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EIGENVALUES OF MATRICES AND DISCRETE LAGUERRE–FOURIER COEFFICIENTS

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Abstract: The discrete Laguerre-functions play an important role in system identification. In this paper we investigate the Fourier coefficients of the matrix function $F(z) = (I - zA)^{-1}$ with respect to the discrete Lagueerre system. Among others an explicit form is given for the Laguerre Fourier coefficients of F. With the help of this formula we introduce a map Q_A which can be used to compute the eigenvalues of the matrix A. The domain of the transformation in question can be defined in the term of hyperbolic distance.

1. Blaschke functions and the discrete Laguerre system

Let $\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}$ be the unit disc on the complex plain $\mathbb{C}$ and denote $\mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\}$ the torus and $\overline{\mathbb{D}} := \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ the closed unit disc.

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The *Blaschke functions* are defined as

$$B_{\mathfrak{a}}(z) := \frac{z-a}{1-\bar{a}z} \quad (z \in \overline{\mathbb{C}}, a \in \mathbb{D}). \tag{1.1}$$

It can be proved that the map

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \overline{z}_1\, z_2|} = |B_{z_1}(z_2)| \quad (z_1, z_2 \in \mathbb{D}) \tag{1.2}$$

is a metric on $\mathbb{D}$. Moreover the Blaschke functions B_a $(a \in \mathbb{D})$ are *isometries* with respect to this metric [5, 8], i.e.

$$\rho(B_a(z_1), B_a(z_2)) = \rho(z_1, z_2) \quad (a \in \mathbb{D}, z_1, z_2 \in \mathbb{D}).$$

The maps ϵB_a $((\epsilon, a) \in \mathbb{T} \times \mathbb{D})$ are 1-1 on $\mathbb{T}$ and $\mathbb{D}$, respectively. Moreover *they form a group* with respect to composition of functions. This group can be considered as the *transformation group of congruence* in the Poincaré model of the hyperbolic plain [5].

For any $k \in \mathbb{N}$ and any $a \in \mathbb{D}$ we denote by $L_{k,a}$ the discrete Laguerre functions defined by

$$L_{k,a}(z) := \frac{\sqrt{1-|a|^2}}{1-\overline{a}z} B_a^k(z) \quad (z \in \overline{\mathbb{D}},\ a \in \mathbb{D},\ k \in \mathbb{N}) \tag{1.3}$$

(see [1], [4], [6]). It is known that the system $(L_{k,a}, k \in \mathbb{N})$ is orthonormal and complete in $H^2(\mathbb{T})$ with respect to the scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \overline{g}(e^{it})\, dt \quad (f, g \in H^2(\mathbb{T})). \tag{1.4}$$

We denote by $\mathfrak{R}$ the set of rational functions analytic in the closed disc $\overline{\mathbb{D}}$. The rational functions of the form

$$r_{j,a}(z) := \frac{z^j}{(1-\overline{a}z)^{j+1}} \quad (z \in \overline{\mathbb{D}}, a \in \mathbb{D}, j \in \mathbb{N}) \tag{1.3}$$

generates the set $\mathfrak{R}$ (see e.g. [6]). Namely every function $f \in \mathfrak{R}$ can be written in the form

$$f(z) = \sum_{i=1}^{N} \sum_{j=0}^{m_i - 1} \frac{c_{i,j}\, z^j}{(1-\overline{a}_i z)^{j+1}},$$

where $a_i^* := 1/\overline{a}_i$ $(i = 1, 2, \ldots, N)$ are the poles of f with the multiplicity m_i and the $c_{i,j}$'s are complex numbers and $c_{i,m_i-1} \neq 0$.

To get the Fourier coefficients of f with respect to the discrete Laguerre system we shall use the next statement (see [6]).

Lemma 1. *For every function* $G \in \mathfrak{R}$

$$\langle G, r_{j,a}\rangle = \frac{G^{(j)}(a)}{j!} \quad (j \in \mathbb{N}, a \in \mathbb{D}). \tag{1.5}$$

The derivative of $L_{k,a}$ can be expressed of the form

$$L'_{k,a} = \alpha_a L_{k,a} + k\beta_a L_{k-1,a} \quad (k \in \mathbb{N}, a \in \mathbb{D}), \tag{1.6}$$

where

$$\alpha_a(z) := \frac{\overline{a}}{1-\overline{a}z}, \beta_a(z) := B'_a(z) \ (z \in \overline{\mathbb{D}}, a \in \mathbb{D}, k \in \mathbb{N}).$$

For the second derivative we get

$$\begin{aligned} L''_{k,a} &= \alpha'_a L_{k,a} + k\beta'_a L_{k-1,a} + \\ &\quad + \alpha_a(\alpha_a L_{k,a} + k\beta_a L_{k-1,a}) + k\beta_a(\alpha_a L_{k-1,a} + (k-1)\beta_a L_{k-2,a}) = \\ &= (\alpha'_a + \alpha_a^2) L_{k,a} + \binom{k}{1}(\beta'_a + 2\alpha_a\beta_a) L_{k-1,a} + 2\binom{k}{2}\beta_a^2 L_{k-2,a}. \end{aligned}$$

It was shown in [6] that for the derivative of higher order the following recursion holds.

Lemma 2. *For any* $j, k \in \mathbb{N}$

$$L_{k,a}^{(j)} = \sum_{\ell=0}^{j} \gamma_{j,\ell,a}\binom{k}{\ell} L_{k-\ell,a}, \tag{1.7}$$

where the functions $\gamma_{j,\ell,a} : \mathbb{D} \to \mathbb{C}$ $(0 \le \ell \le j, j \in \mathbb{N}^*)$ *do not depend on* k *and can be computed by the equations*

$$\begin{aligned} \gamma_{1,0,a} &= \alpha_a, \gamma_{1,1,a} = \beta_a, \\ \gamma_{j+1,\ell,a} &= \alpha_a\gamma_{j,\ell,a} + \gamma'_{j,k,a} + \ell\beta_a\gamma_{j,\ell-1,a} \quad (\ell = 0, 1, \ldots, j), \\ \gamma_{j+1,j+1,a} &= (j+1)\beta_a\gamma_{j,j,a} \quad (j \in \mathbb{N}). \end{aligned}$$

2. The transfer function of matrices

Let $A \in \mathbb{C}^{n\times n}$ be complex matrix and suppose that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A are in $\mathbb{D}$. In this case there exist a norm $\|\cdot\|$ in $\mathbb{C}^n$ such that the matrix norm induced by this vector norm satisfies $\|A\| < 1$. Starting with an arbitrary vector $x_0 \in \mathbb{C}^n$ we introduce the sequence $(x_k \in \mathbb{C}^n, k \in \mathbb{N})$ by

$$x_{k+1} = Ax_k \quad (k \in \mathbb{N}). \tag{2.1}$$

Recursion (2.1) is called von Mises iteration and can be considered as a special linear time-invariant system (see [2], [3], [7]). The transfer function of this system is defined by

$$F(z) := \sum_{k=0}^{\infty} x_k z^k \quad (z \in \mathbb{D}). \tag{2.2}$$

On the basis of the relations $\|x_k\| \le \|A\|^k \|x_0\| (k \in \mathbb{N})$ it follows that the series (2.2) converges on the disc $\mathbb{D}_R := \{z \in \mathbb{C} \,|\, |z| < R := 1/\|A\|\}$ and the function $F : \mathbb{D}_R \to \mathbb{C}^n$ is analytic. From (2.2) we get

$$F(z) - x_0 = \sum_{k=0}^{\infty} x_{k+1} z^{k+1} = zA\left(\sum_{k=0}^{\infty} x_k z^k\right) = zAF(z),$$

and consequently F satisfies

$$(I - zA)F(z) = x_0 \quad (z \in \mathbb{D}),$$

where $I \in \mathbb{C}^{n\times n}$ is the unit matrix. Hence for F we get

$$F(z) = (I - zA)^{-1} x_0 \quad (z \in \mathbb{D}). \tag{2.3}$$

Using the minimal polynomial P of the matrix A the matrix function $(I - zA)^{-1}$ can be written in an explicit form. Namely let

$$P(\lambda) = \prod_{j=1}^{s} (\lambda - \lambda_j)^{m_j} \quad (\lambda \in \mathbb{C}, \lambda_i \ne \lambda_j \text{ if } i \ne j)$$

and denote by $m := m_1 + \dots + m_s \le n$ the degree of P. We introduce the basic polynomials of Hermite interpolation process generated by the system of roots (λ_j, m_j) $(j = 1, 2, \dots, s)$. The polynomials h_{ij} $(j = 1, 2, \dots, m_i,\ i = 1, 2, \dots, s)$ with degree less then m, are defined by the conditions

(2.4)
$$h_{ij}^{(j_1-1)}(\lambda_{i_1}) = \delta_{i,i_1}\delta_{j,j_1} \quad (1 \le j \le m_i,\ 1 \le i \le s,\ 1 \le j_1 \le m_{i_1},\ 1 \le i_1 \le s).$$

Using the notation $g(w) := (1 - zw)^{-1}$ the matrix function in question can be written in the form

$$g(A) = (I - zA)^{-1} = \sum_{i=1}^{s}\sum_{j=0}^{m_i-1} \frac{z^j}{(1-\lambda_i z)^{j+1}} h_{ij}(A) \quad (z \in \mathbb{D}).$$

This implies that the scalar product of $F(z)$ and the vector $y_0 \in \mathbb{C}^n$ is equal to

$$f(z) := [F(z), y_0] = \sum_{i=1}^{s}\sum_{j=0}^{m_i-1} \frac{z^j}{(1-\lambda_i z)^{j+1}} [h_{ij}(A)x_0, y_0] \quad (z \in \mathbb{D}). \tag{2.5}$$

Denote by

$$f_k(a) := \langle L_{k,a}, f\rangle \quad (k \in \mathbb{N}, a \in \mathbb{D}) \tag{2.6}$$

the conjugate of the discrete Laguerre–Fourier coefficients of f. Then by Lemma 1 and (2.5)

$$f_k(a) = \sum_{i=1}^{s}\sum_{j=0}^{m_i-1} c_{ij}\langle L_{k,a}, r_{j,\overline{\lambda}_i}\rangle = \sum_{i=1}^{s}\sum_{j=0}^{m_i-1} \frac{c_{ij}}{j!} L_{k,a}^{(j)}(\overline{\lambda}_i), \tag{2.7}$$

where

$$c_{ij} := [y_0, h_{ij}(A)x_0].$$

Using Lemma 2 we get

$$\begin{aligned} f_k(a) &= \sum_{i=1}^{s}\sum_{j=0}^{m_i-1} \frac{c_{ij}}{j!} \sum_{\ell=0}^{j} \binom{k}{\ell} \gamma_{j,\ell,a}(\overline{\lambda}_i) L_{k-\ell,a}(\overline{\lambda}_i) = \\ &= \sum_{i=1}^{s}\sum_{\ell=0}^{m_i-1} L_{k-\ell,a}(\overline{\lambda}_i)\binom{k}{\ell} \sum_{j=\ell}^{m_i-1} \frac{c_{ij}}{j!}\gamma_{j,\ell,a}(\overline{\lambda}_i) = \\ &= \sum_{i=1}^{s}\sum_{\ell=0}^{m_i-1} b_{i\ell}(a)\binom{k}{\ell} L_{k-\ell,a}(\overline{\lambda}_i), \end{aligned}$$

where

$$b_{i\ell}(a) = \sum_{j=\ell}^{m_i-1} \frac{c_{ij}}{j!}\gamma_{j,\ell,a}(\overline{\lambda}_i).$$

For every $k \geq \max\{m_1, \ldots, m_s\}$ the coefficients $f_k(a)$ can be written in the form (see (1,3))

$$f_k(a) = \sum_{i=1}^{s} B_a^{k-m_i}(\overline{\lambda}_i) \sum_{\ell=0}^{m_i-1} b_{i\ell}(a) L_{m_i-\ell,a}(\overline{\lambda}_i) \binom{k}{\ell} = \tag{2.8}$$

$$= \sum_{i=1}^{s} B_a^{k-m_i}(\overline{\lambda}_i) P_{i,a}(k),$$

where the function

$$P_{i,a}(k) := \sum_{\ell=0}^{m_i-1} b_{i\ell}(a) L_{m_i-\ell,a}(\overline{\lambda}_i) \binom{k}{\ell} \tag{2.9}$$

is a polynomial of degree $(m_i - 1)$ of the variable k, with coefficients, depending on the parameter a.

In the next section we show that (2.8) can be used to compute the eigenvalues of A.

3. Algorithm to compute eigenvalues

Let us fix the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ of A and set $a_1 := \overline{\lambda}_1$, $a_2 := \overline{\lambda}_2, \ldots, a_s := \overline{\lambda}_s$. Depending on this set of eigenvalues and using the hyperbolic distance ρ defined in (1.2) for $i = 1, 2, \ldots, s$ we introduce the following domains of $\mathbb{D}$:

$$\begin{aligned} D_{ij} &: = \{a \in \mathbb{D} : \rho(a, a_i) > \rho(a, a_j)\}, \quad D_i := \bigcap_{1 \leq j \leq s, i \neq j} D_{ij} \\ D_0 &: = \bigcup_{i=1}^{s} D_i. \end{aligned} \tag{3.1}$$

Obviously on the set D_i

$$q_i(a) := \max_{j \neq i} \frac{\rho(a_j, a)}{\rho(a_i, a)} < 1 \quad (a \in D_i) \tag{3.2}$$

is satisfied.

We show that on set D_0 the limit

$$(\mathcal{Q}_A)(a) := \lim_{k\to\infty} \frac{f_{k+1}(a)}{f_k(a)} \quad (a \in D_0) \tag{3.3}$$

exists and the function $\mathcal{Q}_A$ can be used to compute the eigenvalues of A.

Theorem. *Suppose that the eigenvalues of the matrix $A \in \mathbb{C}^{n\times n}$ belong to $\mathbb{D}$ and the polynomial $P_{i,a}$ in* (2.9) *is not identically zero. Then the limit in* (3.3) *exists and*

$$(\mathcal{Q}_A)(a) = B_a(\overline{\lambda}_i), \quad \text{if} \quad a \in D_i \ (i = 1, 2, \dots, s). \tag{3.4}$$

Proof. According to the condition we take the following decomposition:

$$\frac{f_{k+1}(a)}{f_k(a)} =$$

$$= B_a(\overline{\lambda}_i)\frac{P_{i,a}(k+1) + \sum_{j=1,j\neq i}^{s} P_{j,a}(k+1)B_a^{k+1-m_j}(\overline{\lambda}_j)/B_a^{k+1-m_i}(\overline{\lambda}_i)}{P_{i,a}(k) + \sum_{j=1,j\neq i}^{s} P_{j,a}(k)B_a^{k-m_j}(\overline{\lambda}_j)/B_a^{k-m_i}(\overline{\lambda}_i)}.$$

Applying

$$|P_{j,a}(k+1)B_a^{k+1-m_j}(\overline{\lambda}_j)/B_a^{k+1-m_i}(\overline{\lambda}_i)| = |P_{j,a}(k+1)|\frac{\rho(a,\overline{\lambda}_j)^{k+1-m_j}}{\rho(a,\overline{\lambda}_i)^{k+1-m_i}} =$$

$$= O(|P_{j,a}(k+1)||q_i(a)|^k) \to 0 \quad (k \to \infty),$$

we get

$$\lim_{k\to\infty} \frac{f_{k+1}(a)}{f_k(a)} = \lim_{k\to\infty} B_a(\overline{\lambda}_i)\frac{P_{i,a}(k+1)}{P_{i,a}(k)} = B_a(\overline{\lambda}_i)$$

and Theorem is proved. ♢

It is easy to see that the inverse of the map B_a is B_{-a} and consequently we get

Corollary 1. *For any matrix $A \in \mathbb{C}^{n\times n}$ with eigenvalues in $\mathbb{D}$ in the case $a \in D_i$ with $P_{i,a} \neq 0$ we have*

$$B_{-a}((\mathcal{Q}_A)(a)) = \overline{\lambda}_i.$$

To check the condition $P_{i,a} \neq 0$ in general case is not so easy. In the special case $m_1 = m_2 = \dots = m_s = 1$ by (2.8) we have

$$f_k(a) = \sum_{i=1}^{s} c_i L_{k,a}(\overline{\lambda}_i), \ P_{i,a} = c_i := [y_0, h_{i1}(A)x_0] \ \ (i = 1, \dots, s),$$

where

$$h_{i1}(z) := \prod_{j=1,j\neq i}^{s} \frac{z - \lambda_j}{\lambda_i - \lambda_j} \quad (z \in \mathbb{C}, i = 1, 2, \dots, s)$$

are the Lagrange interpolation polynomials. In this case $c_i \neq 0, a \neq \overline{\lambda}_i$ implies

$$\frac{f_{k+1}(a)}{f_k(a)} = B_a(\overline{\lambda}_i)\frac{c_i + \sum_{j=1,j\neq i}^{s} c_j L_{k+1,a}(\overline{\lambda}_j)/B_a^{k+1}(\overline{\lambda}_i)}{c_i + \sum_{j=1,j\neq i}^{s} c_j L_{k,a}(\overline{\lambda}_j)/B_a^{k}(\overline{\lambda}_i)} = \tag{3.5}$$

$$= B_a(\overline{\lambda}_i)\frac{1+\epsilon_{k+1,i}}{1+\epsilon_{k,i}},$$

where

$$\epsilon_{k,i} := \frac{1}{c_i}\sum_{j=1,j\neq i}^{s} c_j L_{k,a}(\overline{\lambda}_j)/B_a^k(\overline{\lambda}_i).$$

By (3.3) for $a \in D_i$ we have

$$\left|\frac{L_{k,a}(\overline{\lambda}_j)}{B_a^k(\overline{\lambda}_i)}\right| = \frac{\sqrt{1-|a|^2}}{|1-\overline{a}\overline{\lambda}_j|}\frac{\rho^k(a,\overline{\lambda}_j)}{\rho^k(a,\overline{\lambda}_i)} \leq \frac{\sqrt{1-|a|^2}}{|1-\overline{a}\overline{\lambda}_j|}q_i^k(a),$$

and consequently

$$|\epsilon_{k,i}| \leq \kappa_i q_i^k(a), \tag{3.6}$$

where

$$\kappa_i := \frac{\sqrt{1-|a|^2}}{|c_i|}\sum_{j=1,j\neq i}\frac{|c_j|}{|1-\overline{a}\overline{\lambda}_j|}.$$

Thus by (3.5) and (3.6) we get

$$\left|\frac{f_{k+1}(a)}{f_k(a)} - B_a(\overline{\lambda}_i)\right| = |B_a(\overline{\lambda}_i)|\left|1 - \frac{1+\epsilon k+1, i}{1+\epsilon_{k,i}}\right| \leq$$

$$\leq \frac{|\epsilon_{k+1,i}| + |\epsilon_{k,i}|}{1-|\epsilon_{k,i}|} \leq \frac{2\kappa_i}{1-\kappa_i q_i^k(a)}\, q_i^k(a)$$

that concludes in an estimation of the the convergence rate as it is stated in the corollary as follows:

Corollary 2. *If the multiplicity of every eigenvalue is* 1 *then in the case if $a \in D_i$ and $c_i \neq 0$ we have*

$$\left|\frac{f_{k+1}(a)}{f_k(a)} - B_a(\overline{\lambda}_i)\right| = O(q_i^k(a)) \quad (k \to \infty).$$

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