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FACTORING ABELIAN GROUPS WHOSE ORDERS ARE PRODUCTS OF SIX PRIMES

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Abstract: The paper answers a particular case of an open problem which attempts to extend Rédei's theorem on decomposing a finite abelian group into a direct product of its subsets.

1. Introduction

Let G be a finite abelian group written multiplicatively. Let A_1,..., A_n be subsets of G. The product $A_1 \cdots A_n$ is defined to be the set

$$\{a_1 \cdots a_n : \ a_1 \in A_1, \dots, a_n \in A_n\}.$$

The product $A_1 \cdots A_n$ is called direct if

$$a_1 \cdots a_n = a'_1 \cdots a'_n, \quad a_1, a'_1 \in A_1, \dots, a_n, a'_n \in A_n$$

imply that $a_1 = a'_1, \dots, a_n = a'_n$. If the product $A_1 \cdots A_n$ is direct and is equal to G, then we say that the equation $G = A_1 \cdots A_n$ is a factorization of G into the subsets $A_1, \dots, A_n$. A subset A of G is called normalized if $e \in A$, where e is the identity element of G. The factorization

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$G = A_1 \cdots A_n$ is termed normalized if each of its factors is a normalized subset of G.

A subset A of G is defined to be periodic if there is an element $g \in G \setminus \{e\}$ such that $gA = A$. We call the element g a period of A. A factorization $G = A_1 \cdots A_n$ is called periodic if at least one of its factors is a periodic subset of G.

Let a be an element of G and let r be an integer such that $2 \leq \leq |a| \leq r$. Here $|a|$ stands for the order of the element a. We will call the set of elements C in the form

$$e, a, a^2, \ldots, a^{r-1}$$

a cyclic subset of G. Clearly, C is a subgroup of G if and only if $a^r = e$. The element a^r is called the terminating element of the cyclic subset C. In order to solve a long standing open geometry problem, G. Hajós [4] proved the following theorem in 1941.

Theorem 1. *If $G = A_1 \cdots A_n$ is a factorization of the finite abelian group G, where each A_i is a cyclic subset, then the factorization is periodic.*

Further investigations reveal that Hajós's theorem is equivalent to its special case when each cyclic factor has a prime cardinality. This is why the next result of L. Rédei [5] can be considered as a generalization of Hajós's theorem.

Theorem 2. *Let $G = A_1 \cdots A_n$ be a normalized factorization of the finite abelian group G such that each $|A_i|$ is a prime, then the factorization is periodic.*

Let $H = \{h_1, h_2, \ldots, h_s\}$ be a subgroup of G with $h_1 = e$, $s \geq 3$. A subset A of G in form

$$A = \{h_1, h_2, \ldots, h_{s-1}, h_s d\}$$

is called a simulated subset, where d is an element of G such that $h_s d \notin \notin \{h_1, h_2, \ldots, h_{s-1}\}$. In Hajós's theorem simulated subsets may appear beside the cyclic subsets. The following theorem was proved in [11].

Theorem 3. *If $G = A_1 \cdots A_n$ is a factorization of the finite abelian group G, where each A_i is a cyclic subset or a simulated subset, then the factorization is periodic.*

For cyclic groups a more general result than Rédei's theorem holds as it was shown by A. D. Sands [8] in 2004.

Theorem 4. *If $G = A_1 \cdots A_n$ is a factorization of the finite cyclic group G such that each $|A_i|$ is a prime power or a product of two primes, then the factorization is periodic.*

Hajós's theorem admits a similar generalization. Namely, in 2008 A. D. Sands [9] proved the following theorem.

Theorem 5. *Let $G = BA_1 \cdots A_n$ be a normalized factorization of the finite abelian group G such that each A_i is a cyclic subset or a simulated subset of G and $|B| = pq$, where p, q are distinct primes. Then the factorization is periodic.*

An example of [1] shows that the conditions that the A_i factors are all cyclic or simulated cannot be dropped from Th. 5. Motivated by the above results [13] advanced the next problem.

Problem 1. *Let p, q be not necessarily distinct primes and let G be a finite abelian group whose p-component and q-component are cyclic. Suppose that $G = BA_1 \cdots A_n$ is a normalized factorization of G such that $|B| = pq$ and each $|A_i|$ is a prime. Does it imply that the factorization is periodic?*

In [13] it was proved that if the primes p and q are equal, then the answer for Problem 1 is "yes". When the primes p and q are distinct then the answer for Problem 1 is "yes" for $n \leq 3$. This paper will extend the above result for $n = 4$. We spell out this assertion formally as a theorem.

Theorem 6. *Let p, q primes and let G be a finite abelian group with cyclic p-component and with cyclic q-component. Let $G = BA_1 \cdots A_n$ be a normalized factorization of G such that $|B| = pq$ and each $|A_i|$ is a prime. If $n \leq 4$, then the factorization is periodic.*

2. Preliminaries

In this section we present some preliminary results. For easier reference we cite a result of [13].

Theorem 7. *Let p be a prime and let G be a finite abelian group whose p-component is cyclic. Let $G = BA_1 \cdots A_n$ be a normalized factorization of G such that $|B| = p^\alpha$ and each $|A_i|$ is a prime. Then the factorization is periodic.*

Let $G = AB$ be a normalized factorization of G, where $|A| = p$ is a prime. Choose an element $a \in A \setminus \{e\}$ and set

$$C = \{e, a, a^2, \ldots, a^{p-1}\}. \tag{1}$$

By Lemma 3 of [12] in the factorization $G = AB$ the factor A can be replaced by C to get the factorization $G = CB$. Since $|C| = p$ is a prime and since C is normalized, it follows that if C is periodic then it is a subgroup of G and so $a^p = e$. Conversely if $a^p = e$, then C is a subgroup of G and so it is a periodic subset of G. In other words C is periodic if and only if $|a| = p$.

If $|a| = p$ for each $a \in A \setminus \{e\}$, then we say that A is a type 1 subset of G. A type 1 subset A of G can be represented in the form

$$A = \{e, a, a^2\rho_2, \ldots, a^{p-1}\rho_{p-1}\}, \quad |\rho_i| = p.$$

If $|a| \neq p$ for some $a \in A \setminus \{e\}$, then we say that A is a type 2 subset of G. In this case the cyclic subset (1) is not periodic. For a type 2 subset A we distinguish two subtypes.

In a typical situation $|a|$ can be written in the form $|a| = p^\alpha m$, where p does not divide m. Choose an integer t which is not divisible by p and set

$$C' = \{e, a^t, (a^t)^2, \ldots, (a^t)^{p-1}\}.$$

By Prop. 3 of [7] in the factorization $G = CB$ the factor C can be replaced C' to get the factorization $G = C'B$.

If $\alpha \geq 2$, then we can choose t such that $|a^t| = p^\alpha$ and so C' is not a periodic subset of G. In this case we call A a type 2a subset of G.

If $\alpha = 1$, then we can choose t such that $|a^t| = pq$, where q is a prime distinct from p. Now C' is not a periodic subsets of G. In this case we call A a type 2b subset of G.

If $\alpha = 0$, then we can choose t such that $a^t = e$. Now C' is a multiset that contains the identity element e with multiplicity p and so the product $C'B$ cannot form a factorization of G. Therefore the $\alpha = 0$ case cannot arise in connection with factorizations.

We may sum up our considerations in the following way. Suppose that $G = AB$ is a normalized factorization, where $|A| = p$ is a prime. If A is a type 1 subset of G, then in the factorization $G = AB$ we do not replace A. If A is a type 2 subset of G, then in the factorization $G = AB$ we replace A by the non-periodic cyclic subset (1) to get the factorization $G = CB$. If A is a type 2a subset of G, then we may assume that $|a| = p^\alpha$, where $\alpha \geq 2$ holds in (1). If A is a type 2b subset of G, then we may assume that $|a| = pq$ holds in (1), where q is a prime distinct from p.

Assume that $G = BA_1 \cdots A_n$ is a normalized factorization such that each $|A_i| = p_i$ is a prime and each A_i is a non-periodic subset. Set

$$D_i = \begin{cases} A_i, & \text{if} \quad A_i \quad \text{is a type 1 subset,} \\ C_i, & \text{if} \quad A_i \quad \text{is a type 2 subset,} \end{cases}$$

where $a_i \in A_i \setminus \{e\}$

$$\begin{aligned} A_i &= \{e, a_i, a_i^2\rho_{i,2}, \ldots, a_i^{p_i-1}\rho_{i,p_i-1}\}, \quad |\rho_{i,j}| = p_i, \\ C_i &= \{e, a_i, a_i^2, \ldots, a_i^{p_i-1}\}. \end{aligned}$$

In the factorization $G = BA_1 \cdots A_n$ we replace each A_i by D_i to get the factorization $G = BD_1 \cdots D_n$. Since A_i is not periodic we know that D_i is not periodic either. We call the subset D_i the standardized version of A_i.

For a subset A of G the notation $\langle A\rangle$ stands for the smallest subgroup of G that contains A. The subgroup $\langle A\rangle$ is referred to as the span of A in G or as the generatum of A in G. One also may say that A spans or generates the subgroup $\langle A\rangle$ in G. For a subset A and for a character χ of G the notation $\chi(A)$ stands for the sum

$$\sum_{a \in A} \chi(a).$$

Let p, q be distinct primes and let G be a finite abelian group whose p-component and q-component are cyclic. Let $G = BA_1 \cdots A_n$ be a normalized factorization of G, where $|B| = pq$ and each $|A_i|$ is a prime. Suppose that the factors $A_1, \ldots, A_s$ are cyclic and the terminating element of A_i has order p^α, $\alpha \geq 1$ or q^β, $\beta \geq 1$ for each i, $1 \leq i \leq s$. Consider the subgroup $H = \langle A_{s+1} \cup \cdots \cup A_n\rangle$ of G. We will use the next lemma later several times.

Lemma 1. *If $|H| = |A_{s+1}| \cdots |A_n|$, then one of the factors $A_{s+1}, \ldots, A_n$ is periodic. If $|H| = p^\gamma |A_{s+1}| \cdots |A_n|$, $\gamma \geq 1$ or $|H| = q^\delta |A_{s+1}| \cdots |A_n|$, $\delta \geq 1$, then B is periodic.*

Proof. From $|H| = |A_{s+1}| \cdots |A_n|$ it follows that the product $A_{s+1} \cdots A_n$ forms a factorization of H. By Th. 2 at least one of the factors $A_{s+1}, \ldots, A_n$ is periodic. This completes the proof of the first statement of the lemma.

Next assume that $|H| = p^\gamma |A_{s+1}| \cdots |A_n|$, $\gamma \geq 1$ and try to show that B is periodic. Setting $C = BA_1 \cdots A_s$ and $D = A_{s+1} \cdots A_n$ from the factorization $G = BA_1 \cdots A_s A_{s+1} \cdots A_n$ we get the normalized factorization $G = CD$ of G. Choose an element $c \in C$. Multiplying both sides of the factorization $G = CD$ by c^{-1} we get the normalized factorization $G = Gc^{-1} = (Cc^{-1})D$ of G. Restricting the factorization $G = (Cc^{-1})D$ to H we end up with the normalized factorization $H = G \cap H = [(Cc^{-1}) \cap H]D$. It follows that $|H| = |(Cc^{-1}) \cap H||D|$.

Using $|H| = p^\gamma |A_{s+1}| \cdots |A_n| = p^\gamma |D|$ we get that $|(Cc^{-1}) \cap H| = p^\gamma$. From the factorization

$$H = [(Cc^{-1}) \cap H] A_{s+1} \cdots A_n,$$

by Th. 7, we get that at least one of the factors $(Cc^{-1}) \cap H, A_{s+1}, \dots, A_n$ is periodic. Only $(Cc^{-1}) \cap H$ can be periodic.

The p-component of G is cyclic and so it has a unique subgroup K of order p. The elements of $K \setminus \{e\}$ are periods of $(Cc^{-1}) \cap H$. As $(Cc^{-1}) \cap H$ is a normalized subset, $K \subseteq [(Cc^{-1}) \cap H]$ must hold. In particular

$$\bigcap_{c \in C} Cc^{-1} \neq \{e\}.$$

From this, by Lemma 3 of [2], C is periodic.

By Th. 1 of [10], the fact that the elements of $K \setminus \{e\}$ are periods of C can be expressed equivalently using characters of G. Namely, $\chi(K) = 0$ implies $\chi(C) = 0$ for each character χ of G.

Since the q-component of G is cyclic, there is a unique subgroup L of G of order q. We claim that $\chi(K) = 0$, $\chi(L) = 0$ implies $\chi(B) = 0$ for each character χ of G.

In order to prove the claim choose a character χ of G for which $\chi(K) = 0$ and $\chi(L) = 0$. Using $\chi(K) = 0$ we get

$$0 = \chi(C) = \chi(BA_1 \cdots A_s) = \chi(B)\chi(A_1) \cdots \chi(A_s).$$

Thus $\chi(B) = 0$ or $\chi(A_i) = 0$ for some i, $1 \leq i \leq s$. If $\chi(B) = 0$, then there is nothing left to prove. Therefore we may assume that $\chi(A_i) = 0$ for some i, $1 \leq i \leq s$. Set $A_i = \{e, a, a^2, \dots, a^{r-1}\}$. The terminating element of A_i is a^r and we have assumed that the order of a^r is either p^α, $\alpha \geq 1$ or q^β, $\beta \geq 1$. If $|a^r| = p^\alpha$, then set $b = a^{rp^{\alpha-1}}$ and if $|a^r| = q^\beta$, then set $b = a^{rq^{\beta-1}}$.

This means that $b \in K$ or $b \in L$. Note that $\chi(A_i) = 0$ is equivalent to that $\chi(a) \neq 1$ and $\chi(a^r) = 1$. But $\chi(a^r) = 1$ cannot hold as $\chi(K) = 0$ and $\chi(L) = 0$ imply $\chi(b) \neq 1$. This contradiction proves the claim.

By Th. 2 of [10], there are subsets S, T of G such that B can be represented in the form

$$B = SK \cup TL, \tag{2}$$

where the products are direct and the union is disjoint. Considering the cardinalities in (2) we get

$$pq = |B| = |S|p + |T|q.$$

It follows that q divides $|S|$ and p divides $|T|$. Using $|S| \geq 0$, $|T| \geq 0$ we can draw the conclusion that either $S = \emptyset$ and so the elements of $L \setminus \{e\}$ are periods of B or $T = \emptyset$ and so the elements of $K \setminus \{e\}$ are periods of B. $\diamondsuit$

Let $q_1, \dots, q_n$ be (not necessarily distinct) prime powers. The direct product of cyclic subgroups of orders $q_1, \dots, q_n$ respectively must be a commutative group G. We refer to G as a group of type $(q_1, \dots, q_n)$.

Suppose that the answer for Problem 1 is "no". This means that there is a finite abelian group G whose p-component and q-component are cyclic, where p, q are primes. Further there is a normalized factorization $G = BA_1 \cdots A_n$ such that $|B| = pq$, each $|A_i|$ is a prime and none of the factors $B, A_1 \dots, A_n$ is periodic. Let us call such a factorization simply a counter-example.

From Th. 7 we know that in a counter-example p and q must be distinct primes. From Th. 4 we know that in a counter-example G cannot be a cyclic group.

Lemma 2. *In a counter-example the p-component of G is of order p and the q-component of G is of order q.*

Proof. Let $G = BA_1 \cdots A_n$ be a counter-example. Suppose that the p-component of G is of order p^λ and the q-component of G is of order q^μ. We know that $\lambda \geq 1$, $\mu \geq 1$. By symmetry we may assume that $\lambda \geq \mu$. If $\lambda = \mu = 1$, then there is nothing to prove. Thus we may assume that $\lambda \geq 2$.

Let $A_1, \dots, A_s$ be all the factors among $A_1, \dots, A_n$ whose cardinality is equal to p. Clearly, $s = \lambda - 1$.

Note that A_i cannot be a type 1 subset of for each i, $1 \leq i \leq s$. Indeed, if A_i is a type 1 subset of G for some i, $1 \leq i \leq s$, then each element of $A_i \setminus \{e\}$ has order p and consequently A_i is equal to the unique subgroup of G of order p. This is an outright contradiction.

Therefore A_i is a type 2a or type 2b subset of G for each i, $1 \leq i \leq s$. If A_i is a type 2a subset of G, then A_i is a cyclic subset of G in the form

$$A_i = \{e, a_i, a_i^2, \dots, a_i^{p-1}\}, \quad |a_i| = p^{\alpha_i}, \quad \alpha_i \geq 2.$$

We set

$$C_i = \{e, c_i, c_i^2, \dots, c_i^{p-1}\}, \quad a_i = c_i.$$

Plainly, A_i can be replaced by C_i in the factorization $G = BA_1 \cdots A_n$ as $A_i = C_i$. If A_i is a type 2b subset of G, then A_i is a cyclic subset of G in the form

$$A_i = \{e, a_i b_i, (a_i b_i)^2, \ldots, (a_i b_i)^{p-1}\}, \quad |a_i| = p$$

and $|b_i|$ is a prime. In this case we can write $|a_i|$ in the form p^{α_i} with $\alpha_i = 1$. We set

$$C_i = \{e, c_i, c_i^2, \ldots, c_i^{p-1}\}, \quad a_i = c_i.$$

In the factorization $G = BA_1 \cdots A_n$ the factor A_i can be replaced by C_i as there is an integer t relatively prime to p such that $A_i^t = C_i$. Namely, the choice $t = |b_i|$ is suitable.

We claim that the numbers $\alpha_1, \ldots, \alpha_s$ are distinct. In order to prove the claim assume on the contrary that there are i, j, $1 \leq i < j \leq s$ such that $\alpha_i = \alpha_j$. For the sake of definiteness we assume that $i = 1$, $j = 2$.

In the factorization $G = BA_1A_2A_3 \cdots A_n$ we replace the factors A_1, A_2 by C_1, C_2 to get the factorization $G = BC_1C_2A_3 \cdots A_n$. As $|c_1| = |C_2|$, there is an integer t relatively prime to p such that $c_i^t = c_2$. In the factorization $G = BC_1C_2A_3 \cdots A_n$ we replace the factor C_1 by C_1^t to get the factorization $G = BC_1^tC_2A_3 \cdots A_n$. Now $C_1^t = C_2$ and so the product $C_1^tC_2$ cannot be direct. This contradiction proves the claim.

We may assume that $\lambda \geq \alpha_1 > \cdots > \alpha_s \geq 1$. Set $H = \langle A_1 \cup \cup \cdots \cup A_n \rangle$. If $\alpha_1 < \lambda$, then $|G : H| = p^\alpha$, $\alpha \geq 1$. So Lemma 1 is applicable and it gives that the factorization $G = BA_1 \cdots A_n$ is periodic. Thus we may assume that $\alpha_1 = \lambda$. Set $H = \langle A_2 \cup \cdots \cup A_n \rangle$. From the factorization $G = (BA_1)A_2 \cdots A_n$, by Lemma 1, we get that the factorization $G = BA_1 \cdots A_n$ is periodic. (If $|H| = |A_2| \cdots |A_n|$, then one of the factors $A_2, \ldots, A_n$ is periodic. If $|H| \neq |A_2| \cdots |A_n|$, then B is periodic.) $\Diamond$

Let m be an integer such that the answer for Problem 1 is "yes" for each n, $n \leq m - 1$. Let $G = BA_1 \cdots A_m$ be a normalized factorization such that the p-component of G is of order p and the q-component of G of is of order q. Further $|B| = pq$ and each A_i is a non-periodic standardized subset of G of prime cardinality. We assume that A_m is a type 1 subset of G in the form

$$A_m = \{e, a, a^2\rho_2, \ldots, a^{r-1}\rho_{r-1}\},$$

$|\rho_i| = r$ is a prime $r \geq 3$. We may assume that at least one of $\rho_2, \ldots, \rho_{r-1}$ is distinct from e. For the sake of simplicity we assume that $\rho_2 \neq e$. In addition we assume that the r-component of G is an elementary r-group, that is, the r-component of G is of type $(r, \ldots, r)$.

In the factorization $G = BA_1 \cdots A_m$ the factor A_m can be replaced by the subgroup $H = \langle a \rangle$ to get the factorization $G = BA_1 \cdots A_{m-1}H$.

By considering the factor group G/H we get the factorization

$$G/H = [(BH)/H][(A_1H)/H]\cdots[(A_{m-1}H)/H] \tag{3}$$

of the factor group G/H.

Multiplying the factorization $G = BA_1 \cdots A_m$ by a^{-1} we get the factorization $G = ga^{-1} = BA_1 \cdots A_{m-1}(A_m a^{-1})$. In this factorization the factor $(A_m a^{-1})$ can be replaced by the subgroup $K = \langle a\rho_2 \rangle$ to get the factorization $G = BA_1 \cdots A_{m-1}K$. Passing to the factor group G/K gives the factorization

$$G/K = [(BK)/K][(A_1K)/K]\cdots[(A_{m-1}K)/K] \tag{4}$$

of the factor group G/K.

Lemma 3. *Suppose that $(A_iH)/H$ is not a periodic subset of G/H in the factorization* (3) *and $(A_iK)/K$ is not a periodic subset of G/K in the factorization* (4)*. Then in the factorization $G = BA_1 \cdots A_m$ the factor B is periodic.*

Proof. The group G is a direct product of its subgroups L, M, N, where $|L| = pq$, N is the r-component of G and none of the primes p, q, r divides $|M|$. Let x, y be a basis for L with $|x| = p$, $|y| = q$. The element $a \in N$ can be augmented by suitable elements of N to form a basis for N. Similarly, the element $a\rho_2 \in N$ can be augmented by suitable elements to form a basis for N. In fact there are elements $z_1, \ldots, z_{s-1} \in MN$ such that both $z_1, \ldots, z_{s-1}, a$ and $z_1, \ldots, z_{s-1}, a\rho_2$ form a basis for MN. Thus $x, y, z_1, \ldots, z_s$ form a basis for G, where z_s is either a or $a\rho_2$.

Let us choose z_s to be a. Each $b \in B$ can be represented uniquely in the form

$$b = x^i y^j z_1^{\alpha(1,i,j)} \cdots z_s^{\alpha(s,i,j)},$$

where

$$0 \le i \le p-1, \quad 0 \le j \le q-1, \quad 0 \le \alpha(k,i,j) \le |z_k| - 1.$$

The factorization (3) is periodic, by the choice of m and by the assumption of the lemma only $(BH)/H$ can be periodic. We may assume that xH is a period of $(BH)/H$ since this is only a matter of exchanging the roles of the elements x, y. It follows that

$$\alpha(k,0,j) = \alpha(k,1,j) = \cdots = \alpha(k,p-1,j)$$

for each j, $0 \le j \le q-1$ and for each k, $1 \le k \le s-1$. In particular

$$0 = \alpha(k,0,0) = \alpha(k,1,0) = \cdots = \alpha(k,p-1,0)$$

as B is a normalized subset of G.

The element ρ_2 can be represented uniquely in the form

$$\rho_2 = z_1^{\beta(1)} \cdots z_s^{\beta(s)}, \quad 0 \leq \beta(i) \leq |z_i| - 1.$$

Using this b can be written in the form

$$b = x^i y^j \Big[\prod_{k=1}^{s-1} z_k^{\alpha(k,i,j)-\beta(k)\alpha(s,i,j)}\Big] a^{\alpha(s,i,j)-\beta(s)\alpha(s,i,j)} \rho_2^{\alpha(s,i,j)}.$$

From $\rho_2 \neq e$, it follows that one of $\beta(1), \ldots, \beta(s)$ is not zero. For the sake of definiteness we assume that $\beta(1) \neq 0$.

From the factorization (4) it follows that $(BK)/K$ is a periodic subset in G/K. We may assume that either xK or yK is a period of $(BK)/K$. Let us first assume that xK is a period of $(BK)/K$. This implies that

$$\alpha(1,0,j) - \beta(1)\alpha(s,0,j) =$$
$$\alpha(1,1,j) - \beta(1)\alpha(s,1,j) = \cdots = \alpha(1,p-1,j) - \beta(1)\alpha(s,p-1,j).$$

As $\beta(1) \neq 0$ we get

$$\alpha(s,0,j) = \alpha(s,1,j) = \cdots = \alpha(s,p-1,j)$$

and so B is periodic.

Let us assume next that yK is a period of $(BK)/K$ in G/K. This implies that

$$\alpha(1,i,0) - \beta(1)\alpha(s,i,0) =$$
$$\alpha(1,i,1) - \beta(1)\alpha(s,i,1) = \cdots = \alpha(1,i,q-1) - \beta(1)\alpha(s,i,q-1) = 0$$

for each i, $0 \leq i \leq p-1$. In other words

$$0 = \alpha(1,i,j) - \beta(1)\alpha(s,i,j), \quad 0 \leq i \leq p-1, \quad 0 \leq j \leq q-1.$$

From

$$\alpha(1,0,j) = \alpha(1,1,j) = \cdots = \alpha(1,p-1,j)$$

we get

$$\alpha(s,0,j) = \alpha(s,1,j) = \cdots = \alpha(s,p-1,j)$$

and so B is periodic. $\Diamond$

Lemma 4. *In a counter-example for $n = 4$ the type of G can only be one of the following*

$$(p,q,r,r,r,r), \quad (p,q,r^2,r,r), \quad (p,q,r^3,r), \quad (p,q,r^2,r^2),$$
$$(p,q,r,r,r,s), \quad (p,q,r^2,r,s), \quad (p,q,r,r,s,s), \quad (p,q,r,r,s^2),$$
$$(p,q,r,r,s,t),$$

where p, q, r, s, t are distinct primes.

Proof. By Lemma 2 we may assume that p-component of G has order p and the q-component of G has order q. As the order of G is a product of six not necessarily distinct primes, we need all non-cyclic finite abelian group whose order is a product of four not necessarily distinct primes. $\Diamond$

3. A special case

This section is devoted to a very special case of Problem 1. Suppose $G = HK$ is a factorization of the finite abelian group G, where H, K are subgroups of G. Each element $g \in G$ can be represented uniquely in the form

$$g = ab, \quad a \in H, \quad b \in K.$$

The element a will be called the H-part of g and the element b will be referred to as the K-part of g. Suppose p is a prime divisor of $|G|$. If H is a p-group and $|K|$ is not divisible by p, then H is called the p-component of G and K is called the p'-component of G. The H-part of an element $g \in G$ is referred to as the p-part of g and the K-part of g is referred to as the p'-part of g.

Let A be a normalized subset of G such that $|A| = p$ is a prime. The height of A is defined to be the product of the orders of the p'-parts of the elements of A. Let $A_1, \ldots, A_n$ be normalized subsets of G with prime cardinalities. The height of a factorization $G = BA_1 \cdots A_n$ is defined to be the product of the heights of the factors $A_1, \ldots, A_n$.

Theorem 8. *Let G be a group of type $(p, q, r, \ldots, r)$, where p, q, r are distinct primes. Let $G = BA_1 \cdots A_n$ be a normalized factorization of G such that $|B| = pq$, $|A_i| = r$ for each i, $1 \leq i \leq n$. Then at least one of the factors $B, A_1, \ldots, A_n$ is periodic.*

Proof. We divide the proof into four steps.

Step (1): Suppose there is a counter-example $G = BA_1 \cdots A_n$. We choose a counter-example with minimal n. For a fixed n we choose one with a minimal height. Let $x, y, u_1, \ldots, u_n$ be basis elements of G with $|x| = p$, $|y| = q$, $|u_1| = \cdots = |u_n| = r$. Set $L = \langle u_1, \ldots, u_n \rangle$. Let A_i' be the set of the L-parts of the elements of A_i. It is a corollary of Prop. 3 of [7] that in the factorization $G = BA_1 \cdots A_n$ each A_i can be replaced by A_i' to get the normalized factorization $G = BA_1' \cdots A_n'$. In particular the product $A_1' \cdots A_n'$ is direct. The cardinalities give that $L = A_1' \cdots A_n'$ is a factorization of L. Thus $G = BL$ is a normalized factorization of G. Therefore B is a complete set of representatives in G modulo L. The elements of B are in the form

$$x^i y^j l_{i,j}, \quad l_{i,j} \in L, \quad 1 \leq i \leq p-1, \quad 1 \leq j \leq q-1.$$

Step (2): If $A_i \subseteq L$ for each i, $1 \leq i \leq n$, then $L = A_1 \cdots A_n$ is a normalized factorization of L and by Th. 2, one of the factors $A_1, \ldots, A_n$ is periodic. This is a contradiction. Thus $A_i \not\subseteq L$ for some i, $1 \leq i \leq n$, say $A_1 \not\subseteq L$. There is an element $a \in A_1$ whose p-part or q-part is not e. Set $A_1' = \{e, a, a^2, \ldots, a^{r-1}\}$. By Lemma 3 of [12], in the factorization $G = BA_1 \cdots A_n$ the factor A_1 can be replaced by A_1' to get the normalized factorization $G = BA_1'A_2 \cdots A_n$. The element a can be represented in the form $a = a_1 d_1$, where $|a_1| = r$, $|d_1| \in \{p, q\}$. Set $A_1'' = \{e, a_1, a_1^2, \ldots, a_1^{r-2}, a_1^{r-1}d_1\}$. By Lemma 2 of [11], in the factorization $G = BA_1'A_2 \cdots A_n$ the factor A_1' can be replaced by A_1'' to get the factorization $G = BA_1''A_2 \cdots A_n$. In general if $A_i \not\subseteq L$, then A_i can be replaced by a non-periodic simulated subset. We assume that in the starting counter-example these replacements have already been done. We call a factor A_i a type α factor if $A_i \not\subseteq L$ and we call A_i a type β factor if $A_i \subseteq L$.

Step (3): If each A_i is a type α factor, then by Th. 5, one of the factors $B, A_1, \ldots, A_n$ is periodic. This is a contradiction and so there are type β factors among $A_1, \ldots, A_n$. We may assume that $A_1, \ldots, A_s$ type α factors and $A_{s+1}, \ldots, A_n$ are type β factors. If $r = 2$, then a type β factor is obviously a subgroup and consequently it is periodic. Thus in a counter-example $r \geq 3$ must hold. A type β factor A_i can be expressed in the form

$$A_i = \{e, a_i, a_i^2 d_{i,2}, \ldots, a_i^{r-1} d_{i,r-1}\},$$

where $d_{i,j} \notin \langle a_i \rangle$. We also use the representation $A_i = \{a_{i,0}, \ldots, a_{i,r-1}\}$ for A_i, where

$$a_{i,0} = e,\ a_{i,1} = a_i,\ a_{i,2} = a_i d_{i,2}, \ldots, a_{i,r-1} = a_i d_{i,r-1}.$$

For notational convenience temporarily we introduce the notation $A_0 = B$. By Lemma 3 of [11], in the factorization $G = A_0 \cdots A_n$ the factor A_n can be replaced by $H_{n,k,m} = \langle a_{n,k} a_{n,m}^{-1} \rangle$ to get the normalized factorization $G = A_0 \cdots A_{n-1} H_{n,k,m}$ for each k, m, $k \neq m$. Considering the factor group $G/H_{n,k,m}$ we get the normalized factorization

$$G/H_{n,k,m} = (A_0/H_{n,k,m})/H_{n,k,m} \cdots (A_{n-1}H_{n,k,m})/H_{n,k,m}.$$

The minimality of n in the counter-example gives that one of the factors $(A_i/H_{n,k,m})/H_{n,k,m}$ is periodic. In the $i \neq 0$ case $A_i H_{n,k,m}$ is a subgroup of G. In the $1 \leq i \leq s$ case $d_i \in H_{n,k,m} \subseteq L$ follows. This is a contradiction. Thus $(A_i/H_{n,k,m})/H_{n,k,m}$ can be periodic only in the $i \in \{0, s+1, \ldots, n\}$ case. In other words for each i, $s+1 \leq i \leq n$ there is an $f(i,k,m) \in \{0, s+1, \ldots, n\}$ such that $(A_{f(i,k,m)} H_{i,k,m})/H_{i,k,m}$ is periodic. We record this data by constructing a graph Γ on the nodes $\{0, s+1, \ldots, n\}$. For each i, k, m we draw an directed edge from i to $f(i,k,m)$.

If for each i, $s+1 \le i \le n$ there are k, m such that $f(i,k,m) \in \in \{s+1,\dots,n\}$, then Γ contains a cycle. Let $\Omega \subseteq \{s+1,\dots,n\}$ be the nodes of this cycle. Note that the product $\prod_{i\in\Omega} A_i$ forms a factorization of the group $\prod_{i\in\Omega}\langle a_i\rangle$. By Th. 2, at least one of the factors A_i, $i \in \Omega$ is periodic. This is a contradiction. Thus there is an i, $s+1 \le i \le n$ such that $(BH_{i,k,m})/H_{i,k,m}$ is periodic for each possible choice of k, m. We assume that $i = n$. The elements of $(BH_{n,k,m})/H_{n,k,m}$ are the following

$$x^i y^j l'_{i,j} H_{n,k,m},\quad l'_{i,j} \in \langle a_1,\dots,a_{n-1}\rangle,\quad 0 \le i \le p-1,\quad 0 \le j \le q-1.$$

This set is periodic with period $xH_{n,k,m}$ or $yH_{n,k,m}$.

Step (4): Suppose first that $(BH_{n,k,m})/H_{n,k,m}$ is periodic with period $xH_{n,k,m}$. It follows that

$$l'_{0,j} = l'_{1,j} = \cdots = l'_{p-1,j}$$

for each j, $0 \le j \le q-1$. Let l'_j be the common value. Therefore the elements of B are the following

$$x^i y^j l'_{i,j} a_n^{\beta(i,j)},\quad l'_{i,j} \in \langle a_1,\dots,a_{n-1}\rangle,\quad 0 \le \beta(i,j) \le r-1.$$

Here we set $k = 1$, $m = 0$ and used the representation

$$A_n = \{e, a_n, a_n^2 d_{n,2},\dots,a_n^{r-1} d_{n,r-1}\}$$

of A_n. One of $d_{n,2},\dots,d_{n,r-1}$ is not equal to e, since otherwise A_n is periodic. We may assume that $d_{n,r-1} \ne e$ as A_n can be replaced by A_n^t for each integer t that is relatively prime to r. (A little reflection will convince the reader that replacing A_n by A_n^t is not changing the family of subsets $H_{n,k,m}$ originally assigned to A_n.) Plainly $d_{n,r-1} \in \in \langle a_1,\dots,a_{n-1}\rangle\backslash\{e\}$. For notational simplicity temporarily set $d = d_{n,r-1}$. There is a γ, $1 \le \gamma \le r-1$ for which $a^\gamma \in A_n^\gamma$ holds. In the factorization $G = BA_1\cdots A_n$ the factor A_n can be replaced by $H = \langle d^\gamma a_n\rangle$ to get the normalized factorization $G = BA_1\cdots A_{n-1}H$. In the factor group G/H the factor $(BH)/H$ must be periodic because of the choice of A_n. One can write the elements of B in the following form

$$x^i y^j l'_j d^{-\gamma\beta(i,j)} (d^\gamma a_n)^{\beta(i,j)}.$$

Here $l'_j d^{-\gamma\beta(i,j)} \in \langle a_1,\dots,a_{n-1}\rangle$ and $(d^\gamma a_n)^{\beta(i,j)} \in H$. If $(BH)/H$ is periodic with period xH, then it follows that

$$l'_j d^{-\gamma\beta(0,j)} = l'_j d^{-\gamma\beta(1,j)} = \cdots = l'_j d^{-\gamma\beta(p-1,j)}$$

for each j, $0 \le j \le q-1$. Therefore

$$\beta(0,j) = \beta(1,j) = \cdots = \beta(p-1,j).$$

It implies that B is periodic with period x.

To finish the proof note that A_n can be replaced by $H_{n,k,m}$ such that there are at least three distinct among the subgroups $H_{n,k,m}$ as $r \geq 3$. By the pigeon-hole principle there are at least two choices of the k, m values for which $(BH_{n,k,m})/H_{n,k,m}$ is periodic with period $xH_{n,k,m}$ or there are at least two choices of the k, m values for which $(BH_{n,k,m})/H_{n,k,m}$ is periodic with period $yH_{n,k,m}$. For the sake of definiteness suppose that the first possibility occurs. We then carry out the argument above with these particular choices of k and m.

This completes the proof. $\diamondsuit$

4. Eight propositions

In this long section we deal with the eight group types left open by Lemma 4 and Th. 8.

Proposition 1. *Let G be a group of type (p, q, r, r, s, t), where p, q, r, s, t are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = r$, $|A_3| = s$, $|A_4| = t$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_4 = A_4$, then each element of $A_4 \setminus \{e\}$ has order t and so A_4 is equal to the unique subgroup of G of order t. This gives the contradiction that A_4 is periodic. Thus we may assume that $D_4 = C_4$. A similar argument shows that we may assume that $D_3 = C_3$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_1 = A_1$.

If $D_2 = A_2$, then the product A_1A_2 forms a factorization of the r-component of G which is a group of type (r, r). By Th. 2, either A_1 or A_2 is periodic. Thus we may assume that $D_1 = A_1$, $D_2 = C_2$, $D_3 = C_3$, $D_4 = C_4$. The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

Table 1: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 1

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\|a_1\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp
$\|a_3\|$	sp	sp	sp	sp	sq	sq	sq	sq	sr	sr	sr	sr	st	st	st	st
$\|a_4\|$	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts
Case	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\|a_1\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq
$\|a_3\|$	sp	sp	sp	sp	sq	sq	sq	sq	sr	sr	sr	sr	st	st	st	st
$\|a_4\|$	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts
Case	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
$\|a_1\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs	rs
$\|a_3\|$	sp	sp	sp	sp	sq	sq	sq	sq	sr	sr	sr	sr	st	st	st	st
$\|a_4\|$	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts
Case	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
$\|a_1\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt	rt
$\|a_3\|$	sp	sp	sp	sp	sq	sq	sq	sq	sr	sr	sr	sr	st	st	st	st
$\|a_4\|$	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts	tp	tq	tr	ts

$$
\begin{aligned}
|a_1| &\in \{r\},\\
|a_2| &\in \{rp, rq, rs, rt\},\\
|a_3| &\in \{sp, sq, sr, st\},\\
|a_4| &\in \{tp, tq, tr, ts\}.
\end{aligned}
$$

This leaves 64 cases to consider. These are depicted in Table 1. Set $H = \langle A_1 \cup C_2 \cup C_3 \cup C_4 \rangle$. In case 64 $|H| = |A_1||C_2||C_3||C_4|$ holds. It follows that $H = A_1C_2C_3C_4$ is a factorization. By Th. 2, the factorization is periodic. The same holds in cases 43, 44, 47, 48, 59, 60, 63.

In case 1 $|H| = p|A_1||C_2||C_3||C_4|$ holds. By Lemma 1, it follows that B is periodic. The same holds in cases 1, 3, 4, 9, 11, 12, 13, 15, 16, 33, 35, 36, 41, 45, 49, 51, 52, 57, 61.

In case 32 $|H| = q|A_1||C_2||C_3||C_4|$ holds. By Lemma 1, it follows that B is periodic. The same holds in cases 22, 23, 24, 26, 27, 28, 30, 31, 32, 38, 39, 40, 42, 46, 54, 55, 56, 58, 62.

Set $H = \langle A_1 \cup C_2 \cup C_3 \rangle$. In case 2 $|H| = p|A_1||C_2||C_3|$ holds. From the factorization $G = (BC_4)A_1C_2C_3$, by Lemma 1, it follows that B is periodic. The same applies in cases 10, 34.

In case 21 $|H| = q|A_1||C_2||C_3|$ holds. From the factorization $G = (BC_4)A_1C_2C_3$, by Lemma 1, it follows that B is periodic. The same applies in cases 25, 37.

Set $H = \langle A_1 \cup C_2 \cup C_4 \rangle$. In cases 5, 7, 53 $|H| = p|A_1||C_2||C_4|$. From the factorization $G = (BC_3)A_1C_2C_4$, by Lemma 1, it follows that B is periodic. In cases 18, 19, 50 $|H| = q|A_1||C_2||C_4|$. From the factorization $G = (BC_3)A_1C_2C_4$, by Lemma 1, it follows that B is periodic.

Set $H = \langle A_1 \cup C_2 \rangle$. In case 6 $|H| = p|A_1||C_2||C_4|$ and in case 17 $|H| = q|A_1||C_2||C_4|$. From the factorization $G = (BC_3C_4)A_1C_2$, by Lemma 1, it follows that B is periodic.

In the remaining cases 8, 14, 20, 29 Lemma 3 is applicable with the type 1 factor A_1. $\Diamond$

Proposition 2. *Let G be a group of type (p, q, r^3, r), where p, q, r are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = |A_3| = |A_4| = r$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_4 = A_4$.

If $D_3 = A_3$, then each element of $A_3A_4 \setminus \{e\}$ has order p. Note that G has a unique subgroup of type (r, r). Thus the product A_3A_4 forms a factorization of this subgroup. By Th. 2, either A_3 or A_4 is periodic. Thus we may assume that $D_3 = C_3$. A similar argument shows that we may assume that $D_1 = C_1$, $D_2 = C_2$.

Therefore we may assume that $D_1 = C_1$, $D_2 = C_2$, $D_3 = C_3$, $D_4 = A_4$. The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

$$|a_1| \in \{r^3, r^2, rp, rq\},$$
$$|a_2| \in \{r^3, r^2, rp, rq\},$$
$$|a_3| \in \{r^3, r^2, rp, rq\},$$
$$|a_4| \in \{r\}.$$

Table 2: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 2

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\|a_1\|$	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3	r^3
$\|a_2\|$	r^3	r^3	r^3	r^3	r^2	r^2	r^2	r^2	rp	rp	rp	rp	rq	rq	rq	rq
$\|a_3\|$	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
Case	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\|a_1\|$	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2
$\|a_2\|$	r^3	r^3	r^3	r^3	r^2	r^2	r^2	r^2	rp	rp	rp	rp	rq	rq	rq	rq
$\|a_3\|$	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
Case	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
$\|a_1\|$	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp	rp
$\|a_2\|$	r^3	r^3	r^3	r^3	r^2	r^2	r^2	r^2	rp	rp	rp	rp	rq	rq	rq	rq
$\|a_3\|$	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
Case	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
$\|a_1\|$	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq	rq
$\|a_2\|$	r^3	r^3	r^3	r^3	r^2	r^2	r^2	r^2	rp	rp	rp	rp	rq	rq	rq	rq
$\|a_3\|$	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq	r^3	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r

This leaves 64 cases to consider. These are depicted in Table 2.

Let us consider case 64. In the factorization $G = BC_1C_2C_3A_4$ the factors C_1, C_2, C_3, A_4 can be replaced by $\langle a_1^q\rangle$, $\langle a_2^q\rangle$, $\langle a_3^q\rangle$, $\langle a_4\rangle$ to get the factorization $G = B\langle a_1^q\rangle\langle a_2^q\rangle\langle a_3^q\rangle\langle a_4\rangle$. This shows that the product $\langle a_1^q\rangle\langle a_2^q\rangle\langle a_3^q\rangle\langle a_4\rangle$ is direct. It follows the contradiction that G has a subgroup of type (r, r, r). The same argument applies in cases 11, 12, 15, 16, 27, 28, 31, 32, 35, 36, 40, 41, 42, 43, 44, 45, 46, 47, 48, 51, 52, 55, 56, 57, 58, 59, 60, 61, 62, 63.

Set $H = \langle C_1 \cup C_2 \cup C_3 \cup A_4\rangle$. In cases 1, 2, 5, 6, 17, 18, 21, 22 $|H| = |C_1||C_2||C_3||A_4|$. Thus $H = C_1C_2C_3A_4$ is a factorization and, by Th. 2, the factorization is periodic. In cases 3, 7, 9, 10, 19, 23, 25, 26, 33, 34, 37, 38 $|H| = p|C_1||C_2||C_3||A_4|$. In the factorization $G = BC_1C_2C_3A_4$, by Lemma 1, the factor B is periodic. In cases 4, 8, 13, 14, 20, 24, 29, 30, 49, 50, 53, 54 $|H| = q|C_1||C_2||C_3||A_4|$. In the factorization $G = BC_1C_2C_3A_4$, by Lemma 1, the factor B is periodic. $\Diamond$

Proposition 3. *Let G be a group of type (p, q, r^2, r^2), where p, q, r are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = |A_3| = |A_4| = r$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_4 = A_4$.

If $D_3 = A_3$, then each element of $A_3A_4 \setminus \{e\}$ has order p. Note that G has a unique subgroup of type (r, r). Thus the product A_3A_4 forms a factorization of this subgroup. By Th. 2, either A_3 or A_4 is periodic. Thus we may assume that $D_3 = C_3$. A similar argument shows that we may assume that $D_1 = C_1$, $D_2 = C_2$.

Therefore we may assume that $D_1 = C_1$, $D_2 = C_2$, $D_3 = C_3$, $D_4 = A_4$. The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

$$|a_1| \in \{r^2, rp, rq\},$$
$$|a_2| \in \{r^2, rp, rq\},$$
$$|a_3| \in \{r^2, rp, rq\},$$
$$|a_4| \in \{r\},$$

This leaves 27 cases to consider. These are depicted in Table 3.

Set $H = \langle C_1 \cup C_2 \cup C_3 \cup A_4 \rangle$. In cases 1 C_1, C_2, C_3, A_4 is in the r-component of G. Hence $H = C_1C_2C_2A_4$ is a factorization and by Th. 2 the factorization is periodic.

In cases 2, 4, 5, 10, 11, 13, 14 $|H| = p|C_1||C_2||C_3||A_4|$ and so by Lemma 1, B is periodic. In cases 3, 7, 9, 19, 21, 25, 27 $|H| = q|C_1||C_2||C_3||A_4|$ and so by Lemma 1, B is periodic.

Consider case 6. In the factorization $G = BC_1C_2C_3A_4$ the factors C_2, C_3, A_4 can be replaced by $\langle a_2^p \rangle$, $\langle a_3^q \rangle$, $\langle a_4 \rangle$. From the factorization $G = BC_1\langle a_2^p \rangle\langle a_3^q \rangle\langle a_4 \rangle$ one can draw the conclusion that the product $\langle a_2^p \rangle\langle a_3^q \rangle\langle a_4 \rangle$ is direct. This leads to the contradiction that G has a subgroup of type (r, r, r). A similar argument can be used in all the remaining cases. $\Diamond$

Table 3: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 3

Case	1	2	3	4	5	6	7	8	9
$\|a_1\|$	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2
$\|a_2\|$	r^2	r^2	r^2	rp	rp	rp	rq	rq	rq
$\|a_3\|$	r^2	rp	rq	r^2	rp	rq	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r
Case	10	11	12	13	14	15	16	17	18
$\|a_1\|$	rp	rp	rp	rp	rp	rp	rp	rp	rp
$\|a_2\|$	r^2	r^2	r^2	rp	rp	rp	rq	rq	rq
$\|a_3\|$	r^2	rp	rq	r^2	rp	rq	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r
Case	19	20	21	22	23	24	25	26	27
$\|a_1\|$	rq	rq	rq	rq	rq	rq	rq	rq	rq
$\|a_2\|$	r^2	r^2	r^2	rp	rp	rp	rq	rq	rq
$\|a_3\|$	r^2	rp	rq	r^2	rp	rq	r^2	rp	rq
$\|a_4\|$	r	r	r	r	r	r	r	r	r

Proposition 4. *Let G be a group of type (p, q, r^2, r, r), where p, q, r are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = |A_3| = |A_4| = r$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_4 = A_4$.

If $D_2 = A_2$, $D_3 = A_3$, then each element of $A_2A_3A_4 \setminus \{e\}$ has order p. Note that G has a unique subgroup of type (r, r, r). Thus the product $A_2A_3A_4$ forms a factorization of this subgroup. By Th. 2, one of A_2, A_3, A_4 is periodic. Thus by symmetry we may assume that $D_2 = C_2$.

Therefore we may assume that one of the following situations holds

$$D_1 = C_1, \quad D_2 = C_2, \quad D_3 = A_3, \quad D_4 = A_4,$$
$$D_1 = C_1, \quad D_2 = C_2, \quad D_3 = C_3, \quad D_4 = A_4.$$

Table 4: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 4

Case	1	2	3	4	5	6	7	8	9
$\lvert a_1\rvert$	r^2	r^2	r^2	rp	rp	rp	rq	rq	rq
$\lvert a_2\rvert$	r^2	rp	rq	r^2	rp	rq	r^2	rp	rq
$\lvert a_3\rvert$	r	r	r	r	r	r	r	r	r
$\lvert a_4\rvert$	r	r	r	r	r	r	r	r	r

The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

$$\begin{array}{ll} |a_1| \in \{r^2, rp, rq\}, & |a_1| \in \{r^2, rp, rq\}, \\ |a_2| \in \{r^2, rp, rq\}, & |a_2| \in \{r^2, rp, rq\}, \\ |a_3| \in \{r\}, & |a_3| \in \{r^2, rp, rq\}, \\ |a_4| \in \{r\}, & |a_4| \in \{r\}. \end{array}$$

This leaves 9 and 27 cases to consider, respectively. These are depicted in Table 4 and Table 3.

Let us deal with Table 4 first. Set $H = \langle C_1 \cup C_2 \cup A_3 \cup A_4\rangle$. In case 1 $|H| = |C_1||C_2||A_3||A_4|$ and so $H = C_1C_2A_3A_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 2, 4, 5 $|H| = p|C_1||C_2||A_3||A_4|$. By Lemma 1, B is periodic. In cases 3, 7, 9 $|H| = q|C_1||C_2||A_3||A_4|$. By Lemma 1, B is periodic.

Consider case 6. In the factorization $G = BC_1C_2A_3A_4$ the factors C_1, C_2, A_3, A_4 can be replaced by $\langle a_1^p\rangle$, $\langle a_2^q\rangle$, $\langle a_3\rangle$, $\langle a_4\rangle$. This means that the product $\langle a_1^p\rangle\langle a_2^q\rangle\langle a_3\rangle\langle a_4\rangle$ is direct. This leads to the contradiction that G has a subgroup of type (r, r, r, r). Case 8 can be settled in a similar way.

Next let us deal with Table 3. Set $H = \langle C_1 \cup C_2 \cup C_3 \cup A_4\rangle$. In case 1 $|H| = |C_1||C_2||C_3||A_4|$ and so $H = C_1C_2C_3A_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 2, 4, 5, 10 11, 13 $|H| = p|C_1||C_2||C_3||A_4|$. By Lemma 1, B is periodic. In cases 3, 7, 9, 19, 21, 25 $|H| = q|C_1||C_2||A_3||A_4|$. By Lemma 1, B is periodic.

Let us consider case 27. In the factorization $G = BC_1C_2C_3A_4$ the factors C_1, C_2, C_3, A_4 can be replaced by $\langle a_1^q\rangle$, $\langle a_2^q\rangle$, $\langle a_3^q\rangle$, $\langle a_4\rangle$ to get the factorization $G = B\langle a_1^q\rangle\langle a_2^q\rangle\langle a_3^q\rangle\langle a_4\rangle$. This shows that the product $\langle a_1^q\rangle\langle a_2^q\rangle\langle a_3^q\rangle\langle a_4\rangle$ is direct. It follows the contradiction that G has a subgroup of type (r, r, r, r). The same argument applies in cases 14, 15 17, 18, 23, 24, 26.

We are left with cases 6, 8, 12, 16, 20, 22. By symmetry it is enough to settle case 8. In this case we carry out a more detailed analysis.

In the factorization $G = BC_1C_2C_3A_4$ the factors C_2, C_3, A_4 can be replaced by $\langle a_2^q \rangle$, $\langle a_3^p \rangle$, $\langle a_4 \rangle$ to get the factorization $G = BC_1\langle a_2^q \rangle\langle a_3^p \rangle\langle a_4 \rangle$. The product $C_1\langle a_2^q \rangle\langle a_3^p \rangle\langle a_4 \rangle$ forms a factorization of the r-component of G which is a group of type (r^2, r, r). It follows that one of

$$\{a_1, a_2^q, a_3^p\}, \quad \{a_1, a_3^p, a_4\}, \quad \{a_1, a_2^q, a_4\}$$

is a basis for the r-component of G. The elements of G of order r together with the identity element form a unique subgroup of G of type (r, r, r). One of

$$\{a_1^r, a_2^q, a_3^p\}, \quad \{a_1^r, a_3^p, a_4\}, \quad \{a_1^r, a_2^q, a_4\}$$

is a basis of this subgroup.

The subgroup $N = \langle A_4 \rangle$ is of type (r, r) or (r, r, r). Suppose first that N is of type (r, r). If $a_2^q \notin N$, then set $H = \langle C_1 \cup C_3 \cup A_4 \rangle$. Now $|H| = q|C_1||C_3||A_4|$ and so by Lemma 1, B is periodic. If $a_3^p \notin N$, then set $H = \langle C_1 \cup C_2 \cup A_4 \rangle$. Now $|H| = p|C_1||C_2||A_4|$ and so by Lemma 1, B is periodic. It remains that $a_2^q, a_3^p \in N$. It follows that a_2^q, a_3^p form a basis for N. But then the product $\langle a_2^q \rangle\langle a_3^p \rangle\langle a_4 \rangle$ cannot be direct.

Suppose next that N is of type (r, r, r). Let

$$K = \langle a_2^q \rangle, \quad L = \langle a_3^p \rangle, \quad M = \langle a_4 \rangle.$$

Consider the factorizations

$$\begin{aligned}
(5) \qquad G/K &= [(BK)/K][(C_1K)/K][(C_3K)/K][(A_4K)/K],\\
(6) \qquad G/L &= [(BL)/L][(C_1L)/L][(C_2L)/L][(A_4L)/L],\\
(7) \qquad G/M &= [(BM)/M][(C_1M)/M][(C_2M)/M][(C_3M)/M].
\end{aligned}$$

If a_1^r, a_2^q, a_3^p is a basis for N, then $a_1^r \notin K$ and so $(C_1K)/K$ cannot be periodic in (5). Plainly, $a_3^r \notin K$ and so $(C_3K)/K$ cannot be periodic in (5). As N is of type (r, r, r), $(A_4K)/K$ cannot be periodic in (5). Thus $(BK)/K$ must be periodic in (5). An analogous argument gives that $(BL)/L$ is periodic in (6). In the way we have seen in the proof of Lemma 3 we can conclude that B is periodic.

If a_1^r, a_3^p, a_4 is a basis for N, then $a_1^r \notin L$ and so $(C_1L)/L$ cannot be periodic in (6). Plainly, $a_2^r \notin L$ and so $(C_2K)/K$ cannot be periodic in (6). As N is of type (r, r, r), $(A_4L)/L$ cannot be periodic in (6). Thus $(BL)/L$ must be periodic in (6). An analogous argument gives that $(BM)/M$ is periodic in (7). Again we can conclude that B is periodic.

The case when a_1^r, a_2^q, a_4 is a basis for N can be settled in a similar way. $\Diamond$

Table 5: The choices for $|a_3|$, $|a_4|$ in Prop. 5

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert a_3\rvert$	s^2	s^2	s^2	s^2	sp	sp	sp	sp	sq	sq	sq	sq	sr	sr	sr	sr
$\lvert a_4\rvert$	s^2	sp	sq	sr	s^2	sp	sq	sr	s^2	sp	sq	sr	s^2	sp	sq	sr

Proposition 5. *Let G be a group of type (p, q, r, r, s^2), where p, q, r, s are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = r$, $|A_3| = |A_4| = s$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_4 = A_4$, then each element of $A_4 \setminus \{e\}$ has order s and so A_4 is equal to the unique subgroup of G of order s. This gives the contradiction that A_4 is periodic. Thus we may assume that $D_4 = C_4$. A similar argument shows that we may assume that $D_3 = C_3$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_1 = A_1$.

If $D_2 = A_2$, then the product A_1A_2 forms a factorization of the r-component of G which is a group of type (r, r). By Th. 2, either A_1 or A_2 is periodic. Thus we may assume that $D_1 = A_1$, $D_2 = C_2$, $D_3 = C_3$, $D_4 = C_4$. The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

$$
\begin{aligned}
|a_1| &\in \{r\},\\
|a_2| &\in \{rp, rq, rs\},\\
|a_3| &\in \{s^2, sp, sq, sr\},\\
|a_4| &\in \{s^2, sp, sq, sr\}.
\end{aligned}
$$

There are 16 choices for $|a_3|$, $|a_4|$ which are depicted in Table 5. In case 1 the product C_3C_4 forms a factorization of the s-component of G. By Th. 2, we get the contradiction that one of C_3, C_4 is periodic.

In case 6 in the factorization $G = BA_1C_2C_3C_4$ the factors C_3, C_4 can be replaced by $\langle a_3^p \rangle$, $\langle a_4^p \rangle$. This leads to the contradiction that G has a subgroup of type (s, s). Using a similar argument we can sort out the cases 7, 8, 10, 11, 12, 14, 15, 16.

Table 6: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 5

Case	1	2	3	4	5	6	7	8	9
$\|a_1\|$	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\|a_3\|$	s^2	s^2	s^2	s^2	s^2	s^2	s^2	s^2	s^2
$\|a_4\|$	sp	sq	sr	sp	sq	sr	sp	sq	sr

By symmetry we may assume that $|a_3| \in \{s^2\}$, $|a_4| \in \{sp, sq, sr\}$. So there are 9 choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ to consider. These cases are depicted in Table 6.

Set $H = \langle A_1 \cup C_2 \cup C_3 \cup C_4 \rangle$. In case 9 $|H| = |A_1||C_2||C_3||C_4|$ and so $H = A_1C_2C_3C_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 1, 3, 7 $|H| = p|A_1||C_2||C_3||C_4|$ and in cases 5, 6, 8 $|H| = q|A_1||C_2||C_3||C_4|$. By Lemma 1, B is periodic.

In cases 2, 4 Lemma 3 is applicable with the type 1 set A_1. $\Diamond$

Proposition 6. *Let G be a group of type (p, q, r^2, r, s), where p, q, r, s are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = |A_3| = r$, $|A_4| = s$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_4 = A_4$, then each element of $A_4 \setminus \{e\}$ has order s and so A_4 is equal to the unique subgroup of G of order s. This gives the contradiction that A_4 is periodic. Thus we may assume that $D_4 = C_4$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_1 = A_1$.

If $D_2 = A_2$, then each element of $A_1A_2 \setminus \{e\}$ has order r. The elements of G of order r together with e form a unique subgroup of G of type (r, r). Therefore the product A_1A_2 forms a factorization of this subgroup of G. By Th. 2, either A_1 or A_2 is periodic. Thus we may assume that $D_1 = A_1$, $D_2 = C_2$, $D_3 = C_3$, $D_4 = C_4$. The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

Table 7: The choices for $|a_1|$, $|a_2|$, $|a_3|$ in Prop. 6

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert a_1\rvert$	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
$\lvert a_2\rvert$	r^2	r^2	r^2	r^2	rp	rp	rp	rp	rq	rq	rq	rq	rs	rs	rs	rs
$\lvert a_3\rvert$	r^2	rp	rq	rs	s^2	rp	rq	rs	r^2	rp	rq	sr	r^2	sp	rq	rs

Table 8: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 6

Case	1	2	3	4	5	6	7	8	9
$\lvert a_1\rvert$	r	r	r	r	r	r	r	r	r
$\lvert a_2\rvert$	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2	r^2
$\lvert a_3\rvert$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\lvert a_4\rvert$	sp	sq	sr	sp	sq	sr	sp	sq	sr

$$\begin{aligned}|a_1| &\in \{r\},\\ |a_2| &\in \{r^2, rp, rq, rs\},\\ |a_3| &\in \{r^2, rp, rq, rs\},\\ |a_4| &\in \{sp, sq, sr\}.\end{aligned}$$

There are 16 choices for $|a_1$, $|a_2|$, $|a_3|$ which are depicted in Table 7. In case 1 the product $A_1C_2C_3$ forms a factorization of the r-component of G. By Th. 2, we get the contradiction that one of A_1, C_2, C_3 is periodic.

In case 6 in the factorization $G = BA_1C_2C_3C_4$ the factors A_1, C_2, C_3 can be replaced by $\langle a_1\rangle$, $\langle a_2^p\rangle$, $\langle a_3^p\rangle$. This leads to the contradiction that G has a subgroup of type (r, r, r). Using a similar argument we can sort out the cases 7, 8, 10, 11, 12, 14, 15, 16.

By symmetry we may assume that $|a_2| \in \{r^2\}$, $|a_3| \in \{rp, rq, rs\}$. So there are 9 choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ to consider. These cases are depicted in Table 8.

Set $H = \langle A_1 \cup C_2 \cup C_3 \cup C_4\rangle$. In case 9 $|H| = |A_1||C_2||C_3||C_4|$ and so $H = A_1C_2C_3C_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 1, 3, 7 $|H| = p|A_1||C_2||C_3||C_4|$ and in cases 5, 6, 8 $|H| = q|A_1||C_2||C_3||C_4|$. By Lemma 1, B is periodic.

Set $H = \langle A_1 \cup C_2 \cup C_3\rangle$. In case 2 $|H| = p|A_1||C_2||C_3|$ and in case 4 $|H| = q|A_1||C_2||C_3|$. From the factorization $G = (BC_4)A_1C_2C_3$ by Lemma 1, it follows that B is periodic. ♢

Proposition 7. *Let G be a group of type (p, q, r, r, s, s), where p, q, r,*

Table 9: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 7

Case	1	2	3	4	5	6	7	8	9
$\|a_1\|$	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\|a_3\|$	s	s	s	s	s	s	s	s	s
$\|a_4\|$	sp	sq	sr	sp	sq	sr	sp	sq	sr

s are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = r$, $|A_3| = |A_4| = s$. Then the factorization is periodic.

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is periodic since otherwise there nothing to prove. In the factorization $G = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_1 = A_1$.

If $D_2 = A_2$, then the product A_1A_2 forms a factorization of the r-component of G. By Th. 2, either A_1 or A_2 is periodic. Thus we may assume that $D_2 = C_2$.

If $D_3 = A_3$, $D_4 = A_4$, then the product A_3A_4 forms a factorization of the s-component of G. It follows that either A_3 or A_4 is periodic. By symmetry we may assume that $D_4 = C_4$.

Therefore we may assume that one of the following situations holds

$$D_1 = A_1, \quad D_2 = C_2, \quad D_3 = A_3, \quad D_4 = C_4,$$
$$D_1 = A_1, \quad D_2 = C_2, \quad D_3 = C_3, \quad D_4 = C_4.$$

The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

$$|a_1| \in \{r\}, \qquad |a_1| \in \{r\},$$
$$|a_2| \in \{rp, rq, rs\}, \qquad |a_2| \in \{rp, rq, rs\},$$
$$|a_3| \in \{s\}, \qquad |a_3| \in \{sp, sq, sr\},$$
$$|a_4| \in \{sp, sq, sr\}, \qquad |a_4| \in \{sp, sq, sr\}.$$

This leaves 9 and 27 cases to consider, respectively. These are depicted in Table 9 and Table 10.

Let us deal with Table 9 first. Set $H = \langle A_1 \cup C_2 \cup A_3 \cup C_4 \rangle$. In case 9 $|H| = |A_1||C_2||A_3||C_4|$ and so $H = A_1C_2A_3C_4$ is a factoriza-

Table 10: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 7

Case	1	2	3	4	5	6	7	8	9
$\|a_1\|$	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rp	rp	rp	rp	rp	rp	rp	rp	rp
$\|a_3\|$	sp	sp	sp	sq	sq	sq	sr	sr	sr
$\|a_4\|$	sp	sq	sr	sp	sq	sr	sp	sq	sr
Case	10	11	12	13	14	15	16	17	18
$\|a_1\|$	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rq	rq	rq	rq	rq	rq	rq	rq	rq
$\|a_3\|$	sp	sp	sp	sq	sq	sq	sr	sr	sr
$\|a_4\|$	sp	sq	sr	sp	sq	sr	sp	sq	sr
Case	19	20	21	22	23	24	25	26	27
$\|a_1\|$	r	r	r	r	r	r	r	r	r
$\|a_2\|$	rs	rs	rs	rs	rs	rs	rs	rs	rs
$\|a_3\|$	sp	sp	sp	sq	sq	sq	sr	sr	sr
$\|a_4\|$	sp	sq	sr	sp	sq	sr	sp	sq	sr

tion. By Th. 2, the factorization is periodic. In cases 1, 3, 7 $|H| = p|A_1||C_2||A_3||C_4|$ and in cases 5, 6, 8 $|H| = q|A_1||C_2||A_3||C_4|$. By Lemma 1, B is periodic.

In cases 2, 4 Lemma 3 is applicable with the type 1 subset A_3.

Next let us deal with Table 10. Set $H = \langle A_1 \cup C_2 \cup C_3 \cup C_4 \rangle$. In case 27 $|H| = |A_1||C_2||C_3||C_4|$ and so $H = A_1C_2C_3C_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 1, 3, 7, 9, 19, 21, 25 $|H| = p|A_1||C_2||C_3||C_4|$ and in cases 14, 15, 17, 18, 23, 24, 26 $|H| = q|A_1||C_2||C_3||C_4|$. By Lemma 1, B is periodic.

Set $H = \langle A_1 \cup C_2 \cup C_4 \rangle$. In case 6 $|H| = p|A_1||C_2||C_4|$ and in case 12 $|H| = p|A_1||C_2||C_4|$. From the factorization $G = (BC_3)A_1C_2C_4$, by Lemma 1, it follows that B is periodic.

In the remaining cases Lemma 3 is applicable with the type 1 subset A_1. $\Diamond$

Proposition 8. *Let G be a group of type (p, q, r, r, r, s), where p, q, r, s are distinct primes. Suppose that $G = BA_1A_2A_3A_4$ is a normalized factorization such that $|B| = pq$, $|A_1| = |A_2| = |A_3| = r$, $|A_4| = s$. Then the factorization is periodic.*

Proof. We may assume that none of the factors A_1, A_2, A_3, A_4 is pe-

Table 11: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 8

Case	1	2	3	4	5	6	7	8	9
$\|a_1\|$	r	r	r	r	r	r	r	r	r
$\|a_2\|$	r	r	r	r	r	r	r	r	r
$\|a_3\|$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\|a_4\|$	sp	sq	sr	sp	sq	sr	sp	sq	sr

riodic since otherwise there nothing to prove. In the factorization $G = = BA_1A_2A_3A_4$ we replace each A_i by D_i to get the normalized factorization $G = BD_1D_2D_3D_4$.

If $D_4 = A_4$, then each element of $A_4 \setminus \{e\}$ has order s and so A_4 is equal to to the unique subgroup of G of order s. Thus we may assume that $D_4 = C_4$.

If $D_i = C_i$ for each i, $1 \leq i \leq 4$, then from the factorization $G = BC_1C_2C_3C_4$, by Th. 5, it follows the contradiction that at least one of the factors B, C_1, C_2, C_3, C_4 is periodic. By symmetry we may assume that $D_1 = A_1$.

If $D_i = A_i$ for each i, $1 \leq i \leq 3$, then the product $A_1A_2A_3$ forms a factorization of the r-component of G. By Th. 2, one of A_1, A_2, A_3 is periodic. Therefore we may assume that one of the following situations holds

$$D_1 = A_1, \quad D_2 = A_2, \quad D_3 = C_3, \quad D_4 = C_4,$$
$$D_1 = A_1, \quad D_2 = C_2, \quad D_3 = C_3, \quad D_4 = C_4.$$

The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ are the following

$$\begin{array}{ll} |a_1| \in \{r\}, & |a_1| \in \{r\}, \\ |a_2| \in \{r\}, & |a_2| \in \{rp, rq, rs\}, \\ |a_3| \in \{rp, rq, rs\}, & |a_3| \in \{rp, rq, rs\}, \\ |a_4| \in \{sp, sq, sr\}, & |a_4| \in \{sp, sq, sr\}. \end{array}$$

This leaves 9 and 27 cases to consider, respectively. These are depicted in Table 11 and Table 12.

Let us settle Table 11 first. Set $H = \langle A_1 \cup A_2 \cup C_3 \cup C_4 \rangle$. In case 9 $|H| = |A_1||A_2||C_3||C_4|$ and so $H = A_1A_2C_3C_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 1, 3, 7 $|H| = = p|A_1||A_2||C_3||C_4|$ and in cases 5, 6, 8 $|H| = q|A_1||A_2||C_3||C_4|$. By Lemma 1, B is periodic.

Table 12: The choices for $|a_1|$, $|a_2|$, $|a_3|$, $|a_4|$ in Prop. 8

Case	1	2	3	4	5	6	7	8	9
$\lvert a_1\rvert$	r	r	r	r	r	r	r	r	r
$\lvert a_2\rvert$	rp	rp	rp	rp	rp	rp	rp	rp	rp
$\lvert a_3\rvert$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\lvert a_4\rvert$	sp	sq	sr	sp	sq	sr	sp	sq	sr
Case	10	11	12	13	14	15	16	17	18
$\lvert a_1\rvert$	r	r	r	r	r	r	r	r	r
$\lvert a_2\rvert$	rq	rq	rq	rq	rq	rq	rq	rq	rq
$\lvert a_3\rvert$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\lvert a_4\rvert$	sp	sq	sr	sp	sq	sr	sp	sq	sr
Case	19	20	21	22	23	24	25	26	27
$\lvert a_1\rvert$	r	r	r	r	r	r	r	r	r
$\lvert a_2\rvert$	rs	rs	rs	rs	rs	rs	rs	rs	rs
$\lvert a_3\rvert$	rp	rp	rp	rq	rq	rq	rs	rs	rs
$\lvert a_4\rvert$	sp	sq	sr	sp	sq	sr	sp	sq	sr

Set $H = \langle A_1 \cup A_2 \cup C_3 \rangle$. In case 2 $|H| = p|A_1||A_2||C_3|$ and in cases 5, 6, 8 $|H| = q|A_1||A_2||C_3|$. From the factorization $G = (BC_4)A_1A_2C_3$, by Lemma 1, B is periodic.

Finally let us turn to Table 12. Set $H = \langle A_1 \cup C_2 \cup C_3 \cup C_4 \rangle$. In case 27 $|H| = |A_1||C_2||C_3||C_4|$ and so $H = A_1C_2C_3C_4$ is a factorization. By Th. 2, the factorization is periodic. In cases 1, 3, 7, 9, 19, 21, 25 $|H| = p|A_1||C_2||C_3||C_4|$ and in cases 14, 15, 17, 18, 23, 24, 26 $|H| = q|A_1||C_2||C_3||C_4|$. By Lemma 1, B is periodic.

In cases 4, 5, 8, 10, 11, 16, 20, 21 Lemma 3 is applicable with type 1 subset A_1. Thus we left with cases 6, 12. These are symmetric cases so it is enough to settle case 6.

In case 6 we carry out a more detailed analysis. Let $K = \langle a_1 \rangle$, $L = \langle a_4^r \rangle$. Consider the factorizations

$$G/K = [(BK)/K][(C_2K)/K][(C_3K)/K][(C_4K)/K], \tag{8}$$

$$G/L = [(BL)/L][(A_1L)/L][(C_2L)/L][(C_3L)/L]. \tag{9}$$

In (9) only $(BL)/L$ can be periodic. In (8) $(BK)/K$ or $(C_4K)/K$ can be periodic. If in (8) $(BK)/K$ is periodic, then the argument we used in the proof of Lemma 3 provides that B is periodic. Thus we may assume that in (8) $(C_4K)/K$ is periodic. This implies that $a_4^r \in K$.

The subset A_1 can be written in the form

$$A_1 = \{e, a_1, a_1^2\rho_2, \ldots, a_1^{r-1}\rho_{r-1}\}, \quad |\rho_i| = r.$$

If $\rho_2, \ldots, \rho_{r-1} \in \langle a_1 \rangle$, then A_1 is periodic. Therefore we may assume that one of $\rho_2, \ldots, \rho_{r-1}$ is not an element of $\langle a_1 \rangle$. For the sake of definiteness we assume that $\rho_2 \notin \langle a_1 \rangle$.

Multiplying the factorization $G = BA_1C_2C_3C_4$ by a_1^{-1} we get the factorization $G = Ga_1^{-1} = B(A_1a_1^{-1})C_2C_3C_4$. Set $M = \langle a_1\rho_2 \rangle$ and consider the factorization

$$G/M = [(BM)/L][(C_2M)/M][(C_3M)/M][(C_4M)/M]. \tag{10}$$

In (10) only $(BM)/L$ or $(C_4M)/M$ can be periodic. If $(BM)/L$ is periodic, then using the fact that $(BL)/L$ is periodic in (9) we get that B is periodic. Thus we may assume that $(C_4M)/M$ is periodic in (10). This implies that $a_4^r \in M$. Now $a_4^r \in K \cap M = \{e\}$ and so $a_4^r = e$. This means that C_4 is periodic. $\Diamond$

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