g(x_0) \in \mathbb{R}$ for any ω (recall $g(x_0) > -\infty$), so that we get $g(\Im_\mu(X)) \le \Im_\mu(g\circ X)$. If g is not decreasing, then there is $a \in I$ such that $g'_+(a) > 0$. Consider a sequence $(x_n)_{n\ge1}$ in I such that $a < x_n \uparrow x_0$. Consequently, given n, we have $g(X(\omega)) \ge g(x_n) + g'_+(x_n)(X(\omega) - x_n)$ for any ω and $g'_+(x_n) > 0$; therefore, recalling that $\Im_\mu(X) = i_1 > x_n$ for all n, we have $\Im_\mu(g\circ X) \ge$ $\ge g(x_n) + g'_+(x_n)(\Im_\mu(X) - x_n) \ge g(x_n)$. Now, by carrying out the passage to the limit, we get $g(\Im_\mu(X)) = g(i_1^-) = g(x_0^-) \le \Im_\mu(g\circ X)$.

Finally, if μ is additive ($\mu = \overline{\mu}$!), we get the usual Jensen inequality. ◊

Remark 2.6. (i) Given a non-null monotone set function μ (not necessarily a probability), we have $\|\mu\| = \|\overline{\mu}\| > 0$, so that $\frac{\mu}{\|\mu\|}$ and $\frac{\overline{\mu}}{\|\mu\|}$ are monotone probabilities. Consequently, in the same hypotheses for I, X and g, the thesis of the previous theorem holds by putting $\frac{\mu}{\|\mu\|}$, $\frac{\overline{\mu}}{\|\mu\|}$ instead of μ and $\overline{\mu}$, respectively. In particular, we get

$$g\left(\mathrm{C}\int_\Omega X\, d\frac{\mu}{\|\mu\|}\right) \le \max\left(\mathrm{C}\int_\Omega g\circ X\, d\frac{\mu}{\|\mu\|}, \mathrm{C}\int_\Omega g\circ X d\frac{\overline{\mu}}{\|\mu\|}\right)$$

and hence, by positive homogeneity of $\Im_\mu(X)$ w.r.t. μ, the following inequality holds:

$$\|\mu\|\, g\left(\frac{1}{\|\mu\|}\Im_\mu(X)\right) \le \max(\Im_\mu(g\circ X), \Im_{\overline{\mu}}(g\circ X)).$$

(ii) With reference to Ex. 2.4, note that $\overline{\mu}_S(A) = 1$, if $A\cap S \ne \emptyset$, and $\overline{\mu}_S(A) = 0$, otherwise so that $\Im_{\overline{\mu}_S}(Y) = \sup Y(S)$ for any $Y \ge 0$. Therefore, we have $\Im_{\overline{\mu}_S}(g\circ X) = 1 > \frac{1}{4} = g(\Im_{\mu_S}(X))$.

The following result immediately follows from the previous theorem and generalizes Cor. 4.1 in [8] which is stated for Ω finite.

Corollary 2.7. *Given $I =]i_0, i_1[\subset \mathbb{R}$ (bounded or not), let $X : \Omega \to I$ and $g : I \to \mathbb{R}$ such that $\Im_\mu(X)$ and $\Im_\mu(g\circ X)$ exist. If g is an increasing convex function and μ is a monotone probability, then $g(\Im_\mu(X)) \le$ $\le \Im_\mu(g\circ X)$, where $g(\Im_\mu(X)) = g(i_1^-)$, if $\Im_\mu(X) = i_1$, and $g(\Im_\mu(X)) =$ $= g(i_0^+)$, if $\Im_\mu(X) = i_0$.*

We conclude the paper by a theorem which supplies a characterization of normalized positive linear functionals on real vector spaces of measurable functions, including all bounded ones, in terms of Chebyshev inequality or Jensen inequality. Note that, Lebesgue integral (in a countable additive context) and Choquet integral, Dunford–Schwartz integral (D-integral) and Stieltjes type integral (S-integral) (in an additive context) are positive linear functionals on the corresponding real vector space of integrable functions (for the Choquet integral see Lemma 2.1 and for the last two integrals see Theorems 4.5.7 $\div$ 4.5.9 and 4.4.13 in [4]). In this way, we find again Chebyshev and Jensen inequalities for Lebesgue and Choquet integrals and state their validity for D-integral and S-integral, as well.

Theorem 2.8. *Let $\mathbb{D}$ be a real vector space of $\mathcal{F}$-Borel measurable functions including all bounded ones. Moreover, let $\Im : \mathbb{D} \to \mathbb{R}$ be such that:*

(a) $\Im(I_\Omega) = 1$ *(normality);*

(b) $\Im(\alpha X + \beta Y) = \alpha\Im(X) + \beta\Im(Y)$ *for any $X, Y \in \mathbb{D}$ and any real α, β (linearity).*

Then, the following statements are equivalent:

(i) *Chebyshev inequality: Let X, Y be comonotonic such that $X, Y, XY \in \mathbb{D}$. Then, the inequality $\Im(XY) \geq \Im(X)\Im(Y)$ holds.*

(ii) *Jensen inequality: Let $I \subset \mathbb{R}$ be an open interval (bounded or not) and $X : \Omega \to I$ such that $\Im(X) \in I$. Moreover, let $g : I \to \mathbb{R}$ be a convex function such that $g \circ X, R^{(g)}_{\Im(X)} \circ X \in \mathbb{D}$. Then, the inequality $g(\Im(X)) \leq \Im(g \circ X)$ holds.*

(iii) *Positivity: Let $X \in \mathbb{D}$ and $X \geq 0$. Then, $\Im(X) \geq 0$.*

Proof. (i) $\Rightarrow$ (ii). Letting $x_0 = \Im(X)$, we have $g \circ X - g(x_0) = (R^{(g)}_{x_0} \circ X)(X - x_0)$ and, by normality and linearity, $\Im(X - x_0) = \Im(X) - x_0\Im(I_\Omega) = \Im(X) - x_0 = 0$. Consequently, we have

$$\Im(g \circ X) - g(x_0) = \Im(g \circ X) - g(x_0)\Im(I_\Omega) = \Im(g \circ X - g(x_0)) = \\ = \Im((R^{(g)}_{x_0} \circ X)(X - x_0)) \geq \Im(R^{(g)}_{x_0} \circ X)\Im(X - x_0) = 0,$$

where the inequality follows from (i), on noting that $R^{(g)}_{x_0} \circ X$ and $X - x_0$ are comonotonic ($R^{(g)}_{x_0}$ is increasing!). Therefore, $g(\Im(X)) \leq \Im(g \circ X)$.

(ii) $\Rightarrow$ (iii). Let $I = \mathbb{R}$ and $g(x) = |x|$. On noting that $g \circ X = X$, $|R^{(g)}_{\Im(X)}| \le 1$ and $R^{(g)}_{\Im(X)}$ is increasing, we have $g \circ X, R^{(g)}_{\Im(X)} \circ X \in \mathbb{D}$ and hence, by (ii), $0 \le |\Im(X)| \le \Im(|X|) = \Im(X)$.

(iii) $\Rightarrow$ (i). Plainly, by (iii) and linearity, the functional $\Im$ is monotone. Then, (i) immediately follows from Prop. 2.4 in [7]. $\Diamond$

Remark 2.9. Note that the hypothesis $R^{(g)}_{\Im(X)} \circ X \in \mathbb{D}$, assumed in (ii) of the previous theorem, can be dropped whenever $R^{(g)}_{\overline{x}}$ is bounded for some $\overline{x} \in I$.

References

[1] AGAHI, H. and ESLAMI, E.: A general inequality of Chebyshev type for semi(co)normed fuzzy integrals, *Soft Comput.* **15** (2011), 771–780.

[2] AGAHI, H., MESIAR, R. and OUYANG, Y.: New general extension of Chebyshev type inequalities for Sugeno integrals, *Internat. J. of Approx. Reason.* **51** (2009), 135–140.

[3] AGAHI, H., MESIAR, R. and OUYANG, Y.: Further development of Chebyshev type inequalities for Sugeno integrals and T-(S-) evaluators, *Kybernetika* **46** (2010), 83–95.

[4] BHASKARA RAO, K. P. S. and BHASKARA RAO, M.: *Theory of Charges*, Academic Press, London, 1993.

[5] DENNEBERG, D.: *Non-Additive Measure and Integral*, Kluwer Academic Publisher, Dordrecht, 1994.

[6] GIROTTO, B. and HOLZER, S.: A Chebyshev type inequality for Sugeno integral and comonotonicity, *Internat. J. of Approx. Reason.* **52** (2011), 444–448.

[7] GIROTTO, B. and HOLZER, S.: A Chebyshev type inequality for Choquet integral and comonotonicity, *Internat. J. of Approx. Reason.* **52** (2011), 1118–1123.

[8] MESIAR, R., LI, J. and PAP, E.: The Choquet integral as Lebesgue integral and related inequalities, *Kybernetika* **46** (2010), 1098–1107.

[9] OUYANG, Y. and MESIAR, R.: On the Chebyshev type inequalities for seminormed fuzzy integral, *Appl. Math. Lett.* **22** (2009), 1810–1815.

[10] ROSENMÜLLER, J.: On core and value, *Meth. Oper. Res.* **9** (1971), 84–104.

[11] SHILKRET, N.: Maxitive measure and integration, *Indag. Math.* **8** (1971), 109–116.

[12] WANG, R.-S.: Some inequalities and convergence theorems for Choquet integrals, *J. Appl. Math. Comput.* **35** (2011), 305–321.

[13] WANG, Z. and KLIR, G. J.: *Generalized measure theory*, Springer, New York, 2009.