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CIRCULAR CUBICS AND QUARTICS IN PSEUDO-EUCLIDEAN PLANE OBTAINED BY INVERSION

N. **Kovačević**

University of Zagreb, Faculty of Mining, Geology and Petroleum Engineering, Pierottijeva 6, 10000 Zagreb, Croatia

E. **Jurkin**

University of Zagreb, Faculty of Mining, Geology and Petroleum Engineering, Pierottijeva 6, 10000 Zagreb, Croatia

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Abstract: This paper is an extension of [4] where the inversions in a pseudo-Euclidean plane with an equiform conic as the fundamental conic and a pole at its center or at an isotropic point have been studied. The type of circularity for the curves has also been introduced.
We have shown that by using these transformations the rational cubics of all types of circularity can be constructed as images of conics. Unfortunately, the same does not hold for the quartics.

1. Motivation

The main purpose of this article is to continue solving of the construction problem of one of the most interesting classes of curves in various projective-metric planes, the circular curves. The idea is to compare

E-mail addresses: nkovacev@rgn.hr, ejurkin@rgn.hr

the problem with the similar one in Euclidean and some other projective-metric geometries, and to open the field for further investigation of the circular curves and their properties in the projective-metric planes.

Unlike other new researches of the similar problems, the authors offer different approach in the study of curves by treating the projective-metric plane as an embedded plane of the projective plane based on the famous Erlangen program of F. Klein. This projective approach allows a treatment of a pseudo-Euclidean curve as a point set.

For further details of the method used in this article other similar articles [2]–[4], [7]–[8] can be considered. This paper should be treated as an extension of [4] where the basic equations of the transformation are given together with the inverse image of the line.

2. Introduction

Pseudo-Euclidean plane. A *pseudo-Euclidean (Minkowski) plane* $\mathcal{M}_2$ can be defined as a projective plane where the metric is induced with an absolute $\{f, F_1, F_2\}$ in the sense of Cayley–Klein, consisting of a real line f and two real points F_1 and F_2 incidental with it, [6], [9]. The line f is called the *absolute line* and the points F_1, F_2 are the *absolute points*. All lines through the absolute points are called *isotropic lines* and all points on f are called *isotropic (ideal, infinite) points*.

If the absolute line f is determined by the equation $x_0 = 0$, the absolute points F_1, F_2 are determined by the coordinates $(0, 1, \pm 1)$.

Furthermore, an involution of points on the absolute line f having the absolute points for the fixed points is called the *absolute involution* and in the following text it will be denoted by ω.

Let us begin by summarizing some facts about curves from [4].

Curves in pseudo-Euclidean plane. An algebraic curve k^n of order n will be treated as the totality of points whose coordinates in some assigned allowable coordinate simplex satisfy a homogeneous equation of the n-th degree $g(x_0, x_1, x_2) = 0$. If one of isotropic points of the given curve k^n coincides with one of absolute points, the curve is said to be *circular*, [4]. Furthermore, if F_1 is the intersection point of k^n and f with the intersection multiplicity t, and F_2 is an intersection point of k^n and f with the intersection multiplicity r, than k^n is said to be a curve of the *type of circularity* (t, r) and its *degree of circularity* is defined as $r + t$, [4]. If $r + t = n$, the curve is said to be *entirely circular*.

Depending on their degree of circularity, the conics are classified in [4] into three groups: *non-circular conics* (ellipses, three types of hyperbolas and parabolas), *1-circular conics* (two types of special hyperbolas), and *entirely circular conics* (circles and special parabolas).

Furthermore, in [4] a notion of an equiform conic in the pseudo-Euclidean plane is introduced. A conic c is called an *equiform conic*, if the equality holds $\phi_c \circ \omega = \omega \circ \phi_c$, where ϕ_c determines the involution of the conjugate points on the absolute line induced by the conic c. It is easy to see that either (a) $\phi_c \equiv \omega$ and the given conic c is a circle, or (b) ω and ϕ_c are involutions whose pairs of fixed points separate each other harmonically. Depending on the reality of the fixed points of the induced involution ϕ_c, we further distinguish an *equiform hyperbola* and an *equiform ellipse*.

Classification of conics. By choosing a basic coordinate simplex in the projective plane $\mathbb{P}^2$ every conic c can be represented by the homogeneous equation of the form

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2 = 0$$

and in the affine coordinates by

$$a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0. \tag{1}$$

The conic c intersects the absolute line f in the points whose coordinates satisfy the equality

$$a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 0.$$

Some short calculations lead us to the conclusions:

- c is a hyperbola iff $a_{12}^2 - a_{11}a_{22} > 0$.
- c is a parabola iff $a_{12}^2 - a_{11}a_{22} = 0$.
- c is an ellipse iff $a_{12}^2 - a_{11}a_{22} < 0$.
- c is a special hyperbola iff $a_{11}+a_{22}+2a_{12} = 0$ or $a_{11}+a_{22}-2a_{12} = 0$.
- c is a special parabola iff $a_{11} = a_{22} = -a_{12}$ or $a_{11} = a_{22} = a_{12}$.
- c is a circle iff $a_{12} = 0$ and $a_{11} = -a_{22}$.

The hyperbolas and ellipses are equiform iff $a_{11} = a_{22}$.

3. Inversion

An *inversion* $\sigma(P, q)$ with respect to the *pole* P and the *fundamental conic* q is an involutive quadratic mapping where corresponding points are conjugate points with respect to q and lie on lines of the pencil (P). These lines are called *rays.* The fixed points P_1, P_2 of the involution determined on the polar line p of the pole P with respect to q together with P define *fundamental triangle* of the given birational mapping. Obviously, the quadratic inversion is an one-to-one mapping with an exception of the points lying on the fundamental triangle. The inversion maps the curve k^n of order n, if k^n contains no fundamental point, onto the curve k_q^{2n} of order $2n$. Since k^n intersects every fundamental line in n points, each mapped onto the corresponding fundamental point, k_q^{2n} has three n-fold points in the fundamental points. If k^n contains one fundamental point as r-fold point, then k_q^{2n} splits into the corresponding polar line with multiplicity r and a curve of order $2n - r$, [1]–[5], [8].

It seems natural to consider an extension of the well-known inversion in the Euclidean plane with circle as the fundamental conic. Therefore, in [4] three main types of inversion in pseudo-Euclidean plane were defined, by choosing one of three types of equiform conics for the fundamental conic of transformation. Furthermore, depending on the type of the pole P, whether it is the center of the fundamental conic or an isotropic point, we distinguish two general subtypes of each type. Special cases occur when one of the absolute points coincide with one of the fundamental points.

As the equations of all types and subtypes of the inversion have been presented and the inverse image of any line has been discussed in [4], this article continues with the construction of the circular cubics and quartics as the inverse images of conic. Since conics are rational curves and the inversion preserves the genus of curves, only the rational cubics and quartics can be obtained. For every type of the inversion the conditions that the conic has to fulfill in order to obtain the cubic or quartic of the certain type of circularity will be determined. Also a construction of the isotropic tangents of the obtained curves will be given in some cases. Similar observations have been done for the circular quartics in an isotropic plane in [2].

3.1. Circle as fundamental conic of inversion

Let an inversion $\sigma(\mathcal{C}, P)$ be given in the pseudo-Euclidean plane, where $\mathcal{C}$ is a unit circle given by the equation

$$\mathcal{C} \ldots x^2 - y^2 = 1. \tag{2}$$

Depending on the position of the pole P we further distinguish two types of the circle inversion.

Type (1) – The pole P is the center of the circle. The homogeneous coordinates of P are $(1, 0, 0)$. Obviously, the equality $\omega \equiv \phi_{\mathcal{C}}$ holds, and the absolute points coincide with the two fundamental points $P_{1,2}$. The sides of the fundamental triangle are given by the equations $x_0 = 0$, $x \pm y = 0$.

Furthermore, the point $X(x, y)$ is mapped onto the point $\overline{X}(\overline{x}, \overline{y})$ whose coordinates are according to (3.5) in [4] given by

$$\overline{x} = \frac{x}{x^2 - y^2}, \quad \overline{y} = \frac{y}{x^2 - y^2}.$$

Obviously it is an analogue of the Euclidean ordinary inversion[1], and the non-circular conic k^2 given by (1) is, generally (when $a_{00} \neq 0$, i.e. $P \notin \mathcal{C}$), mapped onto the *entirely circular quartic* $k^4_{\mathcal{C}}$ of type $(2, 2)$ given by the affine equation

$$a_{00}(x^2-y^2)^2+2(a_{01}x+a_{02}y)(x^2-y^2)+(a_{11}x^2+2a_{12}xy+a_{22}y^2) = 0 \tag{3}$$

with three double points at the fundamental points, tangents at which are the lines:

$$(t_{P_1})_{1,2} \ \ldots \ y = x + \frac{a_{01} + a_{02} \pm \sqrt{(a_{01} + a_{02})^2 - a_{00}(a_{11} + 2a_{12} + a_{22})}}{2a_{00}},$$

$$(t_{P_2})_{1,2} \ \ldots \ y = -x + \frac{-a_{01}+a_{02} \pm \sqrt{(a_{01}-a_{02})^2 - a_{00}(a_{11}-2a_{12} + a_{22})}}{2a_{00}},$$

$$(t_P)_{1,2} \ \ldots \ y = \frac{-a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{22}}x.$$

The reality of the tangents and the type of a double point (a node, a cusp or an isolated double point) depends on the reality of intersections of the conic k^2 with the corresponding fundamental line.

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[1] Some examples of this type of inversion can be found in [9] (p. 216).

Fig. 1 displays an entirely circular quartic $k^4_{\mathcal{C}}$ obtained as an inverse image of a hyperbola of type 3. Since the isotropic points $G_{1,2}$ of k^2 are mapped onto the pole P, their rays are the tangents of $k^4_{\mathcal{C}}$ at the double point P. Furthermore, since the inverse image of the pencil (P_1) is the pencil of the lines (P_2), the isotropic tangents at the double point F_1 are determined as the inverse images of the lines connecting the point F_2 and the intersections of p_1 and k^2 (these images also contain the line p_1).

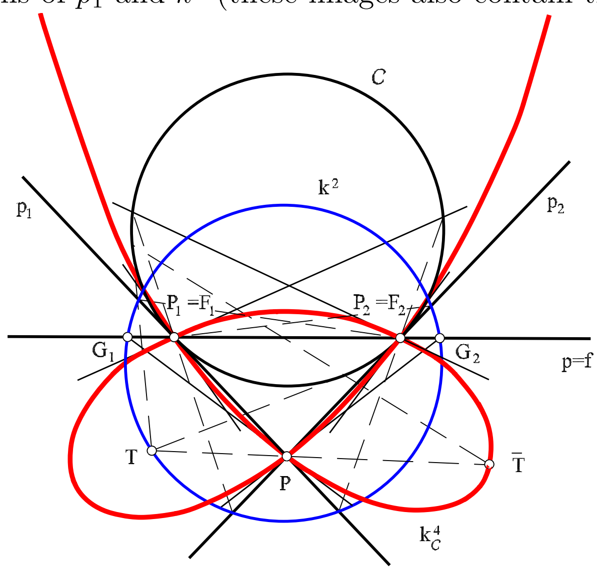

Figure 1

Naturally, for obtaining circular cubics one has to distinguish two cases. First one occurs when $a_{00} = 0$, i.e. k^2 passes through the pole P, and it follows from (3) that the obtained quartic $k^4_{\mathcal{C}}$ splits into f and an at least 2-*circular cubic* $k^3_{\mathcal{C}}$. Besides the absolute points, simple calculation leads to the homogeneous coordinates of the third isotropic point of $k^3_{\mathcal{C}}$ which are of the form $(0, a_{02}, -a_{01})$. Obviously, $k^3_{\mathcal{C}}$ is *entirely circular* iff $|a_{01}| = |a_{02}|$, i.e. k^2 touches one of the fundamental lines p_i, $i = 1, 2$, at the pole P, Fig. 2.

In the second case k^2 passes through one of the other two fundamental points (e.g. P_1), therefore, it is a special hyperbola. For example, substituting $2a_{12} = -(a_{11} + a_{22})$ in (3), an *entirely circular cubic* $k^3_{\mathcal{C}}$ of type $(1, 2)$ with equation

$$a_{00}(x^2 - y^2)(x + y) + 2(x + y)(a_{01}x + a_{02}y) + a_{11}x - a_{22}y = 0$$

is obtained.

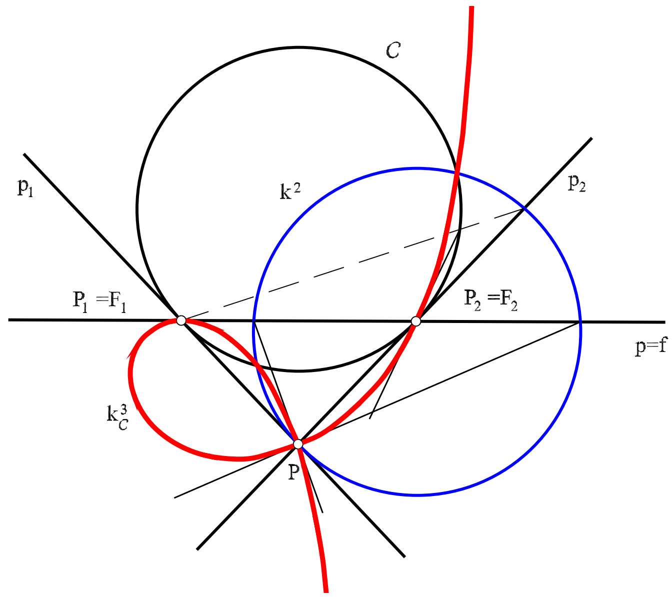

Figure 2

This type of inversion provides a construction of circular cubics of type (r,t) where $r,t \neq 0$, $r+t \geq 2$.

To summarize:

Theorem 1. *Let $\mathcal{C}$ be a circle and P its center in $\mathcal{M}_2$.*

- *The inversion $\sigma(P,\mathcal{C})$ maps a conic k^2 onto the entirely circular quartic $k^4_{\mathcal{C}}$ of type $(2,2)$, if none of the fundamental points lies on k^2.*
- *The inversion $\sigma(P,\mathcal{C})$ maps a conic k^2 passing through one of the fundamental points onto the circular cubic $k^3_{\mathcal{C}}$.*

  $k^3_{\mathcal{C}}$ is an entirely circular cubic of type $(1,2)$ if k^2 is a circular conic or non-circular conic touching p_i, $i=1,2$, at P. Otherwise, the obtained cubic is 2-circular of type $(1,1)$.

Type (2) – The pole P is an isotropic point. Whether or not the pole P coincides with one of the absolute points, two subcases occur.

In a *general case*, when $P \neq F_i$, $i=1,2$, let the coordinates of P be $(0,0,1)$. The other two fundamental points $P_{1,2}(1,\pm1,0)$ and the fundamental lines are $y=0$, $x\pm1=0$. The transformation is described by

$$\overline{x} = x, \quad \overline{y} = \frac{x^2 - 1}{y}.$$

Note that the type of circularity is invariant in this subcase of inversion since the absolute points remain fixed, [4].

The inverse image of a conic k^2 determined by (1) is the quartic $k^4_{\mathcal{C}}$

$$a_{22}(x^2-1)^2+(a_{00}+2a_{01}x+a_{11}x^2)y^2+2a_{02}(x^2-1)y+2a_{12}xy(x^2-1)=0.$$

Its isotropic points, besides P, are given by $(0, -a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}, a_{22})$.

Furthermore, if $a_{11} = 0$, then P is a common point of f and $k^4_{\mathcal{C}}$ with the intersection multiplicity 3. This is the case when one of the isotropic points of k^2 coincides with $Q(0,1,0) = f \cap p$, Fig. 3. If both of the isotropic points of k^2 coincide with Q, $a_{12} = 0$, the pole P is a double point of $k^4_{\mathcal{C}}$ at which quartic osculates and intersects the absolute line.

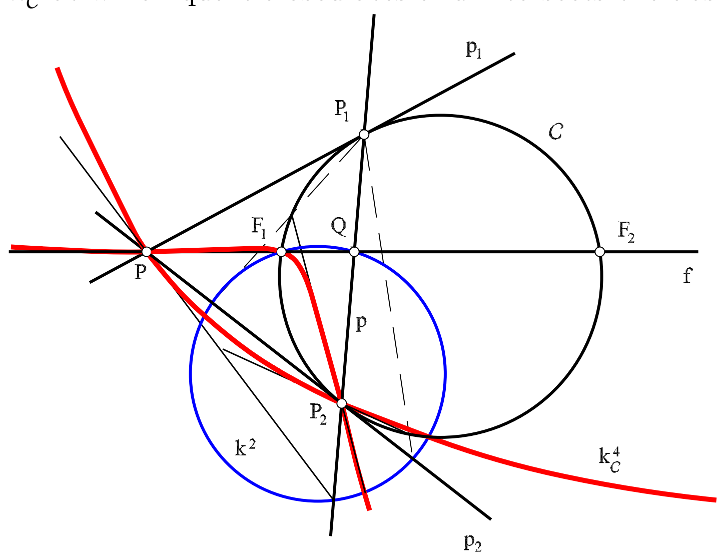

Figure 3

The cubics are obtained if k^2 passes through one of the fundamental points. An example of a 2-circular cubic of type $(2,0)$ is shown in Fig. 4.

A *special case* occurs when $P \in \mathcal{C}$. All three fundamental points coincide with the pole and all three fundamental lines with its polar line p. If $\mathcal{C}$ is given by (2) and e.g. $P = F_1(0,1,1)$, the equation of p is $x - y = 0$. Furthermore, the inversion $\sigma(\mathcal{C}, P)$ is presented by the equations

$$\overline{x} = \frac{-y^2 + xy - 1}{y - x}, \quad \overline{y} = \frac{x^2 - xy - 1}{y - x}.$$

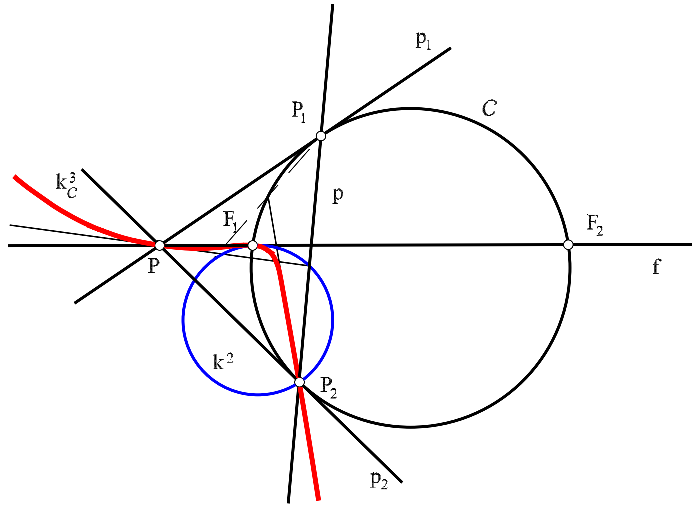

Figure 4

Generally, it maps a conic (1) onto an at least a 2-*circular quartic* $k^4_{\mathcal{C}}$:

$$(4)\quad a_{00}(x-y)^2+a_{11}(y^2-xy+1)^2+a_{22}(1+xy-x^2)^2+ \\ +2a_{01}(x-y)(y^2-xy+1)+2a_{02}(x-y)(1+xy-x^2)+ \\ +2a_{12}(y^2-xy+1)(1+xy-x^2)=0.$$

The homogeneous coordinates of the intersections of $k^4_{\mathcal{C}}$ and f are determined by substituting $x_0=0$ into (4) and they satisfy the equation:

$$(x_1-x_2)^2(a_{11}x_2^2+2a_{12}x_1x_2+a_{22}x_1^2)=0.$$

Obviously, $P=F_1(0,1,1)$ is their common point with the intersection multiplicity 2. For obtaining 3-*circular quartics*, one of the other two intersection points must coincide with $F_2(0,1,-1)$, i.e. $a_{11}-2a_{12}+a_{22}=0$, so $F_2\in k^2$ and $k^4_{\mathcal{C}}$ is of type $(2,1)$, Fig. 5. $k^4_{\mathcal{C}}$ is *entirely circular* of type $(2,2)$ if both intersection points coincide with F_2 which is the case iff $a_{11}=a_{22}=-a_{12}$. This corresponds to the fact that k^2 is a special parabola touching f at F_2.

Every line through the pole P has the equation of the form $y=$ $=x+m$. It intersects $k^4_{\mathcal{C}}$ at the points coordinates of which satisfy

$$\left[a_{11}+2a_{12}+a_{22}-2(a_{01}+a_{02})m+(a_{00}+2a_{11}+2a_{12})m^2-2_{01}m^3+a_{11}m^4\right]+ \\ +2\left[(a_{11}+2a_{12}+a_{22})m-(a_{01}+a_{02})m^2+(a_{11}+a_{12})m^3\right]x+ \\ +(a_{11}+2a_{12}+a_{22})\,m^2x^2=0.$$

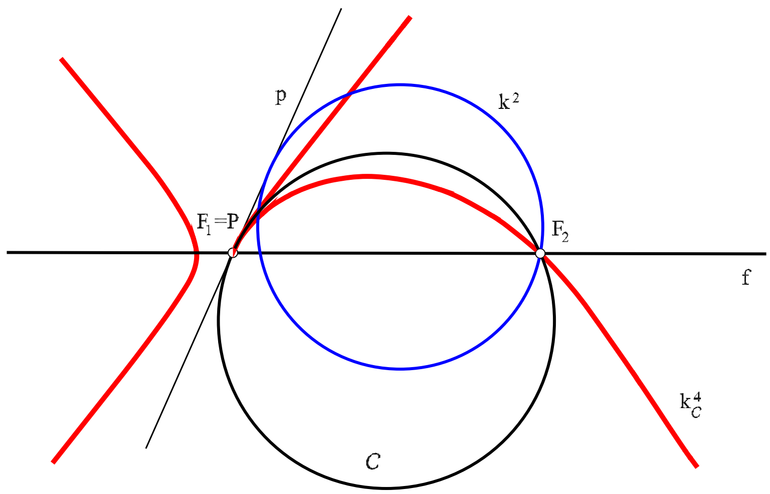

Figure 5

Thus it is obvious that x_0 is a double solution for each m and therefore P is a double point of $k^4_{\mathcal{C}}$. We need to determine m corresponding with the tangent. x_0 is a triple solution if $(a_{11} + 2a_{12} + a_{22})\, m^2$ equals zero. Hence, both tangents of $k^4_{\mathcal{C}}$ at P coincide with p.

For obtaining a circular cubic, the only condition is $P \in k^2$, i.e. $a_{11} + a_{22} + 2a_{12} = 0$. The term (4) turns into

$$(x-y)^2[a_{00} + a_{11}(y^2 - xy + 1) - a_{22}(1 + xy - x^2)] + \\ +2(x-y)[a_{01}(y^2 - xy + 1) + a_{02}(1 + xy - x^2)] = 0.$$

From this it follows that $k^4_{\mathcal{C}}$ splits into the line p and an at least 2-*circular cubic* $k^3_{\mathcal{C}}$ intersecting f at the point $F_1(0,1,1)$ counted twice and an isotropic point coordinates of which satisfy $a_{22}x_1 - a_{11}x_2 = 0$. An *entirely circular cubic* $k^3_{\mathcal{C}}$ of type $(3,0)$ is obtained when that point coincides with F_1 which is the case iff k^2 is a special parabola touching f at F_1. $k^3_{\mathcal{C}}$ is of type $(2,1)$ iff k^2 is a circle. Some calculations similar to those made for the quartic $k^4_{\mathcal{C}}$ deliver the equations of the isotropic tangents at the pole F_1: $y = x$ and $y = x + \dfrac{2(a_{01} + a_{02})}{a_{11} - a_{22}}$. Note that the first one coincides with the polar line p and the second one is identical to the tangent of k^2 at the same point.

We conclude the discussion of this type of inversion by stating the following theorems:

Theorem 2. *Let $\sigma(P, \mathcal{C})$ be an inversion given in $\mathcal{M}_2$ where $\mathcal{C}$ is a circle* *and P an isotropic point. An inverse image of a conic k^2 of type (t, r) not passing through the fundamental points of σ is:*

- *$(t + r)$-circular quartic of type (t, r), if $P \neq F_i$, $i = 1, 2$.*
- *$(r + 2)$-circular quartic of type $(2, r)$, if e.g. $P = F_1$.*

Theorem 3. *Let $\sigma(P, \mathcal{C})$ be an inversion given in $\mathcal{M}_2$ where $\mathcal{C}$ is a circle and P an isotropic point. An inverse image of a conic k^2 of type (t, r) passing through one of the fundamental points of σ is:*

- *$(t + r)$-circular cubic of type (t, r), if $P \neq F_i$, $i = 1, 2$.*
- *$(t + r + 1)$-circular cubic of type $(t + 1, r)$, if e.g. $P = F_1$.*

3.2. Hyperbola as fundamental conic of inversion

Let a hyperbola $\mathcal{H}$ be given by the equation

$$\mathcal{H} \ldots 2xy = 1. \tag{5}$$

Depending whether the pole P of the inversion $\sigma(P, \mathcal{H})$ is the center of $\mathcal{H}$ or an isotropic point, two types are distinguished.

Type (1) – The pole P is the center of the hyperbola. By placing the pole P at the center $(1, 0, 0)$ of $\mathcal{H}$ we get the equations of the inversion

$$\overline{x} = \frac{1}{2y}, \quad \overline{y} = \frac{1}{2x}.$$

The fundamental points are $P(1, 0, 0)$, $P_1(0, 1, 0)$, $P_2(0, 0, 1)$ and the fundamental lines are $x_0 = 0$, $y = 0$, $x = 0$, respectively.

The conic k^2 given by (1) is mapped onto the quartic $k^4_{\mathcal{H}}$

$$4a_{00}x^2y^2 + 4a_{01}x^2y + 4a_{02}xy^2 + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 0. \tag{6}$$

Since the fundamental points $P_1(0, 1, 0)$, $P_2(0, 0, 1)$ are intersections of $k^4_{\mathcal{H}}$ and f with multiplicity 2, it is not possible to obtain a circular quartic.

On the other hand it is possible to construct circular cubics. For obtaining cubics, one of the fundamental points has to lie on k^2.

If k^2 passes through P, $(a_{00} = 0)$, $k^4_{\mathcal{H}}$ splits onto the line p and a cubic $k^3_{\mathcal{H}}$ intersecting f at $P_1(0, 1, 0)$, $P_2(0, 0, 1)$ and an isotropic point coordinates of which satisfy the equation $a_{01}x_1 + a_{02}x_2 = 0$. This point coincides with one of the fundamental points iff k^2 touches the corresponding fundamental line at P. $k^3_{\mathcal{H}}$ is 1-*circular* iff k^2 touches one of the rays PF_i, $i = 1, 2$, at P, Fig. 6.

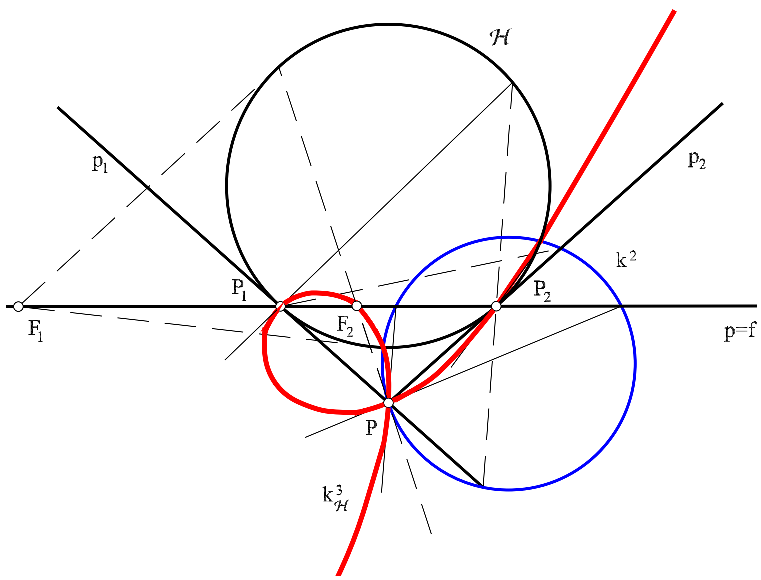

Figure 6

If k^2 passes through e.g. P_1 ($a_{11}=0$), $k^4_{\mathcal{H}}$ splits onto the line p_1 and a non-circular cubic.

We can conclude this study by stating

Theorem 4. *Let $\mathcal{H}$ be a hyperbola in $\mathcal{M}_2$ and let P be its center. In the general case, the inversion $\sigma(P,\mathcal{H})$ maps a conic k^2 onto a non-circular cubic or quartic depending on whether or not k^2 passes through the fundamental points.*

The cubic is 1-circular if k^2 touches one of the rays PF_i, $i=1,2$ at P.

Type (2) – The pole P is an isotropic point. If the fundamental conic $\mathcal{H}$ is given by the eq. (5) and the pole P is an isotropic point with the coordinates $(0,1,p)$, the inversion $\sigma(P,\mathcal{H})$ is determined by the equations

$$\overline{x}=\frac{px^2-xy+1}{px+y},\quad \overline{y}=\frac{y^2-pxy+p}{px+y}.$$

Depending on the position of P we can distinguish three subcases; the pole P not lying on $\mathcal{H}$ and differing from the absolute points ($p\neq\pm1,0$), pole P coinciding with one of the absolute points ($|p|=1$), and P lying on $\mathcal{H}$ ($p=0$).

In the first two cases the coordinates of the fundamental points are of the form $P(0,1,p)$, $P_{1,2}\left(1,\pm\sqrt{-\frac{1}{2p}},\mp\sqrt{-\frac{p}{2}}\right)$ and the fundamental lines are $y=-px$, $y=-px\mp\sqrt{-2p}$, respectively.

In the *general case* $(p\neq 0,\pm 1)$, since f is the ray of the inversion and F_1, F_2 are mapped onto each other, the inversion keeps the absolute figure fixed. Therefore, the degree of the circularity of the curve is an invariant of this transformation. More precisely, the curve k^n of type (t,r) is mapped onto the curve k^{2n} of type (r,t).

$\sigma(P,\mathcal{H})$ maps the conic (1) onto the quartic $k^4_{\mathcal{H}}$:

$$\begin{aligned}(7)\qquad & a_{00}(px+y)^2+a_{11}(px^2-xy+1)^2+a_{22}(y^2-pxy+p)^2+\\ &+2a_{01}(px+y)(px^2-xy+1)+2a_{02}(px+y)(y^2-pxy+p)+\\ &+2a_{12}(px^2-xy+1)(y^2-pxy+p)=0.\end{aligned}$$

The absolute line f meets $k^4_{\mathcal{H}}$ at P with intersection multiplicity 2 and two further points. P is a point of intersection multiplicity 3 iff k^2 passes through $Q=f\cap p$. It has the intersection multiplicity 4 iff k^2 touches f at Q. One of the isotropic points of $k^4_{\mathcal{H}}$ is F_1 iff k^2 passes through F_2, Fig. 7.

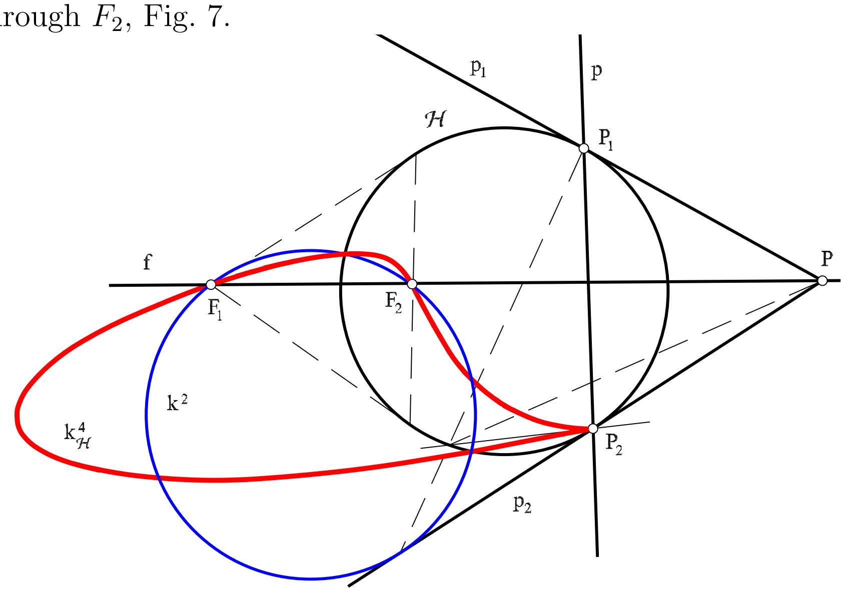

Figure 7

In the *special case*, when $|p| = 1$, the circularity of the curve is not invariant as in the previous one, but the obtained curve is an at least 2-*circular quartic* $k^4_{\mathcal{H}}$. Let us suppose that $P = F_1(0,1,1)$. If k^2 is of type $(0,1)$, f touches $k^4_{\mathcal{H}}$ at the double point F_1 and $k^4_{\mathcal{H}}$ is 3-*circular quartic* of type $(3,0)$, Fig. 8. Furthermore, an *entirely circular quartic* of type $(4,0)$ is obtained iff k^2 touches f at F_2, as then f osculates one branch of $k^4_{\mathcal{H}}$ at F_1.

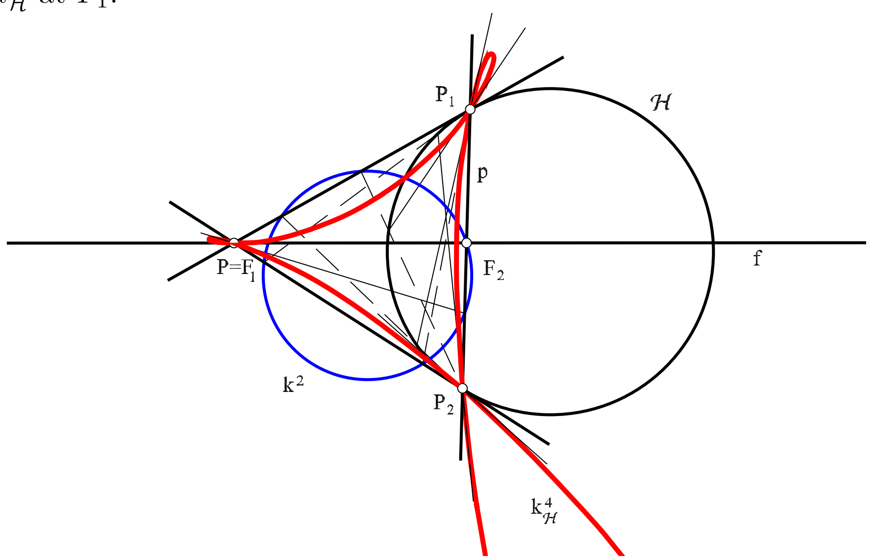

Figure 8

A conic passing through $P = F_1$ is generally mapped onto the 2-*circular cubic* $k^3_{\mathcal{H}}$ having a double point at it. The cubic $k^3_{\mathcal{H}}$ is *entirely circular* of type $(3,0)$ when k^2 is a circle or of type $(2,1)$ when k^2 is a special parabola.

In the case of k^2 passing through P_1 (or P_2) its image $k^3_{\mathcal{H}}$ meets f at the regular point F_1, so cubic is at least 1-*circular* of type $(1,0)$. f touches a 2-*circular cubic* $k^3_{\mathcal{H}}$ at F_1 if $F_2 \in k^2$. It osculates $k^3_{\mathcal{H}}$ at F_1 if f touches k^2 at F_2, therefore the obtained cubic is *entirely circular* of type $(3,0)$, Fig. 9.

The *last case* belonging to this type of inversion shows up when $p = 0$ and all the fundamental points coincide with $P = (0,1,0)$ and all the fundamental lines with the line $y = 0$. The inverse image of a conic k^2 given by (1) is a quartic $k^4_{\mathcal{H}}$ having a double point at P at which both tangents coincide with p. It is easy to see that the degree of circularity is also an invariant in this case, Fig. 10.

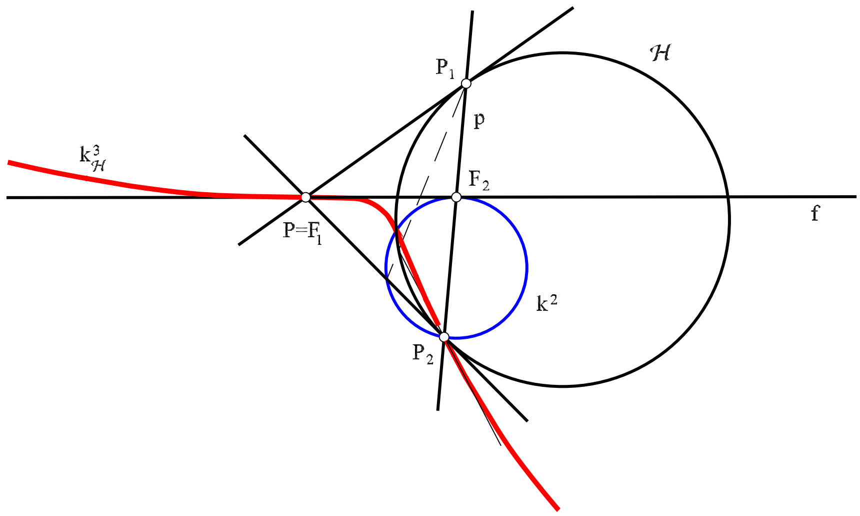

Figure 9

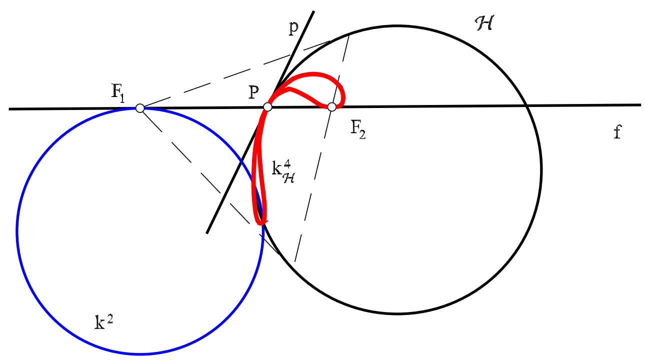

Figure 10

Consequently, the following theorems hold:

Theorem 5. *Let $\sigma(P, \mathcal{H})$ be an inversion given in $\mathcal{M}_2$ where $\mathcal{H}$ is a hyperbola and P an isotropic point. An inverse image of a conic k^2 of type (t, r) not passing through the fundamental points of σ is:*

- *$(t + r)$-circular quartic of type (r, t), if $P \neq F_i$, $i = 1, 2$.*
- *$(r + 2)$-circular quartic of type $(r + 2, 0)$, if e.g. $P = F_1$.*

Theorem 6. *Let $\sigma(P, \mathcal{H})$ be an inversion given in $\mathcal{M}_2$ where $\mathcal{H}$ is a hyperbola and P an isotropic point. An inverse image of a conic k^2 of type (t, r) passing through one of the fundamental points of σ is:*

- *$(t+r)$-circular cubic $k^3_{\mathcal{H}}$ of type (r,t), if $P \neq F_i$, $i = 1, 2$.*
- *$(t+r+1)$-circular cubic $k^3_{\mathcal{H}}$, if $P = F_i$, $i \in \{1, 2\}$.*

If $P = F_1$, $k^3_{\mathcal{H}}$ is of type $(t+r+1, 0)$ for $t < 2$ and of type $(2, 1)$ for $t = 2$.

3.3. Ellipse as fundamental conic of inversion

Let the ellipse $\mathcal{E}$ be given by the equation

$$\mathcal{E} \ldots x^2 + y^2 = 1. \tag{8}$$

As in the previous case, we can differ two main types depending on the position of the pole P.

Type (1) – The pole P is the center of the ellipse. In this case the observations are analogous to those made for the inversion with the pole at the center of the fundamental hyperbola. The basic difference lies in the fact that now the pole is an interior point of the fundamental conic and, therefore, the fundamental points $P_{1,2}$ form a pair of conjugate imaginary points, Fig. 11. The pole is now $P(1, 0, 0)$ and $\sigma(P, \mathcal{E})$ is determined by the equations

$$\overline{x} = \frac{x}{x^2 + y^2}, \quad \overline{y} = \frac{y}{x^2 + y^2}.$$

It maps a conic k^2 given by (1) onto a non-circular quartic $k^4_{\mathcal{E}}$ with the equation

$$a_{00}(x^2 + y^2)^2 + 2(x^2 + y^2)(a_{01}x + a_{02}y) + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy = 0.$$

If k^2 passes through P, its image splits into the fundamental line $p = f$ and, in general, a non-circular cubic $k^3_{\mathcal{E}}$:

$$2(x^2 + y^2)(a_{01}x + a_{02}y) + a_{11}x^2 + a_{22}y^2 + 2a_{12}xy = 0.$$

We can state

Theorem 7. *Let $\mathcal{E}$ be an ellipse in $\mathcal{M}_2$ and let P be its center. The inversion $\sigma(P, \mathcal{E})$ in the general case maps a conic onto a non-circular cubic or quartic depending on whether the conic passes through a fundamental point or not.*

The cubic is $(1, 0)$-circular iff the conic touches the line PF_1 at the pole P.

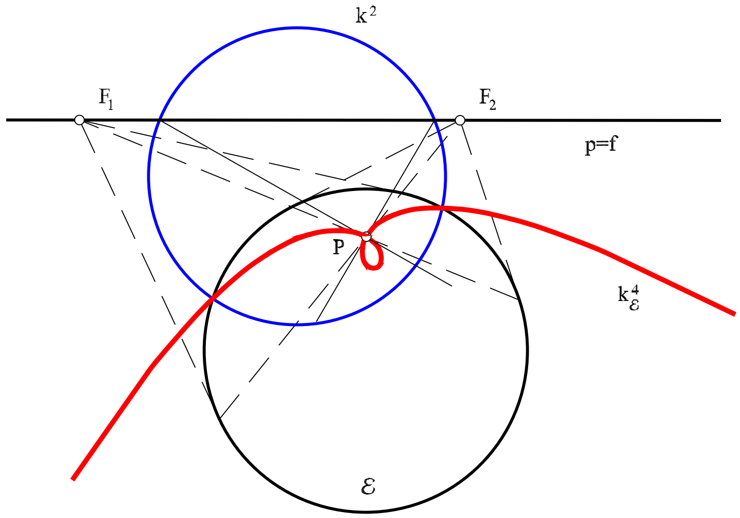

Figure 11

Type (2) – The pole P is an isotropic point. Only two subcases will be distinguished: P coincides or not with one of the absolute points.

In the first, *general case*, by assuming $P(0,1,0)$, the inversion is given by

$$\overline{x} = \frac{1-y^2}{x}, \quad \overline{y} = y$$

and it maps the conic (1) onto the quartic $k^4_{\mathcal{E}}$:

$$x^2(a_{00} + 2a_{02}y + a_{22}y^2) + a_{11}(1-y^2)^2 + 2x(1-y^2)(a_{01} + 2a_{12}y) = 0.$$

The fundamental line p with the equation $x = 0$ meets $\mathcal{E}$ at the fundamental points $P_{1,2}(1,0,\pm 1)$. The other two fundamental lines are $y = \pm 1$.

Since the absolute points are corresponding points of this type of inversion, the degree of circularity is an invariant of the transformation, Fig. 12.

In the *special case*, by assuming $P = F_1(0,1,1)$, we get the following equations of the mapping:

$$\overline{x} = \frac{1+xy-y^2}{x+y}, \quad \overline{y} = \frac{1+xy-x^2}{x+y}.$$

The fundamental points $P_{1,2}$ are given by $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$, while the fundamental lines are of the form $y = -x$, $y = x \mp \sqrt{2}$.

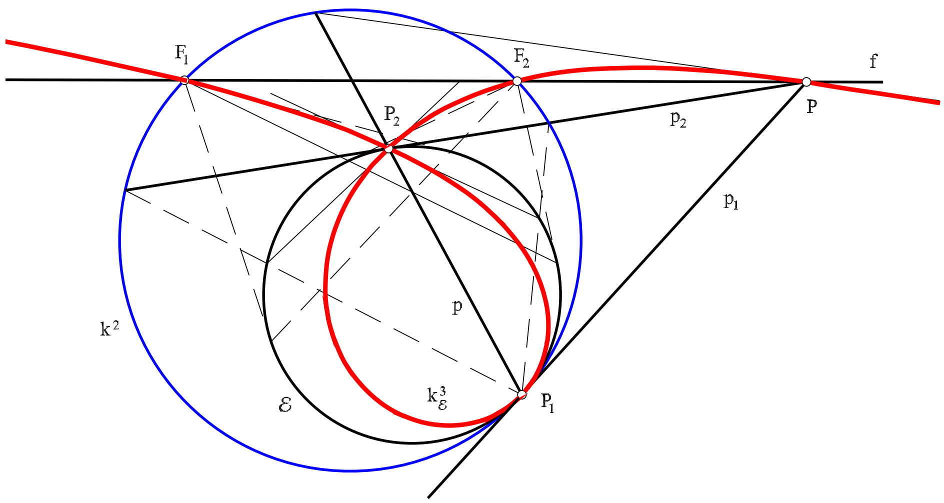

Figure 12

The inverse image of the conic k^2 given by (1) is an at least 2-*circular quartic* $k^4_{\mathcal{E}}$:

$$a_{00}(x+y)^2 + a_{11}(1+xy-y^2)^2 + a_{22}(1+xy-x^2)^2 + \\ + 2(x+y)\big[a_{01}(1+xy-y^2) + a_{02}(1+xy-x^2)\big] + \\ + 2a_{12}(1+xy-x^2)(1+xy-y^2) = 0.$$

In Fig. 13 an entirely circular quartic of type (4,0) is shown.

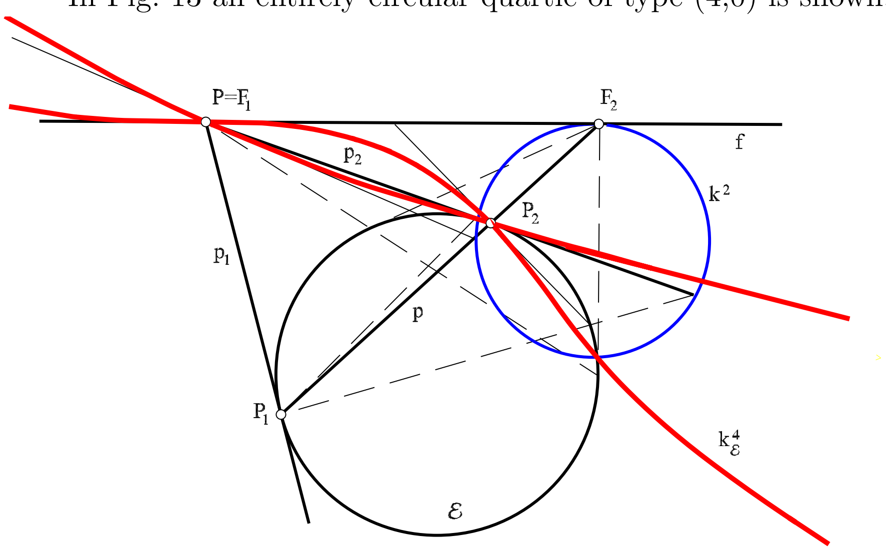

Figure 13

As in the analogous case of the inversion with a hyperbola as the fundamental conic the following theorems holds:

Theorem 8. *Let $\sigma(P, \mathcal{E})$ be an inversion given in the pseudo-Euclidean plane where $\mathcal{E}$ is an ellipse and P an isotropic point. An inverse image of a conic k^2 of type (t, r) not passing through the fundamental points of σ is:*

- *$(t+r)$-circular quartic $k^4_{\mathcal{E}}$ of type (r, t), if $P \neq F_i$, $i = 1, 2$.*
- *$(r+2)$-circular quartic $k^4_{\mathcal{E}}$ of type $(r+2, 0)$, if e.g. $P = F_1$.*

Theorem 9. *Let $\sigma(P, \mathcal{E})$ be an inversion given in the pseudo-Euclidean plane where $\mathcal{E}$ is an ellipse and P an isotropic point. An inverse image of a conic k^2 of type (t, r) passing through one of the fundamental points of σ is:*

- *$(t+r)$-circular cubic $k^3_{\mathcal{E}}$ of type (r, t), if $P \neq F_i$, $i = 1, 2$.*
- *$(t+r+1)$-circular cubic $k^3_{\mathcal{E}}$, if $P = F_i$, $i \in \{1, 2\}$.*

If $P = F_1$, $k^3_{\mathcal{E}}$ is of type $(t+r+1, 0)$ for $t < 2$ and of type $(2, 1)$ for $t = 2$.

4. Conclusions

In this paper we have presented the inversions in the pseudo-Euclidean plane having an equiform conic for the fundamental conic and a pole at its center or at an isotropic point.

It has been shown that by using these transformations the rational cubics of all types of circularity can be constructed as images of conics. Unfortunately, the same does not hold for the quartics. For example we can not obtain any $(3, 1)$-circular quartic. Therefore we conclude this paper with the following theorem while the construction of all types of rational circular quartics is left for further investigation.

Theorem 10. *The rational cubics of all types of circularity in the pseudo-Euclidean plane can be constructed as images of conics by using the inversions having an equiform conic as fundamental conic and its center or an isotropic point as pole.*

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