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A CONSEQUENCE OF THE THEOREM OF BREDIHIN

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Abstract: By using a theorem of Bredihin on Beurling's primes it is proved that

$$\#\{n|\vartheta_2(n) \le x\} = Cx(\log x)^{\tau-1} + \mathcal{O}(x(\log x)^{\tau-1-\varepsilon_1})$$

with some constants $\varepsilon_1 > 0$, $\tau > 0$, where ϑ is completely multiplicative, $\vartheta(p) = p + 1$ for every prime p, and $\vartheta_2(n) = \vartheta(\vartheta(n))$.

1. Introduction and formulation of the theorem

We shall use the following notations: $\mathbb{N}$ = set of natural numbers, $\mathcal{P}$ = set of primes, p with or without suffixes always denote prime numbers, $\pi(x)$ = number of primes up to x, $\pi(x, k, l)$ = number of primes up to x belonging to the arithmetic progression $\equiv l \pmod{k}$. The letters $c, c_1, c_2, \ldots$ denote positive constants not necessarily same at different oc-

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currences. (m,n) denotes the greatest common divisor of $m,n \in \mathbb{N}$. Let $P(n)$ be the largest prime factor of n.

A function $f : \mathbb{N} \to \mathbb{C}$ (= set of complex numbers) is said to be a multiplicative function, if $f(1) = 1$ and $f(mn) = f(m) \cdot f(n)$ holds for every coprime pairs of $m,n \in \mathbb{N}$. We say that f is completely multiplicative, if $f(mn) = f(m) \cdot f(n)$ is satisfied for every $m,n \in \mathbb{N}$.

A function $g : \mathbb{N} \to \mathbb{R}$ (= set of real numbers) is additive, if $g(1) = 0$ and $g(mn) = g(m) + g(n)$, when $(m,n) = 1$.

Let $\varphi(n)$ be Euler's totient function, $\sigma(n)$ be the sum of divisors function, $\omega(n)$ = number of distinct divisors of $n, \mu(n)$ be the Moebius-function. φ, σ, μ are multiplicative, ω is additive. For some prime power $p^\alpha : \varphi(p^\alpha) = p^\alpha - p^{\alpha-1};\ \sigma(p^\alpha) = 1 + p + \ldots + p^\alpha, \mu(p) = -1,\ \mu(p^\alpha) = 0$ if $\alpha \geq 2,\ \omega(p^\alpha) = 1$.

Let $f(n)$ $(n \in \mathbb{N})$ be such a function for which $f(n) \to \infty (n \to \infty)$. A natural question is to find the asymptotic of

$$\#\{n \in \mathbb{N} | f(n) \leq x\} \quad \text{as} \quad x \to \infty. \tag{1.1}$$

In some cases it is harder than to count the asymptotic of $\sum_{n \leq x} f(n)$.

P. T. Bateman investigated (1.1) for $f(n) = \varphi(n)$ by analyzing the Dirichlet series

$$F_0(s) = \sum_{n=1}^{\infty} \frac{1}{\varphi(s)^s} \quad (s = \sigma + it)$$

close to the vertical line $\sigma = 1$, and proved that

$$\#\{n | \varphi(n) \leq x\} = Cx + \mathcal{O}\left(x \exp\left(-(1-\varepsilon)\left(\frac{1}{2} (\log x)(\log\log x) \right) \right)^{\frac{1}{2}} \right)$$

for any $\varepsilon > 0$. Here $C = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$, ζ is the Riemann zeta function.

Similar estimate can be done for $\#\{n | \sigma(n) \leq x\}$, namely

$$\#\{n | \sigma(n) \leq x\} = C_1 x + \mathcal{O}\left(x \exp\left(-(\log x) \right)^{\frac{1}{2}} \right),$$

by using the Dirichlet series,

$$F_1(s) = \sum_{n=1}^{\infty} \frac{1}{\sigma(n)^s},$$

and analyzing its properties at $\sigma = 1$. Here C_1 is a calculable constant. (See also [2], [3], [4]).

Let $a = -1$, or $a \in \mathbb{N}$ be a fixed number, $\kappa_a(n)$ be a completely multiplicative function generated by $\kappa_a(p) = p + a \quad (p \in \mathcal{P})$. Then

$$F^{(a)}(s) = \sum_{n=1}^{\infty} \frac{1}{\kappa_a(n)^s} = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{(p+a)^s}\right)^{-1},$$

and, by using the method of Smati [2] one can prove that

$$\#\{n | \kappa_a(n) \le x\} = c(a)x + \mathcal{O}(x \exp(-c\sqrt{\log x})),$$

where $c(a)$ and c are positive constants.

It would be nice to know the asymptotic of (1.1) for example, if $f(n) = \varphi(\varphi(n))$, $f(n) = \sigma(\varphi(n))$, $f(n) = \varphi(\sigma(n))$, $f(n) = \sigma(\sigma(n))$. There exist some inequalities of (1.1) for these functions in the literature but the asymptotic is unknown.

Similarly, it would be interesting to count the asymptotic of

$$\#\{p \in \mathcal{P}, f(p) < x\} \tag{1.2}$$

where

$$f(p) = \varphi(p+1),\ f(p) = \sigma(p+1),\ f(p) = \kappa_1(p+1). \tag{1.3}$$

Theorem A. *Let f be one of the functions listed in* (1.3)*. Then, there is a positive constant τ such that*

$$\#\{p \in \mathcal{P} | f(p) < x\} = \tau \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^{1+\varepsilon}}\right) \tag{1.4}$$

holds for any constant $\varepsilon < \frac{1}{2}$.

We shall prove it only in the case $f(p) = \kappa_1(p+1)$. In what follows we shall write ϑ instead of κ_1.

Theorem 1. *Let ϑ be completely multiplicative, $\vartheta(p) = p + 1$ for $p \in \mathcal{P}$. Let*

$$R(x) = \#\{p \in \mathcal{P} | \vartheta(p+1) \le x\}.$$

Then

$$R(x) = \tau \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^{1+\varepsilon}}\right),$$

where τ is a positive constant, ε is an arbitrary constant less than 1/2*. $\tau = S(\infty)$, S is defined in* (3.10), (3.11)*.*

Let $1 < \pi_1 \le \pi_2 \le \dots, \quad \pi_j \to \infty \quad (j \to \infty)$ be a sequence of real numbers, $\tilde{\mathcal{P}} = \{\pi_j | j = 1, 2, \dots\}$. Let $\tilde{\mathcal{N}}$ be the semigroup generated by $\tilde{\mathcal{P}}$ under multiplication. Assume that the elements of $\tilde{\mathcal{N}}$ are arranged in ascending order and are denoted by $\{n_i\}_{i=1}^{\infty}$. Let $\Pi_{\tilde{\mathcal{P}}}(x) = \sum_{\pi_j < x} 1$; $N_{\tilde{\mathcal{P}}}(x) = \sum_{n_j < x} 1$.

$\tilde{\mathcal{N}}$ and $\tilde{\mathcal{P}}$ are called the sets of Beurling's type of integers, and that of the set of Beurling's type of primes. These types of semigroups have been introduced by A. Beurling [5]. M. B. Bredihin proved the following assertion which is quoted now as

Lemma 1. *If*

$$\Pi_{\tilde{\mathcal{P}}}(x) = \tau \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^{1+\varepsilon}}\right)$$

with some $\varepsilon > 0$, *then*

$$N_{\tilde{\mathcal{P}}}(x) = Cx(\log x)^{\tau-1} + \mathcal{O}\left(\frac{x(\log x)^{\tau-1}}{(\log x)^{\varepsilon_1}}\right),$$

where $\varepsilon_1 = \min\{1, \varepsilon\}$.

If we choose $\tilde{\mathcal{P}} = \{\vartheta(p) = p+1,\ p \in \mathcal{P}\}$, then $\mathcal{N} = \{\vartheta(n) | n \in \mathbb{N}\}$. Since $\Pi_{\tilde{\mathcal{P}}}(x) = \pi(x-1) = \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^2}\right)$, therefore $T(x)\ \ (= N_{\tilde{\mathcal{P}}}(x)) = = Cx + \mathcal{O}\left(\frac{x}{\log x}\right)$, according to Lemma 1.

Let

$$T_2(x) := \#\{n | \vartheta_2(n) \le x\}.$$

From Lemma 1 and Th. 1 immediately follows

Theorem 2. *We have*

$$T_2(x) = Cx(\log x)^{\tau-1} + \mathcal{O}\left(x(\log x)^{\tau-1-\varepsilon}\right),$$

where $0 < \varepsilon < 1/2$, *C is a positive constant and* τ *is the same as in Th. 1.*

2. Auxiliary results

Let $\pi_k(x) = \#\{n \le x \mid \omega(n) = k\}$.

Lemma 2 (Hardy–Ramanujan [8]). *We have*

$$\pi_k(x) < \frac{c_1 x}{\log x} \frac{(\log\log x + c_2)^{k-1}}{(k-1)!} \quad (k = 1, 2, \ldots)$$

where c_1, c_2 *are suitable explicitly given constants.*

Let, as usual,

$$\mathrm{li}x = \int_2^x \frac{du}{\log u}.$$

According to the Siegel–Walfisz theorem we have

$$\left|\pi(x,k,l) - \frac{\mathrm{li} x}{\varphi(k)}\right| < C\frac{\mathrm{li} x}{\varphi(k)(\log x)^A},$$

uniformly as $(k,l)=1$, $k \le (\log x)^A$, $2 \le x$, A is an arbitrary constant, $C = C(A)$. (See [9].)

Lemma 3 (Sieve results). *We have*

(1)

$$\pi(x+y,k,l) - \pi(x,k,l) < \frac{cy}{\varphi(k)\log\frac{y}{k}} \quad \textit{if} \;\; 1 \le k \le y \le x,$$

especially

(2)

$$\pi(x+y) - \pi(x) < \frac{Cy}{\log y} \quad \textit{if} \;\; 1 < y < x,$$

(3)

$$\pi(x,k,l) < \frac{Cx}{\varphi(k)\log\frac{x}{k}} \quad \textit{if} \;\; k < x,$$

where C is an absolute constant.

(1) is contained in Th. 3.7 in [7], (2) and (3) are special cases of (1).

Lemma 4 (Bombieri–Vinogradov inequality). *Let A be an arbitrary constant, $B \ge 2A+5$. Then*

$$\sum_{k \le \frac{\sqrt{x}}{(\log x)^B}} \max_{y \le x} \max_{(l,k)=1} \left|\pi(x,k,l) - \frac{\mathrm{li} x}{\varphi(k)}\right| \le d\frac{x}{(\log x)^A},$$

where the constant d is ineffective. (See in [9].)

Lemma 5. *For every constant $A(>0)$ there exists a constant B such that*

$$\sum_{\substack{k \le x^{1/3} \\ \omega(k) > B\log\log x}} \frac{2^{\omega(k)}}{\varphi(k)} \le \frac{c}{(\log x)^A} \quad \textit{if} \;\; x \ge 10, \tag{2.1}$$

where c is a constant.

Proof. Let $U_0 = 3$, $U_{j+1} = 2U_j$, $(j = 0,1,\ldots)$. Let $U_T \le x^{1/3} < U_{T+1}$. Let

$$R_h = \sum_{\substack{U_h \le k \le U_{h+1} \\ \omega(k) > B\log\log x}} \frac{2^{\omega(k)}}{\varphi(k)}. \tag{2.2}$$

It is known that $\omega(k) < \log k$, therefore $R_h = 0$ if $\log k < B \log\log x$, i.e. if $k < (\log x)^B$. Assume that $U_{h+1} \geq (\log x)^B$. We know that $\varphi(n) > > \frac{cn}{\log\log n}$ $(n \geq 3)$, therefore

$$\frac{1}{\varphi(k)} \leq \frac{c \log\log U_h}{U_h} \quad \text{if } \ k \in (U_h, U_{h+1}),$$

where c is an absolute constant. Then by Lemma 2,

$$R_h \leq \frac{c \log\log U_h}{U_h} \sum_{\substack{U_h \leq k < U_{h+1} \\ \omega(k) \geq B \log\log x}} 2^{\omega(k)} \leq$$

$$\leq \frac{c(\log\log U_h) U_{h+1}}{U_h (\log U_h)} \sum_{l \geq B \log\log x} 2^l \frac{(\log\log U_h + c_1)^{l-1}}{(l-1)!}.$$

Thus the left-hand side of (2.1) is less than

$$\sum_{\substack{U_{h+1} \geq (\log x)^B \\ h \leq T}} \leq \Sigma_1 \cdot \Sigma_2,$$

where

$$\Sigma_1 = \sum_{h \leq T} \frac{2c(\log\log U_h)}{\log U_h} \leq c_2 \log x,$$

$$\Sigma_2 = \sum_{l \geq B \log\log x} \frac{(2 \log\log x + 2c_1)^{l-1}}{(l-1)!}.$$

Let $\eta_m = \frac{(2\log\log x + 2c)^m}{m!}$. Then, from $\log m! = m \log \frac{m}{e} + \mathcal{O}(1)$ we obtain that

$$\eta_m \leq c_2 \exp\left(m \log \frac{2 \log\log x + 2c}{m} \right).$$

Let $m \geq B \log\log x - 1$. Then $\eta_m \leq c_2 \exp(-m \log \frac{B}{3})$, and so

$$\Sigma_2 = \sum_{m \geq B \log\log x} \eta_m \leq c_2 \sum_{m \geq B \log\log x} \exp\left(-(m-1) \log \frac{B}{3} \right) \leq$$

$$\leq 2 \frac{Bc_2}{3} \exp\left(-B (\log\log x) \log \frac{B}{3} \right)$$

if $x > x(B)$. Thus

$$\Sigma_2 \leq c_3 (\log x)^{-B \log \frac{B}{3}},$$

$c_3 = c_3(B)$. Hence Lemma 5 immediately follows. $\Diamond$

3. Proof of Theorem 1

Let $t = (\log x)^{\varepsilon}$, $0 < \varepsilon < \frac{1}{2}$, ε be fixed, $Q = \prod_{p<t} p$. For some integer n let

$$M(n) := \prod_{\substack{p^a \| n \\ p \le t}} p^a, \qquad E(n) = \prod_{\substack{p^r \| n \\ p > t}} p^r.$$

Let $U(x|D) = \#\{p \le x | M(p+1) = D\}$. $M(p+1) = D$ holds, if and only if $p+1 \equiv 0 \pmod{D}$, and $\left(\frac{p+1}{D}, Q\right) = 1$. Thus

$$U(x, D) = \sum_{\substack{p \le x \\ p+1 \equiv 0 \pmod{D}}} \sum_{\delta | \left(Q, \frac{p+1}{D}\right)} \mu(\delta).$$

Thus

$$U(x|D) = \sum_{\delta | Q} \mu(\delta) \pi(x, \delta D, -1).$$

Let

$$\nu(D) := \sum_{\delta | Q} \frac{\mu(\delta)}{\varphi(\delta D)} = \frac{1}{D} \prod_{\substack{p<t \\ p \nmid D}} \left(1 - \frac{1}{p-1}\right).$$

Hence

$$\max_{y \le x} |U(y|D) - \nu(D) \mathrm{li}\, y| \le \sum_{\delta | Q} \max_{y \le x} \left| \pi(y, \delta D, -1) - \frac{\mathrm{li}\, y}{\varphi(D\delta)} \right|. \tag{3.1}$$

Let $P(n)$ be the largest prime factor of n. Let us sum over those $D \le x^{\frac{1}{4}}$, for which $P(n) \le t$.

Then

(3.2)
$$\sum_{\substack{D \le x^{\frac{1}{4}} \\ P(D) \le t}} \max_{y \le x} |U(y|D) - \nu(D) \mathrm{li}\, y| \le \sum_{\substack{k \le x^{\frac{1}{3}} \\ say}} \max_{y \le x} \left| \pi(y|k, -1) - \frac{\mathrm{li}\, y}{\varphi(k)} \right| \cdot 2^{\omega(k)}.$$

By using Lemma 4, one can deduce that

$$\sum_{\substack{D \le x^{\frac{1}{4}} \\ P(D) \le t}} \max_{y \le x} |U(y|D) - \nu(D) \mathrm{li}\, y| \le \frac{c \mathrm{li}\, x}{(\log x)^A} \tag{3.3}$$

where A is an arbitrary positive constant. It is enough to observe that the right-hand side of (3.3) can be subdivided into two parts according to $\omega(k) \le B \log\log x$ or $\omega(k) > B \log\log x$. The sum under $\omega(k) \le$ $\le B \log\log x$ is $\mathcal{O}\left(\frac{\mathrm{li}\, x}{(\log x)^A}\right)$, for every fixed B. The second sum is less than

$$(\mathrm{li}\, x) \sum_{\substack{k \le x^{\frac{1}{3}} \\ \omega(k) > B \log\log x}} \frac{1}{\varphi(k)} \cdot 2^{\omega(k)} \ll \frac{\mathrm{li}\, x}{(\log x)^A}$$

if B is large enough. See Lemma 5.

3.1.

$Y = (\log x)^{2\varepsilon_2}$, $\quad H = \frac{1}{(\log x)^{\varepsilon_2}}, 0 < \varepsilon_2 < \frac{1}{2}$.

Let g_1, g_2 be completely multiplicative,

$$g_1(q) = \begin{cases} 1 + \frac{1}{q}, & \text{if} \quad q < Y \\ 1, & \text{if} \quad q \ge Y \end{cases}, \qquad g_2(q) = \begin{cases} 1, & \text{if} \quad q < Y \\ 1 + \frac{1}{q}, & \text{if} \quad q \ge Y \end{cases}$$

for every prime q.

$$\kappa(n) = \frac{\vartheta(n)}{n} = \prod_{p^\alpha \| n} \left(1 + \frac{1}{p}\right)^\alpha = g_1(n) \cdot g_2(n).$$

Observe that, from the known inequality $\pi(x, k, l) < C(\mathrm{li}\ x)/\varphi(k)$ if $k \le \sqrt{x}$ (see (3) in Lemma 3), say, we have

$$\sum_{p \le x} \log g_2(p+1) \le c \sum \pi(x, q^a, -1) \frac{a}{q} \ll (\mathrm{li}\ x) \sum_{q > Y} \frac{1}{q^2} \ll$$
$$\ll \frac{\mathrm{li}\ x}{Y \log Y}.$$

Hence

$$\#\left\{p \le x \middle| \log g_2(p+1) > H\right\} \ll \frac{\mathrm{li}\ x}{HY \log Y},$$

thus

$$\#\left\{p \le x \middle| \log g_2(p+1) > H\right\} \ll \frac{\mathrm{li}\ x}{(\log x)^{\varepsilon_2}}.$$

This quantity is not bigger than the error term.

Let us observe:

1.) if $\vartheta(p+1) \le x$, and $A_Y(p+1) = D$, then $(p+1)\kappa(D) \le x$.

2.) if $(p+1)\kappa(D) \le x$, $\quad \log g_2(p+1) \le H$, and $\vartheta(p+1) > x$, then

$$(3.4) \qquad (p+1)\kappa(D) > \frac{x}{g_2(p+1)} \ge x e^{-\log_2(p+1)} \ge x - \frac{cx}{H}$$

and so

$$\frac{x}{\kappa(D)} - \frac{cx}{H\kappa(D)} \le p+1 \le \frac{x}{\kappa(D)}. \tag{3.5}$$

By sieve we obtain that the number of primes satisfying (3.5) is less than

$$\frac{\operatorname{li} x}{H\kappa(D)\varphi(D)}.$$

The sum of this quantity over those D for which $P(D) \le t$ is $\mathcal{O}\left(\frac{(\log t)\operatorname{li}x}{H}\right) = \mathcal{O}\left(\frac{\varepsilon(\log\log x)\operatorname{li}x}{H}\right)$. Indeed,

$$\begin{aligned}\sum_{P(D)\le t} \frac{1}{\kappa(D)\varphi(D)} &\le \prod_{p\le t}\left(1 + \frac{1}{\kappa(p)\varphi(p)} + \frac{1}{\kappa(p^2)\varphi(p^2)} + \cdots\right) \le \\ &\le \prod_{p\le t}\left(1+\frac{1}{p}+\frac{c}{p^2}\right) \le \\ &\le \exp\left(\sum_{p\le t}\frac{1}{p} + c_1\right) \le \\ &\le c_2 \log t.\end{aligned}$$

Here c, c_1, c_2 are absolute constants.

3.) Since $U(x|D) \le \pi(x, D, -1)$, and $\pi(x, D, -1) < C\frac{\operatorname{li}x}{\varphi(D)}$ if $1 \le D \le \sqrt{x}$ (see Lemma 3), therefore

$$\sum_{\substack{D > x^{1/4} \\ P(D)\le t}} U(x|D) < C\operatorname{li}x \sum_{\substack{x^{1/4} < D < x^{1/2} \\ P(D) \le t}} \frac{1}{\varphi(D)} + x \sum_{\substack{x^{1/2} \le D < x \\ P(D)\le t}} \frac{1}{D}. \tag{3.6}$$

Let

$$\Psi(x,y) = \#\{n \le x,\ P(n) \le y\}.$$

It is known (see [9], Th. 1 in Ch. III. 5) that

$$\begin{cases} \Psi(x,y) \le cxe^{-u/2} \quad \text{if} \quad x \ge y \ge 2, \\ u = \frac{\log x}{\log y}. \end{cases} \tag{3.7}$$

Then

$$\sum_{\substack{P(D)\le t \\ V \le D \le 2V}} 1 \le cV \exp\left(-\frac{\log V}{2\log t}\right) \le cV\exp\left(\frac{-\log x}{8\varepsilon \log\log x}\right). \tag{3.8}$$

Subdividing the interval $[x^{1/4}, x^{1/2})$, and $[x^{1/2}, x)$ into intervals of type $[V, 2V)$ and observing that $1/\varphi(D) \le \frac{c \log\log x}{V}$ if $D \in [V, 2V]$, we obtain that

$$\sum_{\substack{D > x^{1/4} \\ P(D) \le t}} U(x|D) \le c \frac{\mathrm{li} x}{H}.$$

From (3.3) we obtain that

$$R(x) = \mathrm{li} x \sum_{\substack{D \le x^{1/4} \\ P(D) \le t}} \frac{\nu(D)}{\kappa(D)} + \mathcal{O}\left(\frac{(\log t)\mathrm{li} x}{H}\right). \tag{3.9}$$

To estimate the right-hand side of (3.9), observe that $\nu(D) = 0$ if D is odd, furthermore that

$$\nu(D) = \frac{1}{D} \prod_{\substack{p<t \\ p \nmid D}} \left(1 - \frac{1}{p-1}\right) = \frac{1}{D} \prod_{\substack{p<t \\ p>2}} \left(1 - \frac{1}{p-1}\right) \prod_{\substack{p|D \\ p>2}} \frac{1}{1 - \frac{1}{p-1}}.$$

Let h be multiplicative,

$$h(2^\alpha) = \frac{1}{2^\alpha (1 + \frac{1}{2})^\alpha} = \frac{1}{3^\alpha}$$

$$h(p^\alpha) = \frac{1}{p^\alpha (1 - \frac{1}{p-1})(1 + \frac{1}{p})^\alpha} = \frac{p-1}{(p+1)^\alpha (p-2)} \quad \text{if } 3 \le p < t \quad (p \in \mathcal{P}),$$

and $h(p^\alpha) = 0$ if $p \ge t$. Thus

$$\frac{\nu(D)}{\kappa(D)} = L(t) h(D), \quad \text{if } P(D) \le t,$$

where

$$L(t) = \prod_{2 < p \le t} \left(1 - \frac{1}{p-1}\right). \tag{3.10}$$

We have

$$\sum_{P(D) \le t} h(D) =$$

$$= \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right) \prod_{3 \le p \le t} \left(1 + \frac{(p-1)}{(p-2)(p+1)} \left(1 + \frac{1}{p+1} + \frac{1}{(p+1)^2} + \cdots\right)\right) =$$

$$= \frac{1}{2} \prod_{3 \le p \le t} \left(1 + \frac{p-1}{p(p-2)}\right),$$

and so

$$S(t) := \sum_{P(D)\le t} h(D)L(t) = \frac{1}{2}\prod_{3\le p\le t}\frac{(p-2)}{(p-1)}\frac{(p^2-p-1)}{p(p-2)} = \tag{3.11}$$

$$= \frac{1}{2}\prod_{3\le p\le t}\left(\frac{p^2-p-1}{p^2-p}\right) =$$

$$= \frac{1}{2}\prod_{3\le p\le t}\left(1-\frac{1}{p(p-1)}\right).$$

Let

$$\tau = S(\infty) = \frac{1}{2}\prod_{3\le p}\left(1-\frac{1}{p(p-1)}\right) \tag{3.12}$$

$S(t)$ is a monoton decreasing function of t,

$$1 \le \frac{S(t)}{S(\infty)} = \prod_{p>t}\left(1-\frac{1}{p(p-1)}\right)^{-1} < \exp\left(2\sum_{p>t}\frac{1}{p(p-1)}\right) <$$

$$< \exp\left(\frac{2}{t}\right) < 1+\frac{4}{t}, \quad \text{and so}$$

$$S(t) = \tau + \mathcal{O}\left(\frac{1}{t}\right).$$

Finally we observe that

$$\sum_{\substack{D>x^{1/4}\\ P(D)<t}}\frac{\nu(D)}{\kappa(D)} \le$$

$$\le \sum_{j=0}^{\infty}\frac{1}{2^j x^{1/4}}\psi(2^j x^{1/4}, t) \le$$

$$\le c\sum_{j=0}^{\infty}\exp\left(-\frac{1}{2}\frac{\log 2^j x^{1/4}}{\log t}\right) = x\exp\left(-\frac{1}{8}\frac{\log x}{\log t}\right)\left(1-e^{-\frac{\log 2}{2\log t}}\right)^{-1} \le$$

$$\le c_2\exp(-\sqrt{\log x}), \quad \text{say.}$$

Thus

$$R(x) = \tau\,\mathrm{li}x + \mathcal{O}\left(\frac{\mathrm{li}x}{t}\right) + \mathcal{O}(\mathrm{li}x\cdot\exp(-\sqrt{\log x})) +$$

$$+ \mathcal{O}\left(\frac{(\log t)\cdot \mathrm{li}x}{H}\right).$$

By choosing $\varepsilon < \varepsilon_2 < \frac{1}{2}$, Th. 1 follows.

Th. 2 is a direct consequence of Lemma 1 and of Th. 1.

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