Mathematica Pannonica
23/1 (2012), 125–134

RINGS IN WHICH EVERY INFINITE SUBSET CONTAINS A PAIR OF ELEMENTS WITH ZERO PRODUCT

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Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

Received: October 2011

MSC 2010: 16 N 40

Keywords: Ring, infinite subset, zero product.

Abstract: B. H. Neumann has shown that every infinite subset of a group G contains a pair of commuting elements if and only if G is finite modulo its centre. Here we consider, analogously, the rings in which each infinite subset contains distinct elements x, y with $xy = 0 = yx$. We show that the rings in question are those which are finite modulo their annihilators provided that they also satisfy the identity $x^2 \approx 0$, which many (and perhaps all) do.

1. Introduction

We consider the following property for rings. (Except in Sec. 4, we mean associative rings.)

$(*)$ *In every infinite subset there exist distinct* a, b *with* $ab = 0$.

Neumann [2] characterized the groups with the property

$(*)_g$ *In every infinite subset there are two commuting elements.*

Though Bell, Klein and Kappe [1] have characterized the rings with the property

$(*)_c$ *In each infinite subset contains a pair of commuting elements*, it seems that in view of the close parallels between parts of ring theory and parts of group theory based on the product and the commutator, there is merit in an investigation of $(*)$.

The groups satisfying $(*)_g$ are the centre by finite ones. One might therefore expect a connection between rings with $(*)$ and rings which are finite modulo their annihilators. These properties do indeed coincide for rings satisfying $x^2 \approx 0$ and the presence of this identity permits a reasonably straightforward adaptation of the group argument. On the other hand we are able to show that rings A with $(*)$ do satisfy $x^2 \approx 0$ except when $A = T_{p_1}(A) \oplus T_{p_2}(A) \oplus \cdots \oplus T_{p_k}(A)$, where $p_1, p_2, \cdots, p_k$ are primes, each $T_{p_i}(A)$ is the maximum p_i-ideal, only one $T_{p_i}(A)$ is infinite and that is not an algebra over an infinite field. This case remains open.

Rings with $(*)$ are nil, and by using the circle operation we can deduce our result from that of Neumann in the case of 2-torsion-free rings. In the final section we briefly consider non-associative rings and deduce the characterisation of rings with $(*)_c$ [1] from our results.

Our condition $(*)$ has a formulation in terms of graphs. We define a directed graph on a ring by taking all its elements as vertices and defining an edge from a to b if and only if $ab \neq 0$. For rings in which $ab = 0$ implies $ba = 0$ we can treat this as an undirected graph and then $(*)$ is equivalent to the statement that the graph of the ring has no infinite complete subgraph (i.e. no infinite subgraph in which every two vertices are joined by an edge). In the group case the graph, being based on commutation, is undirected and the graph version of $(*)_g$ provided the motivation for Erdős, who posed the question answered in [2]. (Graphs play no real part in our results, being mentioned only in connection with Ramsey's Theorem in the proof of **2.1**.) We attempt an argument by analogy with that in [2] to characterize the rings with $(*)$, but get diverted to a different, related question.

While in groups we always have $[x, x] = e$, the analogous ring condition $x^2 \approx 0$, which would allow a relatively straightforward translation of the argument in [2], is of course rather restrictive. Nevertheless, we find that in many cases $(*)$ requires $x^2 \approx 0$. (If this identity is satisfied, then $0 = (a + b)^2 = a^2 + ab + ba + b^2 = ab + ba$ so $ab = -ba$ for all a, b and thus $ab = 0$ implies $ba = 0$ so as noted above we can deal with undirected graphs.)

2. Imitating the group argument

It may seem strange, but we prefer to mimic the group argument first, with the (extravagant) assumption of the identity $x^2 \approx 0$, and use the conclusions reached as motivation for exploring the extent to which $x^2 \approx 0$ *is* valid in the rings in which we are interested. Until further notice, R **is always an infinite (associative) ring satisfying the identity** $x^2 \approx 0$.

We denote by $(0 : a), (0 : S)$ the (two-sided) annihilator of an element a or a subset S, respectively, of a ring, and note that under our current assumptions these coincide with the one-sided versions.

Proposition 2.1. (Cf. [2], Lemma 1.) *If R satisfies $(*)$, then Ra is finite for all $a \in R$.*

Proof. Suppose Ra is infinite for some $a \in R$. Then $R/(0 : a) \cong Ra$ (module isomorphism). Let T be a set of representatives of the cosets of $(0 : a)$. Then T is infinite and for $t, s \in T$, we have $ta = sa$ if and only if $t = s$. The subgraph of the graph of R with vertex set T has no infinite complete subgraph (as the whole graph has none). But then by Ramsey's Theorem (see, for example, [3], Th. A, with $r = 2 = \mu$, C_1 the set of two-element sets $\{x, y\}$ with $xy = 0$, C_2 the set of $\{x, y\}$ with $xy \neq 0$) the graph of T has an infinite independent subset, i.e. an infinite set U of vertices of which no two are joined by an edge. this means that $uv = 0 (= vu)$ for all distinct $u, v \in U$. Now $\{a + u : u \in U\}$ is an infinite subset of R but for all distinct $u, v \in U$ we have

$$(a + u)(a + v) = a^2 + av + ua + uv = av - au \neq 0.$$

This contradicts $(*)$. $\Diamond$

Since (here we mean the additive subgroup index) $[R : (0 : a)] = = |R/(0 : a)| = |Ra|$, we see that Ra is finite for all $a \in R$ if and only if each $(0 : a)$ has finite index and this in turn is equivalent to $(0 : S)$ having finite index for each finite subset S of R.

Proposition 2.2. (Cf. [2], Lemma 2.) *If Ra is finite for every $a \in R$, and if R has a subring A with $A^2 = 0$ and $[R : A]$ finite, then $[R : (0 : R)]$ is finite.*

Proof. Let $b_1, b_2, \ldots, b_n$ be representatives of the (group) cosets of A in R. Then

$$R = (b_1 + A)\dot\cup(b_2 + A)\dot\cup \ldots \dot\cup(b_n + A)$$

so R is generated (additively) by $A \cup \{b_1, b_2, \dots, b_n\}$ and thus $(0 : R) = = (0 : A) \cap (0 : \{b_1, b_2, \dots, b_n\})$. Since $A^2 = 0$ we have $A \subseteq (0 : A)$. As $[R : A]$ is finite, so therefore is $[R : (0 : A)]$. By our remark above,

$$[R : (0 : \{b_1, b_2, \dots, b_n\})]$$

is finite too, whence $[R : (0 : R)]$ is finite. $\Diamond$

It is worthwhile to note explicitly that **2.2** gives us

Proposition 2.3. *If Ra is finite for all $a \in R$ and $(0 : R)$ has infinite index in R, then if a subring A has finite index in R we have $A^2 \neq 0$.*

Proposition 2.4. (Cf. [2], Cor. 5.) *If Ra is finite for all $a \in R$ and $[R : (0 : R)]$ is infinite, then R does not satisfy $(*)$.*

Proof. Since $(0 : R)$ has infinite index in R, certainly $R^2 \neq 0$. Let $c, d \in R$ be such that $cd \neq 0$. Suppose, for some k, there exist $a_1, a_2, ..., a_k, b_1, b_2, ... \dots, b_k \in R$ such that

(i) $a_i a_j \neq 0$ for $i \neq j$,
(ii) $a_i b_j = 0$ for $i \neq j$,
(iii) $a_i b_i \neq 0$ for all i and
(iv) $b_i b_j = 0$ for all i, j.

Note that c, d ensure that the conditions are met for $k = 1$.

Let $A = (0 : \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k\})$. Then $[R : A]$ is finite so by 2.3, $A^2 \neq 0$. Let $a, b \in A$ be such that $ab \neq 0$ Then let

$$a_{k+1} = a + b_1 + b_2 + \cdots + b_k, \qquad b_{k+1} = b.$$

We shall prove that $a_1, a_2, \dots, a_k, a_{k+1}, b_1, b_2, \dots, b_k, b_{k+1}$ satisfy (i)–(iv).

For $i = 1, 2, \dots, k$ we have

$$a_i a_{k+1} = a_i a + a_i b_1 + a_i b_2 + \cdots + a_i b_k = 0 + a_i b_i = a_i b_i \neq 0,$$

so $a_{k+1} a_i \neq 0$. (Anticommutativity and $aa_i = 0$ give us $a_i a = 0$ and then $a_i a_{k+1} \neq 0$ implies $a_{k+1} a_i \neq 0$.) For $i = 1, 2, \dots, k, a_i b_{k+1} = a_i b = 0$ (as $b \in A$) and $a_{k+1} b_i = ab_i + b_1 b_i + b_2 b_i + \cdots + b_k b_i = ab_i + 0 = 0$. This gives us (i) and (ii). Since $a_{k+1} b_{k+1} = ab + b_1 b + b_2 b + \cdots + b_k b = ab \neq 0$, we have (iii), and finally, $b_{k+1} b_i = bb_i = 0 (= b_i b_{k+1})$ for $i = 1, 2, \dots, k$, so (iv) holds too.

By induction there is an infinite set $\{a_n : n = 1, 2, 3, \dots\}$ for which $a_i a_j \neq 0$ whenever $i \neq j$. Thus R does not satisfy $(*)$. $\Diamond$

This gives us the main result.

Proposition 2.5. (Cf. [2], Th. 6.) *Let R be a ring satisfying the identity $x^2 \approx 0$. Then R has $(*)$ if and only if $[R : (0 : R)]$ is finite.*

Proof. If R satisfies $(*)$, then by 2.1 and 2.4, $[R : (0 : R)]$ is finite. Conversely, if $[R : (0 : R)]$ is finite and S is an infinite subset of R, then there exist distinct $c, d \in S$ with $c + (0 : R) = d + (0 : R)$, i.e. $c - d \in (0 : R)$. Then we have $0 = (c - d)d = cd - d^2 = cd$. $\Diamond$

We now drop the assumption that our rings satisfy the identity $x^2 \approx 0$. Both the identity and finiteness of $R/(0 : R)$ are very restrictive.

Proposition 2.6. *If $R/(0 : R)$ is finite and R is either*

(i) *additively torsion-free or*

(ii) *an algebra over an infinite field K,*

then $R^2 = 0$.

Proof. (i) If $n \in \mathbb{Z}$ and $na \in (0 : R)$ then for all $b \in R$ we have $n(ab) = (na)b = 0$ and $n(ba) = b(na) = 0$ so $ab = ba = 0$ and thus $a \in (0 : R)$. It follows that $R/(0 : R)$ is torsion-free and hence zero.

(ii) If $a \in (0 : R)$, $\alpha \in K$, then for every $b \in R$ we have $(\alpha a)b = = \alpha(ab) = 0$ and $b(\alpha a) = \alpha(ba) = 0$. This means that $(0 : R)$ is a K-ideal, whence $R/(0 : R)$ is a finite K-algebra and therefore zero. $\Diamond$

Corollary 2.7. *If $R/(0 : R)$ is finite and R is additively torsion-free or an algebra over an infinite field, then R satisfies both $(*)$ and the identity $x^2 \approx 0$.*

From the above discussion we see that for an infinite ring R, $(*)$ and $x^2 \approx 0$ jointly imply that $R/(0 : R)$ is finite, while $x^2 \approx 0$ and finiteness of $R/(0 : R)$ give $(*)$. The following result completes the pattern.

Proposition 2.8. *If R satisfies $(*)$ and $(0 : R)$ is infinite then R satisfies the identity $x^2 \approx 0$.*

Proof. For each $a \in R$ the coset $a+(0 : R)$ is infinite, so there are distinct elements x, y of $(0 : R)$ such that $0 = (a+x)(a+y) = a^2+ay+xa+xy = = a^2$. $\Diamond$

Generally speaking, the identity $x^2 \approx 0$ is very restrictive, but how much of a restriction is it as far as rings with $(*)$ are concerned? We shall examine this question in the next section.

3. Does $(*)$ imply $x^2 \approx 0$ for infinite rings?

In this section, unless otherwise indicated, all rings are (associative and) infinite.

Proposition 3.1. (i) *If A has $(*)$ and $I \lhd A$, then A/I has $(*)$.*

(ii) *If A has $(*)$, then so does every subring.*

Proof. If $S = \{a_\lambda + I : \lambda \in \Lambda\}$ is infinite, then so is $\{a_\lambda : \lambda \in \Lambda\}$, so $a_\lambda a_\mu = 0$ for some λ, μ. But then $(a_\lambda + I)(a_\mu + I) = 0$. $\Diamond$

Proposition 3.2. *If A has $(*)$ and I is an infinite ideal of A, then $a^2 \in I$ for every $a \in A$ (so A/I satisfies $x^2 \approx 0$).*

Proof. For $a \in A$, $a + I$ is infinite, so there exist $i, j \in I$ such that $0 = (a+i)(a+j) = a^2 + aj + ia + ij$, whence $a^2 = -aj - ia - ij \in I$. $\Diamond$

We denote the *(additive) order of a ring element* a by $0(a)$.

Proposition 3.3. *If A has $(*)$, $a \in A$ and $0(a)$ is infinite, then $a^2 = 0$.*

Proof. Let p, q be distinct primes, $S = \{p^n a : n \in \mathbb{Z}^+\}$, $T = \{q^n a : n \in \mathbb{Z}^+\}$. Then S and T are infinite so there exist $m, n, k, l \in \mathbb{Z}^+$ such that $p^{m+n}a^2 = p^m a \cdot p^n a = 0 = q^k a \cdot q^l a = q^{k+l}a^2$. But then there are integers r, s for which $rp^{m+n} + sq^{k+l} = 1$ and we have $a^2 = rp^{m+n}a^2 + sq^{k+l}a^2 = 0$. $\Diamond$

Proposition 3.4. *If A has $(*)$ and is additively torsion-free, then $A^2 = 0$.*

Proof. Let a, b be in A. If m and n are distinct integers, then $mb \neq nb$, so $a + mb \neq a + nb$. Thus $\{a + nb : n \in \mathbb{Z}\}$ is infinite, so there exist m, n with $(a+mb)(a+nb) = 0$. Thus we have

$$0 = (a+mb)(a+nb) = a^2 + nab + mba + mnb^2 = nab + mba$$

by **3.3**. But also by **3.3** we have

$$0 = (a+b)^2 = a^2 + ab + ba + b^2 = ab + ba,$$

so $0 = nab + mba = nab - mab = (n-m)ab$ and then $ab = 0$. $\Diamond$

We denote the *torsion ideal* of a ring A by $T(A)$, i.e. $T(A) = \{a \in A : 0(a) < \infty\}$.

Proposition 3.5. *If A has $(*)$ then every element of A^2 has finite order.*

Proof. By **3.1** $A/T(A)$ has $(*)$ so by **3.4** $(A/T)^2 = 0$, i.e. $A^2 \subseteq T(A)$. $\Diamond$

Proposition 3.6. *If A has $(*)$ then A is nil.*

Proof. First consider a primitive ring B, which we can suppose is a dense ring of linear transformations of a vector space V over a division ring Δ. If V has an infinite linearly independent set $\{e_1, e_2, \ldots, e_n, \ldots\}$, then for each n there is an $f_n \in B$ with

$$f_n(e_1) = e_1; f_n(e_2) = e_n.$$

Thus $\{f_n : n = 1, 2, \ldots\}$ is infinite, but for all m, n we have $f_n f_m(e_1) = f_n(e_1) = e_1$ so $f_n f_m \neq 0$, so B does not have $(*)$. We conclude that any primitive ring with $(*)$ must be a matrix ring over a division ring. Now we go back to the ring A with $(*)$. By the above argument (and

3.1) any primitive homomorphic image of A must be a full matrix ring $M_k(\Delta)$ over a division ring Δ. By **3.1**, Δ also has $(*)$ so clearly Δ is finite, whence $M_k(\Delta)$ is too. Since then every primitive homomorphic image of A is finite, every primitive *ideal* is infinite. By **3.2** it follows that $a^2 \in J(A)$, the Jacobson radical of A, for all $a \in A$.

Let us suppose A contains a non-nilpotent element a. Then $0 \notin \{a^n : n = 1, 2, \dots\}$ so $a^m a^l \neq 0$ for all m, l and thus $\{a^n : n = 1, 2, \dots\}$ is finite. Hence $a^m = a^{m+t}$ for some m, t. Then $a^m = a^{m+t} = a^m a^t = a^{m+t}a^t = a^m a^{2t}$. Now $-a^{2t} = -(a^2)^t \in J(A)$, so there exists c in $J(A)$ such that $c - a^{2t} - ca^{2t} = 0$. But then $ca^m - a^{2t+m} - ca^{2t+m} = 0$, i.e. $ca^m - a^m - ca^m = 0$, so $a^m = 0$, a contradiction. $\Diamond$

We now examine elements of finite order. We adapt some terminology from abelian groups to rings. For a prime p, a ring is a *p-ring* if each of its elements has p power order. For a ring A, $T_p(A)$ denotes the ideal $\{a \in A : 0(a) \text{ is a power of } p\}$.

Proposition 3.7. *If A has $(*)$ and is an algebra over an infinite field K of prime characteristic p, then A satisfies $x^2 \approx 0$.*

(Note that the case of characteristic 0 is covered by **3.4**.)

Proof. For $a \in A$, $a \neq 0$, the set $(K \setminus \{0\})a$ is infinite so for some $u, v \in K \setminus \{0\}$ we have $uva^2 = ua \cdot va = 0$ so $a^2 = 0$. $\Diamond$

Proposition 3.8. *If A has $(*)$ and $T_p(A), T_q(A)$ are infinite for two distinct primes p, q, then A satisfies $x^2 \approx 0$.*

Proof. By **3.2** $a^2 \in T_p(A) \cap T_q(A) = 0$ for each $a \in A$. $\Diamond$

Proposition 3.9. *If A has $(*)$ and $T_p(A) \neq 0$ for infinitely many primes p, then A satisfies $x^2 \approx 0$.*

Proof. Let $T_{p_i}(A) \neq 0$ for $i = 1, 2, \dots$. If $a \in T_{p_1}(A)$, then as $\bigoplus_{i>1} T_{p_i}(A)$ is an infinite ideal, **3.2** says that $a^2 \in T_{p_1}(A) \cap \bigoplus_{i>1} T_{p_i}(A) = 0$. Similarly $a^2 = 0$ if a is in any $T_{p_i}(A)$. A typical element of $T(A)$ has the form $\sum b_i, b_i \in T_{p_i}(A)$ and $\left(\sum b_i\right)^2 = \sum b_i b_j = \sum {b_i}^2 = 0$. If a has infinite order, then $a^2 = 0$ by **3.3**. $\Diamond$

We call a p-ring A *bounded* if $p^n A = 0$ for some n, otherwise we say the ring is *unbounded*. (This is derived from abelian group theory too.)

Proposition 3.10. *If A has $(*)$, $A \neq T(A)$ and $T_p(A)$ is bounded and infinite for some prime p, then A satisfies $x^2 \approx 0$.*

Proof. By **3.8** and **3.9** we can assume that apart from p there are only finitely many primes $p_1, p_2, \ldots, p_n$ for which $T_{p_i}(A) \neq 0$ and that each such p_i, $T_{p_i}(A)$ is finite, and hence bounded. But then there exists an integer m such that $mT(A) = 0$. Since $T(A) \cap mA = mT(A) = 0$ and $T(A)$ and mA are infinite ideals, **3.2** implies that $a^2 = 0$ for every $a \in A$. $\Diamond$

The following result establishes a sort of leading role for p-rings.

Proposition 3.11. *Let A be a ring with $(*)$. If there is a prime p such that $T_p(A)$ is infinite, then A satisfies $x^2 \approx 0$ if and only if $T_p(A)$ satisfies $x^2 \approx 0$.*

Proof. Suppose $T_p(A)$ satisfies $x^2 \approx 0$. If $0(a)$ is finite, then a has the form $b + a_1 + a_2 + \cdots + a_n$, where $b \in T_p(A)$ and $a_i \in T_{p_i}(A)$ for each i. Now $a^2 = b^2 + \sum {a_i}^2$. By **3.2**, each ${a_i}^2$ is in $T_p(A)$ as well as $T_{p_i}(A)$, so each ${a_i}^2$ is zero. But $b^2 = 0$ so $a^2 = 0$. If a has infinite order then $a^2 = 0$ by **3.3**. The converse is clear. $\Diamond$

Proposition 3.12. *If A is an unbounded p-ring with $(*)$, then A satisfies $x^2 \approx 0$.*

Proof. For each $n \in \mathbf{Z}^+$, $p^n A$ is unbounded and therefore infinite. Let a be in A and let $0(a) = p^{n_1}$. Then there exists $b_1 \in p^{n_1}A$ with $b_1 \neq a$. Let $p^{n_2} = \max\{p^{n_1}, 0(b_1)\} = \max\{0(a), 0(b_1)\}$. Then there exists $b_2 \in p^{n_2}A \setminus \{a, b_1\}$. Let $p^{n_3} = \max\{0(a), 0(b_1), 0(b_2)\}$. Then there exists $b_3 \in p^{n_3}A \setminus \{a, b_1, b_2\}$. If $b_1, b_2, \ldots . b_m$ are chosen, all distinct from each other and from a, with each $b_i \in p^{n_i}A$, where $p^{n_i} = \max\{0(a),\ 0(b_1),\ 0(b_2),\ \ldots,\ 0(b_{i-1})\}$, we can still choose $b_{m+1} \in p^{n_{m+1}}A \setminus \{a, b_1, b_2, \ldots, b_m\}$, where

$$p^{n_{m+1}} = \max\{0(a), 0(b_1), 0(b_2), \ldots, 0(b_m)\}.$$

In this way we get an infinite set $\{b_1, b_2, b_3, \ldots\}$. If $r < s$, then $b_r \in = p^{n_r}A \subseteq p^{n_1}A$ so $b_r a = 0 = ab_r$ (as $0(a) = p^{n_1}$), and similarly $b_s a = 0 = ab_s$. Also $p^{n_s} b_r = 0$ so $b_s b_r = 0 = b_r b_s$. Then $(a + b_s)(a + b_r) = a^2 + ab_r + b_s a + b_s b_r = a^2$. But $\{a + b_n : n = 1, 2, 3, \ldots\}$ is infinite so for some r, s we have $a^2 = (a + b_s)(a + b_r) = 0$. $\Diamond$

We can now prove a result which really puts the spotlight on p-rings.

Proposition 3.13. *If A has $(*)$ and is not a torsion ring, then it satisfies $x^2 \approx 0$.*

Proof. By **3.8**, **3.9** and **3.10**, we only need to consider the case where there is a prime p for which $T_p(A)$ is unbounded, there are finitely many primes $p_1, p_2, \ldots, p_n$ with each $T_{p_i}(A)$ finite and non-zero and $T_q(A) = 0$ for every other prime q. Let A be such a ring, $a \in A$. If $0(a)$ is infinite, then $a^2 = 0$ by **3.3**. If a has finite order, then $a = b + \sum a_i$, where $b \in T_p(A)$ and $a_i \in T_{p_i}(A)$ for each i. Then $a^2 = b^2 + \sum {a_i}^2$. By **3.2** each ${a_i}^2 \in T_p(A)$, so ${a_i}^2 = 0$. Thus $a^2 = b^2$. But by **3.12** (and **3.1**) $b^2 = 0$. $\Diamond$

Thus *the only remaining open case, i.e. the only case where a ring A satisfies $(*)$ but for which the status of $x^2 \approx 0$ is uncertain, is where $A = T_{p_1}(A) \oplus T_{p_2} \oplus \ldots \oplus T_{p_k}(A)$ for finitely many primes $p_1, p_2, \ldots, p_k$ where only one $T_{p_i}(A)$ is infinite and that is (bounded and) not an algebra over an infinite field.*

We note further that if A is an infinite bounded p-ring satisfying $(*)$ then its socle $\mathrm{Soc}(A)$ is an infinite ideal and so by **3.2** $a^2 \in \mathrm{Soc}(A)$ for all $a \in A$ so that A satisfies $px^2 \approx 0$ and hence pA satisfies $x^2 \approx 0$.

By **3.6** every ring with $(*)$ is a group with respect to $\circ$, where $a \circ b = a + b + ab$. If the ring also satisfies $x^2 \approx 0$ and is 2-torsion-free, we have an alternative way of getting **2.5**, by using rather than imitating the theorem of Neumann.

Lemma 3.14. *Let A be a 2-torsion-free ring satisfying the identity $x^2 \approx 0$. The following conditions are equivalent for $a, b \in A$.*

(i) $ab = 0$;
(ii) $ab = 0 = ba$;
(iii) $a \circ b = b \circ a$.

Proof. If $ab = 0$ then $ba = -ab = 0$ and if both these are true then $a \circ b = a + b + ab = a + b = b + a = b + a + ba = b \circ a$. If $a \circ b = b \circ a$ then $ab = a \circ b - (a + b) = b \circ a - (b + a) = ba = -ab$ so $2ab = 0$ and thus $ab = 0$. $\Diamond$

If A is not 2-torsion-free, in particular if $2A = 0$, then from $a \circ b = = b \circ a$ we can only conclude that $2ab = 0$.

Proposition 3.15. ($\subset$ **2.5**) *Let A be a 2-torsion-free ring satisfying the identity $x^2 \approx 0$. Then A has $(*)$ if and only if $[A : (0 : A)]$ is finite.*

Proof. By **3.14** A satisfies $(*)$ if and only if $(A, \circ)$ satisfies $(*)_g$. By Th. 6 of [2], the latter is equivalent to the finiteness of $[(A, \circ) : (Z((A, \circ))]$, and by **3.14** again this is equal to $[A : (0 : A)]$. $\Diamond$

4. Non-associative rings

Associativity of multiplication is not needed in Sec. 2 (nor, apart from **3.6** and **3.7**, in Sec. 3) though as we can't assume that annihilators are anything more than subrings, we can only treat $R/(0 : R)$ etc.

as factor *groups*. Much of the time we can just count cosets instead of mentioning factor structures at all. We have already expressed the summarizing **2.5** in terms of an index rather than a factor ring, while in (i) and (ii) of **2.6** we only need to observe that, respectively, $(0 : R)$ is additively a pure subgroup which therefore produces a torsion-free factor group, and a K-subspace which therefore produces a factor *space*.

If we are dealing with rings which satisfy $x^2 \approx 0$ (e.g. Lie, Mal'tsev) then of course there is "no Sec. 3" and we have the following definitive result.

Theorem 4.1. *In a variety of (not necessarily associative) rings satisfying the identity $x^2 \approx 0$, every infinite subset of a ring R contains two distinct elements with zero product if and only if $[R : (0 : R)]$ is finite.*

Every ring R is also a ring, which we'll call $R^\flat$, with respect to its original addition and the multiplicative commutator $\langle , \rangle$: $\langle a, b\rangle = ab - ba$ (this notation being used to avoid confusion with the group commutator). Moreover, $R^\flat$ satisfies $x^2 \approx 0$. Since $ab = ba$ if and only if $\langle a, b\rangle = 0$, so that in particular $Z(R)$ (ring centre) $= (0 : R^\flat)$, **4.1** has the following consequence (cf. **3.15**.)

Corollary 4.2. *Every infinite subring of a (not necessarily associative) ring R contains a pair of commuting elements if and only if $[R : Z(R)]$ is finite.*

This is Th. 2.1 of [1] without the assumption of associativity; the proof in [1] works for non-associative rings though.

Acknowledgements. I am grateful to Michael Parmenter for interesting discussions of this work. The idea for **4.1** came from a conversation with Venta Terauds, whom I thank.

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