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SPECIAL CLASSES OF RIGHT NEAR-RING RIGHT MODULES

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Abstract: Near-rings considered are right near-rings and R is a near-ring. In this paper an equiprime right R-group is introduced. An ideal I of R is equiprime if and only if I is the annihilator of an equiprime right R-group. Using it special classes of near-ring right modules are introduced. A characterization of the special radicals of near-rings in terms of right modules of near-rings is presented which is similar to the characterization of the special radicals of rings developed by Andrunakievich and Rjabuhin. Some special classes of near-ring right modules are also presented.

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1. Introduction

By a near-ring we mean a right near-ring and R is a near-ring. Andrunakievich and Rjabuhin [1] characterized the special radicals of rings using modules. Booth, Groenewald and Veldsman [4] introduced and studied equiprime near-rings. Using equiprime near-rings, Booth and Groenewald [2] developed special radicals of near-rings and in [3] they gave a characterization of special radicals of zero-symmetric near-rings in terms of left modules of near-rings.

Srinivasa Rao and Siva Prasad [9, 10, 11, 12] introduced and studied the right Jacobson radicals of type-0, 1, 2, and s for near-rings and showed that unlike left Jacobson radicals these are relevant for the extension of a form of Wedderburn–Artin theorem of rings involving matrix rings to near-rings. Unlike in rings, the left and right Jacobson radicals of a near-ring are not comparable. For example, in [13, 14] it is shown that the right Jacobson radicals of near-rings of type-0, 1 and 2 are Kurosh–Amitsur radicals (KA-radicals) in the class of all zero-symmetric near-rings but it is well known that the left Jacobson radicals of type-0 and 1 are not KA-radicals in the class of all zero-symmetric near-rings. Moreover, in [15, 8] the right Jacobson radicals of type-$0(e)$, $1(e)$, and $2(e)$ are introduced for near-rings and showed that they are special radicals of near-rings. This shows the important role played by the right modules of near-rings in the development of structure theory of near-rings.

In this paper an equiprime right R-group is introduced. An ideal I of R is equiprime if and only if I is the annihilator of an equiprime right R-group. Using it special classes of near-ring right modules are introduced. A characterization of the special radicals of near-rings in terms of right modules of near-rings is presented which is similar to the characterization of the special radicals of rings developed by Andrunakievich and Rjabuhin [1]. Some special classes of near-ring right modules are also presented.

2. Preliminaries

R stands for a right near-ring (not necessarily zero-symmetric) and all notations and definitions will be as in [7].

We need the following definitions and results of [9] and [10].

A group $(G, +)$ is called a *right R-group* if there is a mapping $((g, r) \to gr)$ of $G \times R$ into G such that

(i) $(g+h)r = gr + hr$, and
(ii) $g(rs) = (gr)s$, for all $g, h \in G$ and $r, s \in R$.

A subgroup (normal subgroup) H of a right R-group of G is called an *R-subgroup (ideal)* of G if $hr \in H$ for all $h \in H$ and $r \in R$.

Let G, H be right R-groups. A mapping $f : G \to H$ is called an *R-homomorphism* if

(i) $f(x+y) = f(x) + f(y)$ and
(ii) $f(xr) = f(x)r$ for all $x, y \in G$ and for all $r \in R$.

G is said to be *R-isomorphic* to H if there is a one-to-one R-homomorphism of G onto H.

An element g in a right R-group G is called *distributive* if $g(r+s) = gr + gs$ for all $r, s \in R$.

Let G be a right R-group. An element $g \in G$ is called a *generator* of G if g is distributive and $gR = G$. G is said to be *monogenic* if G has a generator.

A monogenic right R-group G is said to be a *right R-group of type*-0 if G is simple, that is, G has no non-trivial ideals and $GR \neq \{0\}$.

A right R-group G of type-0 is said to be of *type*-1 if G has exactly two R-subgroups namely, $\{0\}$ and G.

A right R-group G of type-0 is said to be of *type*-2 if $gR = G$ for all $0 \neq g \in G$.

A near-ring R is called an *equiprime near-ring* if $0 \neq a \in R$, $x, y \in R$ and $arx = ary$ for all $r \in R$, implies $x = y$. An ideal I of R is called *equiprime* if R/I is an equiprime near-ring.

It is known that a near-ring R is equiprime if and only if

1. $x, y \in R$ and $xRy = \{0\}$ implies $x = 0$ or $y = 0$.
2. If $\{0\} \neq I$ is an invariant subnear-ring of R, $x, y \in R$ and $ax = ay$ for all $a \in I$ implies $x = y$.

Moreover, an equiprime near-ring is zero-symmetric.

If I is an ideal of R, then we denote it by $I \lhd R$. A subset S of R is *left invariant* if $RS \subseteq S$. By a radical class we mean a radical class in the sense of Kurosh–Amitsur.

Let $\mathcal{E}$ be a class of near-rings. $\mathcal{E}$ is called *regular*, if $\{0\} \neq I \lhd R \in \mathcal{E}$ implies that $0 \neq I/K \in \mathcal{E}$ for some $K \lhd I$. It is known that, if $\mathcal{E}$ is a regular class, then $\mathcal{UE} = \{R \mid R$ has no non-zero homomorphic image in $\mathcal{E}\}$ is a radical class, called the *upper radical* determined by $\mathcal{E}$. The *subdirect closure* of a class of near-rings $\mathcal{E}$ is the class $\overline{\mathcal{E}} = \{R \mid R$ is a subdirect sum of near-rings from $\mathcal{E}\}$. A class $\mathcal{E}$ is called hereditary if

$I \lhd R \in \mathcal{E}$ implies $I \in \mathcal{E}$. $\mathcal{E}$ is called *c-hereditary* if I is a left invariant ideal of $R \in \mathcal{E}$ implies $I \in \mathcal{E}$. It is clear that a hereditary class is a regular class. If $I \lhd R$ and for every non zero ideal J of R, $J \cap I \neq \{0\}$, then I is called an *essential ideal* of R and is denoted by $I \lhd \cdot R$. A class of near-rings $\mathcal{E}$ is called closed *under essential extensions (essential left invariant extensions)* if $I \in \mathcal{E}$, $I \lhd \cdot R$ (I is an essential ideal of R which is left invariant) implies $R \in \mathcal{E}$. A class of near-rings $\mathcal{E}$ is said to satisfy condition (F_l) if $K \lhd I \lhd R$, and I is left invariant in R and $I/K \in \mathcal{E}$, then $K \lhd R$.

In [2], Booth and Groenewald defined special radicals for near-rings. A class $\mathcal{E}$ consisting of equiprime near-rings is called a *special class* if it is hereditary and closed under essential left invariant extensions. If $\mathcal{R}$ is the upper radical in the class of all near-rings determined by a special class of near-rings, then $\mathcal{R}$ is called a *special radical.* If $\mathcal{R}$ is a radical class, then the class $\mathcal{SR} = \{R \mid \mathcal{R}(R) = \{0\}\}$ is called the *semisimple class* of $\mathcal{R}$.

We also need the following theorem:

Theorem 2.1 (Th. 2.4 of [16])**.** *Let $\mathcal{E}$ be a class of zero-symmetric near-rings. If $\mathcal{E}$ is regular, closed under essential left invariant extensions and satisfies condition (F_l), then $\mathcal{R} := \mathcal{UE}$ is c-hereditary radical class in the variety of all near-rings, $\mathcal{SR} = \overline{\mathcal{E}}$ and $\mathcal{SR}$ is hereditary. So, $\mathcal{R}(R) = \cap\{I \lhd R \mid R/I \in \mathcal{E}\}$ for all near-rings R.*

Remark 2.2. Since all ideals in a zero-symmetric near-ring are left invariant, under the hypothesis of Th. 2.1, in the variety of zero-symmetric near-rings both $\mathcal{R}$ and $\mathcal{SR}$ are hereditary and hence the radical is ideal-hereditary, that is, if $I \lhd R$, then $\mathcal{R}(I) = I \cap \mathcal{R}(R)$.

Proposition 2.3 (Prop. 3.3 of [4])**.** *The class of all equiprime near-rings is closed under essential left invariant extensions.*

Proposition 2.4 (Cor. 2.4 of [4])**.** *The class of all equiprime near-rings satisfy condition (F_l).*

3. Equiprime right *R*-groups

Throughout this section R stands for a right near-ring and not necessarily zero-symmetric.

The *annihilator* of a right R-group G, denoted by $(0 : G)$, is defined as $(0 : G) = \{a \in R \mid Ga = \{0\}\}$.

Proposition 3.1. *Let G be a right R-group and $G0 = \{0\}$. Suppose that I, J are ideals of R and $GI = \{0\}$, $GJ = \{0\}$ implies $G(I + J) = \{0\}$. Then there is a largest ideal of R contained in* $(0 : G)$.

Proof. Since $G0 = \{0\}$, the zero ideal of R is contained in $(0 : G)$. Let I and J be ideals of R contained in $(0 : G)$. By our assumption $I + J \subseteq (0 : G)$. From this we get that for any collection of ideals of R contained in $(0 : G)$ their sum is an ideal of R contained in $(0 : G)$. Therefore, the sum K of all ideals T of R such that $T \subseteq (0 : G)$ is the largest ideal of R contained in $(0 : G)$. $\Diamond$

Definition 3.2. A right R-group G is said to be *equiprime* if:

(i) $GR \neq \{0\}$ and $G0 = \{0\}$;

(ii) I, J are ideals of R and $GI = \{0\}$, $GJ = \{0\}$ implies $G(I + J) = \{0\}$;

(iii) $0 \neq g \in G$, $a, b \in R$ and $gxa = gxb$ for all $x \in R$ implies $a - b \in P$, where P is the largest ideal of R contained in $(0 : G)$;

(iv) $r, s \in R$ and $r - s \in P$ implies $gr = gs$ for all $g \in G$.

Note that if R_c is the constant part of R, and G is an equiprime right R-group, then $GR_c = G(R0) = (GR)0 \subseteq G0 = \{0\}$.

Also note that if G is an equiprime right R-group, then from conditions (i) and (ii) it follows that there is a largest ideal P of R contained in $(0 : G)$.

If R is a ring, then an equiprime right R-group is a right prime R-module [6].

Proposition 3.3. *Let G be a right R-group satisfying conditions* (i), (ii) *and* (iii) *of Def. 3.2. Then $(0 : G)$ is an ideal of R.*

Proof. Let P be the largest ideal of R contained in $(0 : G)$. Let $r \in (0 : G)$. Let $0 \neq g \in G$. Now $gxr = (gx)r = 0 = (gx)0 = gx0$ for all $x \in R$. Therefore, $r = r - 0 \in P$. Hence, $P = (0 : G)$ is an ideal of R. $\Diamond$

Proposition 3.4. *Let G be a right R-group satisfying conditions* (i), (ii) *and* (iv) *of Def. 3.2 and P be the largest ideal of R contained in* $(0 : G)$. *Then the following are equivalent:*

(a) *G is an equiprime right R-group.*

(b) (i) *For $0 \neq g \in G$, $c \notin (0 : G)$, $gRc \neq \{0\}$.*

(ii) *If $\{0\} \neq H$ is a right R-subgroup of $G, c, d \in R$ and $hc = hd$ for all $h \in H$, then $c - d \in P$.*

Proof. (a) $\Rightarrow$ (b). Let $0 \neq g \in G$, $c \notin (0 : G)$. Suppose that $gRc = \{0\}$. Now $gxc = 0 = gx0$ for all $x \in R$. Since G is equiprime, $c = c - 0 \in (0 : G)$, a contradiction. So $gRc \neq \{0\}$. Suppose that $\{0\} \neq H$ is a right R-subgroup of $G, a, b \in R$ and $ha = hb$ for all $h \in H$. Let $0 \neq h_0 \in H$. Now $h_0xa = h_0xb$ for all $x \in R$. Since G is equiprime, $a-b \in (0 : G) = P$.

(b) $\Rightarrow$ (a). Let $r \in (0 : G)$. Now $gr = 0 = g0$ for all $g \in G$. So $r = r - 0 \in P$ and hence $P = (0 : G)$. Suppose that $0 \neq g \in G, c, d \in R$ and $gxc = gxd$ for all $x \in R$. Let $s \in$R$\backslash P$. Then $gRs \neq \{0\}$ and hence $gR \neq \{0\}$. Let K be the subgroup of $(G, +)$ generated by gR $:= \{gr \mid r \in R\}$. Now K is a non-zero right R-subgroup of G. Since $gxc =$ $= gxd$ for all $x \in R$, we get that $kc = kd$ for all $k \in K$. Therefore, $c - d \in P$. Hence, G is an equiprime right R-group. $\Diamond$

Proposition 3.5. *Let Q be an equiprime ideal of R. Then $(Q : R) =$ $= \{r \in R \mid Rr \subseteq Q\} = Q$.*

Proof. Since $R_c \subseteq Q$, we have that $RQ \subseteq Q$. So, $Q \subseteq (Q : R)$. Let $y \in (Q : R)$. Now $Ry \subseteq Q$ and $R0 = R_c \subseteq Q$. So, $ry - r0 \in Q$ for all $r \in R$. Since Q is an equiprime ideal of $R, y = y - 0 \in Q$. Therefore, $(Q : R) \subseteq Q$ and hence $(Q : R) = Q$. $\Diamond$

Proposition 3.6. *Let Q be an ideal of R and $Q \neq R$. Then the following are equivalent:*

(i) *Q is an equiprime ideal of R.*

(ii) *There is an equiprime right R-group G such that $Q = (0 : G)$.*

Proof. Let Q be an equiprime ideal of R. We show that the right R-group $G := R/Q$ is equiprime. We have $(0 : G) = (Q : R) = \{r \in R \mid Rr \subseteq$ $\subseteq Q\} = Q$. If $GR = \{0\}$, then $RR \subset Q$. Since an equiprime ideal is a prime ideal, we get that $R \subseteq Q$, a contradiction to $Q \neq R$. So, $GR \neq \{0\}$. Since $R_c \subseteq Q$, $G0 = \{0\}$. Let I, J be ideals of R such that $GI = GJ = \{0\}$. Then $I \subseteq (Q : R)$, $J \subseteq (Q : R)$. Since $(Q : R) = Q$ is an ideal of R, $I + J \subseteq (Q : R)$, that is, $G(I + J) = \{0\}$. Let P be the largest ideal of R contained in $(0 : G)$. Let $0 \neq r + Q \in R/Q$, $a, b \in R$ and $(r + Q)xa = (r + Q)xb$ for all x in R. Now $rxa - rxb \in Q$ for all $x \in R$. Since Q is equiprime and $r \notin Q$, we get that $a - b \in Q$. By Prop. 3.5, $P = Q$. So, $a - b \in P$. Let $r, s \in R$ and $r - s \in (0 : G) = Q$. Let $x + Q \in R/Q$. $xr = (x((r - s) + s) - xs) + xs = q + xs$, where $q := x((r-s)+s)-xs \in Q$. So, $xr-xs \in Q$ and that $(x+Q)r = (x+Q)s$. Therefore, G is an equiprime right R-group. On the other hand suppose that G is an equiprime right R-group. Let $T := (0 : G)$. We show that the ideal T is an equiprime ideal of R. Let $a \in R\backslash T$, $b, c \in R$ and $axb - axc \in T$ for all $x \in R$. We get $g \in G$ such that $ga \neq 0$. Now

$g(axb) = g(axc)$ and hence $(ga)xb = (ga)xc$ for all $x \in R$. Since G is equiprime, $b - c \in T$. Therefore, T is an equiprime ideal of R. $\Diamond$

Proposition 3.7. *Let G be an equiprime right R-group and let $\{0\} \neq H$ be an R-subgroup of G. Then H is an equiprime right R-group and $(0 : G) = (0 : H)$.*

Proof. Obviously, $(0 : G) \subseteq (0 : H)$. Let $a \in (0 : H)\backslash(0 : G)$ and let $0 \neq h \in H$. Now $hra = 0 = hr0$ for all $r \in R$. Since G is an equiprime right R-group, $a = a - 0 \in (0 : G)$, a contradiction to $a \notin (0 : G)$. Therefore, $(0 : G) = (0 : H)$. Let $0 \neq t \in H, a, b \in R$ and $txa = txb$ for all $x \in R$. Since G is an equiprime right R-group, $a - b \in (0 : G) = (0 : H)$. It is an easy verification that the other conditions of an equiprime right R-group are satisfied by H. Therefore, H is an equiprime right R-group. $\Diamond$

Theorem 3.8. *Let I be an essential left invariant ideal of R and let G be an equiprime right I-group. Then $H := \langle GI \rangle_s$, the subgroup of $(G, +)$ generated by GI, is an equiprime right R-group and $(0 : G)_I = (0 : H)_R$.*

Proof. Let H be the subgroup of $(G, +)$ generated by GI. Clearly, H is an I-subgroup of G. So by Prop. 3.7, H is an equiprime right I-group and $(0 : H)_I = (0 : G)_I$. We show now that H is an equiprime right R-group. Let $h \in H$, $r \in R$. Now $h = \delta_1(g_1s_1) + \delta_2(g_2s_2) + + \cdots + \delta_k(g_ks_k)$ for some $s_i \in I$, $g_i \in G$, $\delta_i \in \{1, -1\}$. Define $hr := \delta_1(g_1(s_1r)) + \delta_2(g_2(s_2r)) + \cdots + \delta_k(g_k(s_kr))$. We show that this operation is well defined. Suppose that h has another representation as $h = \lambda_1(h_1t_1) + \lambda_2(h_2t_2) + \cdots + \lambda_n(h_nt_n)$, $t_i \in I$, $h_i \in G$, $\lambda_i \in \{1, -1\}$. Let $c \in I\backslash(0 : G)_I$. Now $((\delta_1(g_1(s_1r)) + \delta_2(g_2(s_2r)) + \cdots + \delta_k(g_k(s_kr))) - (\lambda_1(h_1(t_1r)) + \lambda_2(h_2(t_2r)) + \cdots + \lambda_n(h_n(t_nr))))ac = ((\delta_1(g_1s_1) + +\delta_2(g_2s_2) + \cdots + \delta_k(g_ks_k)) - (\lambda_1(h_1t_1) + \lambda_2(h_2t_2) + \cdots + \lambda_n(h_nt_n)))(ra)c = = 0(ra)c = 0$ for all $a \in I$. Since G is an equiprime right I-group and $c \notin (0 : G)_I$, we get that $\delta_1(g_1(s_1r)) + \delta_2(g_2(s_2r)) + \cdots + \delta_k(g_k(s_kr)) = = \lambda_1(h_1(t_1r)) + \lambda_2(h_2(t_2r)) + \cdots + \lambda_n(h_n(t_nr))$. So the operation is well defined. It is an easy verification that H is a right R-group under this operation. Clearly, the action of R on H is an extension of the action of I on H. Since $GI \neq \{0\}$, we have $ga \neq 0$, for some $g \in G$, $a \in I$. If $(gI)I = \{0\}$, then $(ga)yb = 0 = (ga)y0$ for all $y \in I$, where $b \in I\backslash(0 : G)_I$. Since G is an equiprime right I-group, $b = b - 0 \in (0 : G)_I$, a contradiction. So, $(gI)I \neq \{0\}$ and that $HR \neq \{0\}$. We have $H0 = \{0\}$. Let J and K be ideals of R and

$HJ = HK = \{0\}$. Now $GIJ = GIK = \{0\}$. So $GJ^\star = GK^\star = \{0\}$, where $J^\star$, $K^\star$ are ideals of I generated IJ and IK respectively. Now $G(J^\star + K^\star) = \{0\}$. For $z \in I \setminus (0 : G)_I$, $h \in H$, $j \in J$, $k \in K$, and $x \in I$, we have $h(j+k)xz = h(jx+kx)z = 0z = 0 = h(j+k)x0$. Since G is an equiprime right I-group and $z - 0 \notin (0 : G)_I$, we have that $h(j+k) = 0$. Therefore, $H(J+K) = \{0\}$. Let $0 \neq h \in H$ and $h = \delta_1(g_1s_1) + \delta_2(g_2s_2) + \cdots + \delta_k(g_ks_k)$, $s_i \in I$, $g_i \in G$, $\delta_i \in \{1, -1\}$. Let P be the largest ideal of R contained in $(0 : H)_R$. Let $Q := (0 : G)_I$. Since G is an equiprime right I-group, by Prop. 3.6, Q is an equiprime ideal of I. So, I/Q is an equiprime near-ring. Therefore, by condition F_l, Q is an ideal of R. Now it is clear that $Q \subseteq P$. Since I/Q is an essential ideal of R/Q and I/Q is equiprime, R/Q is an equiprime near-ring. So, Q is an equiprime ideal of R. Suppose that $r, s \in R$ and $hxr = hxs$ for all $x \in R$. Fix $v \in I$. Now $h(av)r = h(av)s$ for all $a \in I$ and hence $ha(vr) = ha(vs)$ for all $a \in I$. Therefore, $vr - vs \in (0 : G)_I = Q$. Since Q is an equiprime ideal of R and I is a left invariant ideal of R, $r - s \in Q \subseteq P$. Let $p \in P$ and $0 \neq g_0 \in H$. Now $g_0xp = 0 = g_0x0$ for all $x \in R$. As seen above $p = p - 0 \in Q$. Therefore, $P = Q$ and $(0 : H)_R = Q$. Finally, let $r_1, r_2 \in R$ and $r_1 - r_2 \in P$. We have $h = \delta_1(g_1s_1) + \delta_2(g_2s_2) + \cdots + \delta_k(g_ks_k)$, $s_i \in I$, $g_i \in G$, $\delta_i \in \{1, -1\}$. Now $hr_1 = hr_2$ if $g_i(s_ir_1) = g_i(s_ir_2)$ for all $i = 1, 2, \ldots, k$. Since $r_1 - r_2 \in Q$, $ar_1 - ar_2 = a((r_1 - r_2) + r_2) - ar_2 \in Q$ for all $a \in I$. Now $gar_1 = gar_2$ for all $a \in I$, $g \in G$. So, $g_i(s_ir_1) = g_i(s_ir_2)$ and hence $hr_1 = hr_2$. Therefore, H is an equiprime right R-group and $(0 : G)_I = (0 : H)_R$. ♢

Theorem 3.9. *Let G be an equiprime right R-group and let I be a left invariant ideal of R. If $GI \neq \{0\}$, then G is an equiprime right I-group.*

Proof. Suppose that $GI \neq \{0\}$. Clearly, G is a right I-group and $G0 = \{0\}$. Moreover, $(0 : G)_I = (0 : G)_R \cap I$ is an ideal of I. Let $0 \neq g \in G$, $a, b \in I$ and $gya = gyb$ for all $y \in I$. If $gI = \{0\}$, then $gxc = 0 = gx0$ for all $x \in R$, $c \in I$ with $Gc \neq \{0\}$. So $c = c - 0 \in (0 : G)$, a contradiction. Therefore, $gI \neq \{0\}$. We have a $d \in I$ such that $gd \neq 0$. Now $(gd)xa = (gd)xb$ for all $x \in R$. Therefore, $a - b \in (0 : G)_I$. Let $u, v \in I$ and $u - v \in (0 : G)_I \subseteq (0 : G)_R$. So, $gu = gv$ for all $g \in G$. Therefore, G is an equiprime right I-group. ♢

Proposition 3.10. *Let G be an equiprime right R-group and let I be an ideal of R with $GI = \{0\}$. Then G is an equiprime right R/I-group.*

Proof. Let $r + I \in R/I$ and $g \in G$. Define $g(r+I) := gr$. If $r + I = s + I$,

$s \in R$, then $r-s \in I \subseteq (0:G)_R$ and hence $hr = hs$ for all $h \in G$. So, the above operation is well defined. Clearly, G is a right R/I-group. Since $GR \neq \{0\}$, we get that $G(R/I) \neq \{0\}$. We have $GI = \{0\}$. Let J/I, K/I be ideals of R/I and $G(J/I) = G(K/I) = \{0\}$. Now $GJ = GK = \{0\}$ and that $G(J+K) = \{0\}$. So, $G(J/I + K/I) = \{0\}$. Let $P := (0:G)_R$. Now P is an ideal of R. So, $(0:G)_{R/I} = P/I$. Let $0 \neq g_0 \in G$, $a, b \in R$ and $g_0(x+I)(a+I) = g_0(x+I)(b+I)$ for all $x \in R$. Since G is equiprime and $g_0xa = g_0xb$ for all $x \in R$, we have that $a - b \in P$. Therefore, $(a+I) - (b+I) \in P/I$. Let $(r+I) - (s+I) \in P/I$. Now $r - s \in P$ and that $gr = gs$ for all $g \in G$. Therefore, $g(r+I) = g(s+I)$ for all $g \in G$. Hence, G is an equiprime right R/I-group. $\Diamond$

The following proposition is easy and its proof is omitted.

Proposition 3.11. *Let I be an ideal of R and G be an equiprime right R/I-group. Then G is an equiprime right R-group, where $gr := g(r+I)$.*

4. Special classes of right modules of near-rings

In [1] Andrunakievich and Rjabuhin described special radicals of rings in terms of modules. A similar characterization for special radicals of zero-symmetric near-rings was given in terms of left modules of near-rings by Booth and Groenewald [2]. In this section we give a characterization for special radicals of near-rings in terms of right modules of near-rings.

Let $\mathcal{N}$ be the class of all near-rings. Suppose that for every near-ring R, there is a class $\mathcal{M}_R$ of right R-groups. Let $\mathcal{M} = \cup_{R \in \mathcal{N}} \mathcal{M}_R$. Then $\mathcal{M}$ is called a *special class of near-ring right modules* if it satisfies the following conditions:

M1. If $G \in \mathcal{M}_R$, then G is an equiprime right R-group.

M2. If $G \in \mathcal{M}_I$, I is an essential left invariant ideal of R, then $\langle GI \rangle_s$, the subgroup of $(G,+)$ generated by GI, is in $\mathcal{M}_R$.

M3. If $G \in \mathcal{M}_R$, I is a left invariant ideal of R and $GI \neq \{0\}$, then $G \in \mathcal{M}_I$.

M4. If $G \in \mathcal{M}_R$, I is an ideal of R and $GI = \{0\}$, then $G \in \mathcal{M}_{R/I}$, where $g(r+I) := gr$ for all $r \in R, g \in G$.

M5. If $G \in \mathcal{M}_{R/I}$, I is an ideal of R, then $G \in \mathcal{M}_R$, where $gr := g(r+I)$ for all $r \in R$, $g \in G$.

Theorem 4.1. *Let $\mathcal{E} := \cup_{R \in \mathcal{N}} \mathcal{E}_R$, where $\mathcal{E}_R$ is the class of all equiprime right R-groups. Then $\mathcal{E}$ is a special class of near-ring right modules.*

Proof. The proof follows from Th. 3.8 and Th. 3.9, and Prop. 3.10 and Prop. 3.11. ♢

Let $\mathcal{M}$ be a special class of near-ring right modules and let R be a near-ring. We define $\mathcal{M}(R) := \cap\{(0:G)_R \mid G \in \mathcal{M}_R\}$ and $\mathcal{S}_{\mathcal{M}} := \{R \in \in \mathcal{N} \mid \text{there is a } G \in \mathcal{M}_R \text{ such that } (0:G)_R = \{0\}\} \cup \{0\}$.

Theorem 4.2. *Let $\mathcal{M}$ be a special class of near-ring right modules. Then $\mathcal{S}_{\mathcal{M}}$ is a special class of near-rings.*

Proof. Let $\{0\} \neq R \in \mathcal{S}_{\mathcal{M}}$. We get a $G \in \mathcal{M}_R$ such that $(0:G)_R = \{0\}$. By M1, G is an equiprime right R-group. Now by Prop. 3.6, $\{0\} = = (0:G)_R$ is an equiprime ideal of R. So R is an equiprime near-ring. Let I be a non-zero (left invariant) ideal of R. Since $(0:G)_R = = \{0\}$, we have that $GI \neq \{0\}$. So by M3, $G \in \mathcal{M}_I$. Now $(0:G)_I = = (0:G)_R \cap I = \{0\} \cap I = \{0\}$. Therefore, $I \in \mathcal{S}_{\mathcal{M}}$ and hence $\mathcal{S}_{\mathcal{M}}$ is hereditary. Now suppose that J is an essential left invariant ideal of a near-ring T and $J \in \mathcal{S}_{\mathcal{M}}$. We get a $H \in \mathcal{M}_J$, such that $(0:H)_J = \{0\}$. We have that H is an equiprime right J-group and $HJ \neq \{0\}$. Since $\mathcal{M}$ is a special class, by M2 we get that K, the subgroup of $(H,+)$ generated by HJ, is in $\mathcal{M}_T$. Now we claim that $(0:K)_T = (0:H)_J = \{0\}$. Let $P := (0:K)_T$. By Prop. 3.3 and Prop. 3.6, P is an equiprime ideal of T. Since $HJ = \{0\}$, $HJP = \{0\}$. Also, since $JP \subseteq J$ and $(0:H)_J = \{0\}$, $JP = \{0\}$. Suppose that $P \neq \{0\}$. Since J is an essential ideal of T, $L := J \cap P \neq \{0\}$. Now $JL = \{0\}$. This is a contradiction to the fact that J is an equiprime near-ring. So $P = \{0\}$. Therefore, $T \in \mathcal{S}_{\mathcal{M}}$. Hence, $\mathcal{S}_{\mathcal{M}}$ is a special class of near-rings. ♢

Proposition 4.3. *Let $\mathcal{M}$ be a special class of near-ring right modules. Suppose that I is an ideal of R. Then $R/I \in \mathcal{S}_{\mathcal{M}}$ if and only if $I = = (0:G)_R$ for some $G \in \mathcal{M}_R$.*

Proof. Suppose that $R/I \in \mathcal{S}_{\mathcal{M}}$. We get a $G \in \mathcal{M}_{R/I}$ and $(0:G)_{R/I} = = \{0\}$. Since $\mathcal{M}$ is a special class, $G \in \mathcal{M}_R$. Also, $(0:G)_R = I$ as $(0:G)_{R/I} = \{0\}$. On the other hand suppose that $I = (0:G)_R$, for some $G \in \mathcal{M}_R$. Since $I \subseteq (0:G)_R$ and $\mathcal{M}$ is a special class, $G \in \mathcal{M}_{R/I}$. Moreover, $(0:G)_{R/I} = \{0\}$ as $I = (0:G)_R$. ♢

Proposition 4.4. *Let $\mathcal{R}$ be a special class of near-ring right modules. Let $\mathcal{R}$ be the upper radical determined by the special class of near-rings $\mathcal{S}_{\mathcal{M}}$. Then $\mathcal{R}(R) = \cap\{(0:G)_R \mid G \in \mathcal{M}_R\}$.*

Proof. Since $\mathcal{R}$ is the upper radical determined by the hereditary class of near-rings $\mathcal{S}_{\mathcal{M}}$, $\mathcal{R}(R) = \cap\{I \mid I \text{ is an ideal of } R \text{ and } R/I \in \mathcal{S}_{\mathcal{M}}\}$. By

Prop. 4.3, we get that $\mathcal{R}(R) = \cap\{(0 : G)_R \mid G \in \mathcal{M}_R\}$. $\diamondsuit$

Theorem 4.5. *Let $\mathcal{A}$ be a special class of near-rings. For any near-ring R, let $\mathcal{M}_R = \{G \mid G$ be an equiprime right R-group and $R/(0 : G)_R \in \mathcal{A}\}$. Let $\mathcal{M} := \cup_{R\in\mathcal{N}} \mathcal{M}_R$. Then $\mathcal{M}$ is a special class of near-ring right modules and $\mathcal{A} = \mathcal{S}_{\mathcal{M}}$.*

Proof. (i) By definition, each $G \in \mathcal{M}_R$ is an equiprime right R-group.

(ii) Let I be an essential left invariant ideal of R and $G \in \mathcal{M}_I$. Let $H := \langle GI\rangle_s$ be the subgroup of $(G, +)$ generated by GI. Since G is an equiprime right I-group and I is an essential left invariant ideal of R, by Th. 3.8, H is an equiprime right R-group and $(0 : G)_I = (0 : H)_R$. We have $I/(0 : G)_I \in \mathcal{A}$. Now $I/(0 : G)_I$ is an essential left invariant ideal of $R/(0 : H)_R$. Therefore, $R/(0 : H)_R \in \mathcal{A}$ and hence $H \in \mathcal{M}_R$.

(iii) Suppose now that $G \in \mathcal{M}_R$, J is a left invariant ideal of R and $GJ \neq \{0\}$. By Th. 3.9, G is an equiprime right J-group. Moreover, $(0 : G)_J = (0 : G)_R \cap J$. Now $J/(0 : G)_J = J/((0 : G)_R \cap J) \simeq$ $\simeq (J + (0 : G)_R)/(0 : G)_R$ and $(J + (0 : G)_R)/(0 : G)_R$ is a left invariant ideal of $R/(0 : G)_R \in \mathcal{A}$. So $J/(0 : G)_J \in \mathcal{A}$ and hence $G \in \mathcal{M}_J$.

(iv) Assume that $G \in \mathcal{M}_R$, K is an ideal of R and $GK = \{0\}$. By Prop. 3.10, G is an equiprime right R/K-group, where $g(r + K) := gr$. Moreover, $(0 : G)_{R/K} = (0 : G)_R/K$. Now $(R/K)/((0 : G)_R/K) \simeq$ $\simeq R/(0 : G)_R \in \mathcal{A}$. Therefore, $G \in \mathcal{M}_{R/K}$.

(v) Suppose now that P is an ideal of R and $G \in \mathcal{M}_{R/P}$. By Prop. 3.11, G is an equiprime right R-group, where $gr := g(r+P)$. Also, $(0 : G)_{R/P} = (0 : G)_R/P$. Now $R/(0 : G)_R \simeq (R/P)/((0 : G)_R/P) =$ $= (R/P)/(0 : G)_{R/P} \in \mathcal{A}$. Therefore, $G \in \mathcal{M}_R$. Hence, $\mathcal{M}$ is a special class of near-ring right modules. Clearly, $\mathcal{S}_{\mathcal{M}} \subseteq \mathcal{A}$. Let $R \in \mathcal{A}$. Since $\mathcal{A}$ is a class of equiprime near-rings, by Prop. 3.6, there is a faithful equiprime right R-group G. We have $R/(0 : G) = R \in \mathcal{A}$. Therefore, $R \in \mathcal{S}_{\mathcal{M}}$ and hence $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{M}}$. So, $\mathcal{A} = \mathcal{S}_{\mathcal{M}}$. $\diamondsuit$

5. Characterizations for some concrete special radicals

In this section we present characterizations for some concrete special radicals of near-rings.

Strongly equiprime near-rings, uniformly strongly equiprime near-rings, and bounded strongly equiprime near-rings of bound one are introduced and studied in [5] and completely equiprime near-rings are introduced and studied in [2].

An ideal P of a near-ring R is said to be *(right) strongly equiprime* if for each $a \in R\backslash P$, there is a finite subset F_a of R such that $b, c \in R$ and $axb - axc \in P$ for all $x \in F_a$ implies $b - c \in P$. A near-ring R is said to be *(right) strongly equiprime* if $\{0\}$ is a (right) strongly equiprime ideal of R. The *strongly equiprime radical of* R, denoted by $\mathcal{S}(R)$, is the intersection of all strongly equiprime ideals of R. Moreover, $\mathcal{S}$ is a special radical in the class of all near-rings.

Definition 5.1. A right R-group G is said to be *strongly equiprime* if

(i) $GR \neq \{0\}$ and $G0 = \{0\}$;

(ii) I, J are ideals of R and $GI = \{0\}$, $GJ = \{0\}$ implies $G(I + J) = \{0\}$;

(iii) for each $0 \neq g \in G$ there is a finite subset F_g of R such that $a, b \in R$ and $gxa = gxb$ for all $x \in F_g$ implies $a - b \in P$, where P is the largest ideal of R contained in $(0 : G)$;

(iv) $r, s \in R$ and $r - s \in P$ implies $gr = gs$ for all $g \in G$.

Remark 5.2. Trivially, a strongly equiprime right R-group is equiprime. So if G is a strongly equiprime right R-group, then $(0 : G)$ is an equiprime ideal of R.

Proposition 5.3. *Let G be a right R-group. If G is strongly equiprime, then $(0 : G)$ is a strongly equiprime ideal of R.*

Proof. Suppose that G is strongly equiprime. We have that $R \neq (0 : G)$ is an equiprime ideal of R. Let $a \in R\backslash(0 : G)$. Now we get a $g \in G$ such that $ga \neq 0$. So, there is a finite subset F of R such that $b, c \in R$ and $(ga)xb = (ga)xc$ for all $x \in F$ implies $b - c \in (0 : G)$. Suppose that $y, z \in R$ and $axy - axz \in (0 : G)$ for all $x \in F$. Now $(ga)xy = (ga)xz$ for all $x \in F$. Therefore, $y - z \in (0 : G)$. Hence $(0 : G)$ is a strongly equiprime ideal of R. ♢

Proposition 5.4. *Let P be an ideal of R. If P is strongly equiprime, then there is a strongly equiprime right R-group G such that $P = (0 : G)$.*

Proof. Suppose that P is strongly equiprime. Now P is equiprime and hence R/P is an equiprime right R-group under the operation $(r+P)s := rs + P$, $r + P \in R/P$, $s \in R$. Moreover, $(0 : R/P) = (P : R) = P$ as P is equiprime. Let $0 \neq a + P \in R/P$. Now $a \in R\backslash P$. We get a finite subset F of R such that $b, c \in R$ and $axb - axc \in P$ for all $x \in F$ implies

$b-c \in P$. Suppose that $y, z \in R$ and $(a+P)xy = (a+P)xz$ for all $x \in F$. Now $axy - axz \in P$ for all $x \in F$. Therefore, $y - z \in P = (0 : R/P)$. Hence R/P is a strongly equiprime right R-group and $P = (0 : R/P)$. $\Diamond$

Let $\mathbb{H}_R := \{G \mid G$ is a strongly equiprime right R-group$\}$ and $\mathbb{H} := \cup_{R \in \mathcal{N}} \mathbb{H}_R$.

Theorem 5.5. $\mathbb{H}$ *is a special class of near-ring right modules.*

Proof. (i) For each near-ring R, $\mathbb{H}_R$ is the class of strongly equiprime right R-groups. So each $G \in \mathbb{H}_R$ is an equiprime right R-group.

(ii) Let I be an essential left invariant ideal of R and G be a strongly equiprime right I-group. Let H be the subgroup of $(G, +)$ generated by the subset $GI := \{ga \mid g \in G, a \in I\}$. Since G is an equiprime right I-group, by Th. 3.8, H is an equiprime right R-group and $(0 : H)_R = (0 : G)_I$, where $(\delta_1(g_1s_1) + \delta_2(g_2s_2) + \cdots + \delta_k(g_ks_k))r = \delta_1(g_1(s_1r)) + \delta_2(g_2(s_2r)) + \cdots + \delta_k(g_k(s_kr))$, $r \in R$, $g_i \in G$, $s_i \in I$, $\delta_i \in \{1, -1\}$. Let $0 \neq h \in H$. Since G is strongly equiprime, we get a finite subset F of I such that $a, b \in I$ and $hxa = hxb$ for all $x \in F$ implies $a - b \in (0 : G)_I$. Now $F \subseteq R$. Suppose that $r, s \in R$ and $hxr = hxs$ for all $x \in F$. We show that $r - s \in (0 : H)_R = (0 : G)_I$. Suppose that $r - s \notin (0 : G)_I$. By Lemma 3.2 of [4], there is a $b \in I$ such that $(r - s)b \notin (0 : G)_I$ as $I/(0 : G)_I$ is an essential left invariant ideal of $R/(0 : G)_I$ and $I/(0 : G)_I$ is an equiprime near-ring. Now $rb - sb \notin (0 : G)_I$. Since $hxr = hxs$ for all $x \in F$, $hx(rb) = hx(sb)$ for all $x \in F$. So $rb - sb \in (0 : G)_I$, a contradiction to $rb - sb \notin (0 : G)_I$. Hence, $r - s \in (0 : H)_R$. Therefore, H is a strongly equiprime right R-group.

(iii) Suppose now that G is a strongly equiprime right R-group and I is a left invariant ideal of R with $GI \neq \{0\}$. By Th. 3.9, G is an equiprime right I-group and $(0 : G)_I = (0 : G)_R \cap I$. Let $0 \neq g \in G$. Since G is an equiprime right I-group, $gI \neq \{0\}$. So, there is a $c \in I$ such that $gc \neq 0$. Since G is a strongly equiprime right R-group, we get a finite subset F of R such that $y, z \in R$ and $(gc)xy = (gc)xz$ for all $x \in F$ implies $y - z \in (0 : G)_R$. Now $E := cF$ is a finite subset of I. Suppose that $a, b \in I$ and $gxa = gxb$ for all $x \in E$. Now $g(ct)a = g(ct)b$ for all $t \in F$ and $(gc)ta = (gc)tb$ for all $t \in F$. So, $a - b \in (0 : G) \cap I$. Therefore, G is a strongly equiprime right I-group.

(iv) Suppose that G is a strongly equiprime right R-group and I is an ideal of R contained $(0 : G)_R$. We show that G is a strongly equiprime R/I-group. Since G is an equiprime right R-group and $I \subseteq (0 : G)_R$, by Prop. 3.10, G is an equiprime right R/I-group, where $g(r + I) := gr$,

$g \in G$, $r \in R$. Let $0 \neq g \in G$. We get a finite subset F of R such that $r, s \in R$ and $gxr = gxs$ for all $x \in F$ implies $r - s \in (0 : G)$. Let $T = \{x + I \mid x \in F\}$. Let $a + I, b + I \in R/I$ and $g(x + I)(a + I) = = g(x + I)(b + I)$ for all $x + I \in T$. Now $gxa = gxb$ for all $x \in F$. So, $a - b \in (0 : G)$ and that $(a + I) - (b + I) \in (0 : G)_R/I = (0 : G)_{R/I}$. Therefore, G is a strongly equiprime right R/I-group.

(v) Similarly, if H is a strongly equiprime right R/I-group and I is an ideal of R, then we can show that H is a strongly equiprime right R-group. Hence, $\mathbb{H}$ is a special class of near-ring right modules. $\Diamond$

It is clear that $\mathbb{H}(R) = \mathcal{S}(R)$ for all near-rings R.

An ideal P of R is called *uniformly strongly equiprime* if there is a finite subset F of R such that $a \in R\backslash P, b, c \in R$ and $axb - axc \in P$ for all $x \in F$ implies $b - c \in P$. A near-ring R is said to be *uniformly strongly equiprime* if $\{0\}$ is an uniformly strongly equiprime ideal of R. The *uniformly strongly equiprime radical of* R, denoted by $\mathcal{V}(R)$, is the intersection of all uniformly strongly equiprime ideals of R. $\mathcal{V}$ is a special radical in the class of all near-rings.

Definition 5.6. A right R-group G is said to be *uniformly strongly equiprime* if

(i) $GR \neq \{0\}$ and $G0 = \{0\}$;

(ii) I, J are ideals of R and $GI = \{0\}$, $GJ = \{0\}$ implies $G(I + J) = = \{0\}$;

(iii) there is a finite subset F of R such that $0 \neq g \in G$, $a, b \in R$ and $gxa = gxb$ for all $x \in F$ implies $a - b \in P$, where P is the largest ideal of R contained in $(0 : G)$;

(iv) $r, s \in R$ and $r - s \in P$ implies $gr = gs$ for all $g \in G$.

Let $\mathbb{T}_R := \{G \mid G$ is a uniformly strongly equiprime right R-group$\}$ and $\mathbb{T} := \cup_{R \in \mathcal{N}} \mathbb{T}_R$.

By using arguments similar to those used in strongly equiprime right R-groups, we get the following:

Proposition 5.7. *Let G be a right R-group. If G is uniformly strongly equiprime, then $(0 : G)$ is a uniformly strongly equiprime ideal of R.*

Proposition 5.8. *Let P be an ideal of R. If P is uniformly strongly equiprime, then there is a uniformly strongly equiprime right R-group G such that $P = (0 : G)$.*

Theorem 5.9. *$\mathbb{T}$ is a special class of near-ring right modules.*

It is clear that $\mathbb{T}(R) = \mathcal{V}(R)$ for all near-rings R.

An ideal P of R is called *bounded strongly equiprime of bound one* if for each $a \in R\backslash P$ there is a $k \in R$ such that $b, c \in R$ and $akb - akc \in P$ implies $b - c \in P$. A near-ring R is said to be *bounded strongly equiprime of bound one* if the zero ideal $\{0\}$ is bounded strongly equiprime of bound one. The *bounded strongly equiprime radical of R of bound one*, denoted by $\mathcal{W}(R)$, is the intersection of all bounded strongly equiprime ideals of R of bound one. $\mathcal{W}$ is a special radical in the class of all near-rings.

Definition 5.10. A right R-group G is said to be *bounded strongly equiprime of bound one* if

(i) $GR \neq \{0\}$ and $G0 = \{0\}$;

(ii) I, J are ideals of R and $GI = \{0\}$, $GJ = \{0\}$ implies $G(I + J) = \{0\}$;

(iii) for each $0 \neq g \in G$ there is an element $k \in R$ such that $a, b \in R$ and $gka = gkb$ implies $a - b \in P$, where P is the largest ideal of R contained in $(0 : G)$;

(iv) $r, s \in R$ and $r - s \in P$ implies $gr = gs$ for all $g \in G$.

Let $\mathbb{L}_R := \{G \mid G$ is a bounded strongly equiprime right R-group of bound one$\}$ and $\mathbb{L} := \cup_{R \in \mathcal{N}} \mathbb{L}_R$.

By using arguments similar to those used in strongly equiprime right R-groups, we get the following:

Proposition 5.11. *Let G be a right R-group. If G is bounded strongly equiprime of bound one, then $(0 : G)$ is a bounded strongly equiprime ideal of R of bound one.*

Proposition 5.12. *Let P be an ideal of R. If P is bounded strongly equiprime of bound one, then there is a bounded strongly equiprime right R-group G of bound one such that $P = (0 : G)$.*

Theorem 5.13. *$\mathbb{L}$ is a special class of near-ring right modules.*

It is clear that $\mathbb{L}(R) = \mathcal{W}(R)$ for all near-rings R.

An ideal P of R is called *completely equiprime* if $a \in R\backslash P$, $b, c \in R$ and $ab - ac \in P$ implies $b - c \in P$. A near-ring R is said to be *completely equiprime* if $\{0\}$ is a completely equiprime ideal of R. The *completely equiprime radical of R*, denoted by $\mathcal{N}_g(R)$, is the intersection of all completely equiprime ideals of R. $\mathcal{N}_g$ is a KA-radical in the class of all near-rings.

Definition 5.14. A right R-group G is said to be *completely equiprime* if

(i) $GR \neq \{0\}$ and $G0 = \{0\}$;

(ii) I, J are ideals of R and $GI = \{0\}$, $GJ = \{0\}$ implies $G(I + J) = \{0\}$;

(iii) $0 \neq g \in G$, $a, b \in R$ and $ga = gb$ implies $a - b \in P$, where P is the largest ideal of R contained in $(0 : G)$;

(iv) $r, s \in R$ and $r - s \in P$ implies $gr = gs$ for all $g \in G$.

Let $\mathbb{C}_R := \{G \mid G$ is a completely equiprime right R-group$\}$ and $\mathbb{C} := \cup_{R\in\mathcal{N}} \mathbb{C}_R$.

By using arguments similar to those used in strongly equiprime right R-groups, we get the following:

Proposition 5.15. *Let G be a right R-group. If G is completely equiprime, then $(0 : G)$ is a completely equiprime ideal of R.*

Proposition 5.16. *Let P be an ideal of R. If P is completely equiprime, then there is a completely equiprime right R-group G such that $P = (0 : G)$.*

Theorem 5.17. *$\mathbb{C}$ is a special class of near-ring right modules.*

It is clear that $\mathbb{C}(R) = \mathcal{N}_g(R)$ for all near-rings R.

In [15] a right R-group of type-$0(e)$ is introduced and in [8] right R-groups of type-$1(e)$ and $2(e)$ are introduced.

By Prop. 3.7 of [15], if G is a right R-group of type-0 and $G0 = \{0\}$, then there is a largest ideal of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$.

Let G be a right R-group of type-0 and $G0 = \{0\}$. There is a largest ideal P of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. G is said to be a *right R-group of type-$0(e)$* if $0 \neq g \in G$, r_1, $r_2 \in R$ and $gxr_1 = gxr_2$ for all $x \in R$ implies $r_1 - r_2 \in P$.

Let $\nu \in \{1, 2\}$. Let G be a right R-group of type-ν. A right R-group of type-ν is of type-0. By Prop. 3.2 of [8], $G0 = \{0\}$. There is a largest ideal P of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. Then G is said to be a *right R-group of type-$\nu(e)$* if $0 \neq g \in G$, r_1, $r_2 \in R$ and $gxr_1 = gxr_2$ for all $x \in R$ implies $r_1 - r_2 \in P$.

A right R-group of type-$2(e)$ is of type-$1(e)$ and a right R-group of type-$1(e)$ is of type-$0(e)$.

Proposition 5.18. *Let $\nu \in \{0, 1, 2\}$. Let G be a right R-group of type-$\nu(e)$. Then G is an equiprime right R-group.*

Proof. Only the fourth condition in the definition of an equiprime right R-group has to be verified. By Prop. 3.12 of [15], $P := (0 : G) =$

$=\{r \in R \mid Gr = \{0\}\}$ is an ideal of R. Let $r, s \in R$ and $r - s \in P$. Let g_0 be a generator of G. Now $g_0R = G$ and $g_0(x + y) = g_0x + g_0y$ for all $x, y \in R$. Let $g \in G$. We have $g = g_0t$, for some $t \in R$. Now $gr = g_0tr = $ $= g_0(t((r-s) + s)-ts + ts) = g_0(t((r-s) + s)-ts) + g_0ts = 0 + gs = gs$. Therefore, G is an equiprime right R-group. $\diamondsuit$

Let $\nu \in \{0, 1, 2\}$. If G is a right R-group of type-$\nu(e)$, then $(0 : G) = $ $= \{r \in R \mid Gr = \{0\}\}$ is an ideal of R and is called a *right* $\nu(e)$*-primitive ideal* of R. R is *right* $\nu(e)$*-primitive* if $\{0\}$ is a right $\nu(e)$-primitive ideal of R. The intersection of all right $\nu(e)$-primitive ideals of R is *the right Jacobson radical of* R *of type-*$\nu(e)$ and is denoted by $J^r_{\nu(e)}(R)$. In [15] and [8] it is shown that $J^r_{\nu(e)}$ is a special radical in the class of all near-rings.

Let $\mathbb{G}_{\nu,\ R} := \{G \mid G$ is a right R-group of type-$\nu(e)\}$ and $\mathbb{G}_\nu :=$ $:= \cup_{R\in\mathcal{N}}\, \mathbb{G}_{\nu,\ R}$, $\nu \in \{0, 1, 2\}$.

Clearly, M4 and M5 conditions in the definition of a special class of near-ring right modules are satisfied by $\mathbb{G}_\nu$. By Th. 3.28 of [15] and Th. 3.32 of [8] we get that $\mathbb{G}_\nu$ satisfies condition M3.

Proposition 5.19. *Let* $\nu \in \{0, 1, 2\}$*, and* I *be an essential left invariant ideal of* R *and let* G *be a right* I*-group of type-*$\nu(e)$*. Let* H *be the subgroup of* $(G, +)$ *generated by* GI*. Then* H *is a right* R*-group of type-*$\nu(e)$ *and* $(0 : G)_I = (0 : H)_R$*.*

Proof. From the proof of Th. 3.33 of [15] and Th. 3.36 of [8] it follows that a faithful right I-group of type-$\nu(e)$ is a faithful right R-group of type-$\nu(e)$. Since G is monogenic, $H = G$. Now $J = (0 : G)_I$, is an equiprime ideal of I. Clearly, G is a faithful right I/J-group of type-$\nu(e)$, where $g(a + J) := ga$. Since $J \triangleleft I \triangleleft R$, I is left invariant and I/J is equiprime, we get that $J \triangleleft R$. Since I/J is an essential left invariant ideal of R/J, G is a faithful right R/J-group of type-$\nu(e)$. Therefore, H is a right R-group of type-$\nu(e)$ and $(0 : H)_R = J$. $\diamondsuit$

From the above observations we have:

Theorem 5.20. $\mathbb{G}_\nu$ *is a special class of near-ring right modules,* $\nu \in$ $\in \{0, 1, 2\}$*.*

It is clear that $\mathbb{G}_\nu(R) = J^r_{\nu(e)}(R)$ for all near-rings R.

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