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PRIMARY ABELIAN n-Σ-GROUPS REVISITED

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Abstract: We give a systematic study of n-Σ-groups. In particular, we extend a classical result due to Megibben (Pac. J. Math., 1964), concerning when Σ-groups are direct sums of countable groups.

1. Introduction and definitions

Throughout the present paper, suppose all our groups are p-primary abelian, where p is a prime fixed for the duration. All terminology and notation not explicitly defined and stated herein are standard and follow essentially those from the existing literature.

In [2] the following concept was introduced:

Definition 1. A group G is called *an n-Σ-group* if $G[p^n] = \cup_{i<\omega} G_i, G_i \subseteq \subseteq G_{i+1} \leq G[p^n]$ and $G_i \cap p^i G = (p^\omega G)[p^n]$.

These groups were explored intensively in a series of subsequent papers [2], [3], [4] and [6], respectively. Moreover, they were extensively studied also in [7], [8] and [9] as their crucial properties are exhibited. It is well known that this group class is rather large and properly contains the classes of n-summable groups and pillared groups (in particular, totally

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projective groups) – see [8]. Note that when $n = 1$ this notion coincides with the classical definition of a Σ-group, defined as in [12].

On the other hand, Keef gave in [13] the following.

Definition 2. Let $\lambda \leq \omega_1$ be an ordinal. A group G is said to be *a C_λ-group* provided that one (and hence each) p^α-high subgroup is a dsc-group (= a direct sum of countable groups) whenever $\alpha < \lambda$ is limit.

Note that when λ is limit, this is equivalent to $G/p^\alpha G$ is a dsc-group for every $\alpha < \lambda$.

Furthermore, it was proved in [7] (see also [4]) that a group G is an n-Σ-group if and only if G is a $C_{\omega+n}$-group. We shall enlarge below this claim (compare also with Prop. 1).

In [10] was completely described a subclass of the class of n-Σ-groups whose all p^ω-bounded subgroups (= separable subgroups) are n-Σ-groups (i.e., Σ-cyclic group). These are the direct products of a countable group and a Σ-cyclic group.

The purpose of this article is to illustrate that some new interesting things on these groups can be shown as well.

2. Main results

For the readers' convenience, we begin this section with a recollection of some known facts that will be used in the proofs of statements given below.

Proposition A ([7], [9])**.** *Let N be a countable (nice) subgroup of the n-Σ-group G. Then G/N is an n-Σ-group.*

Proposition B ([4], [7])**.** *Let G be a group of length$(G) \leq \omega + n - 1$. Then G is an n-Σ-group if and only if G is a dsc-group.*

The latter assertion can be refined to the following.

Proposition C ([4], [7])**.** *A group G is an n-Σ-group if and only if every $p^{\omega+n-1}$-high subgroup of G is a dsc-group.*

Proposition D ([14], [2])**.** *Any n-Σ-group is $p^{\omega+n-1}$-projective if and only if it is a dsc-group of length at most $\omega + n - 1$. Even more, $p^{\omega+n}$-projective n-Σ-groups are dsc-groups of length not exceeding $\omega + n$.*

We continue by a non-trivial generalization to a result of Megibben from [16] as well as to statements of [7] and [4].

Theorem 2.1. *Suppose $n \geq 1$ and $p^{\omega+n-1}G$ is at most countable. Then G is an n-Σ-group if and only if G is the direct sum of a countable group* *and a dsc-group of length at most $\omega + n - 1$. In particular, n-Σ-groups of length less than or equal to $\omega + n - 1$ are dsc-groups.*

Proof. Since $p^{\omega+n-1}G$ is countable (and nice in G), utilizing Prop. A, $G/p^{\omega+n-1}G$ must be an n-Σ-group. Next, in view of Prop. B, $G/p^{\omega+n-1}G$ is a dsc-group. Finally, since $p^{\omega+n-1}G$ is countable, applying a Nunke's theorem from [17], we deduce that G is a dsc-group as well. Furthermore, the argument concerning the last part is self-evident. $\Diamond$

Remark. When $n = 1$ this is precisely a theorem due to Megibben in [16].

Next we come to a strengthening of an assertion from [10]. Recall that a group G is said to be *$\omega + n$-totally $p^{\omega+n}$-projective* if each of its $p^{\omega+n}$-bounded subgroups is $p^{\omega+n}$-projective.

Corollary 2.2 ($2^{\aleph_0} < 2^{\aleph_1}$)**.** *If G is a proper $\omega+n$-totally $p^{\omega+n}$-projective group, then G is not a Σ-group.*

Proof. If we assume the contrary, then, in virtue of Cor. 3.7 from [10], $p^{\omega}G$ has to be countable. Hence Th. 2.1 is applicable for $n = 1$ to conclude that G is a dsc-group. However, this contradicts Cor. 2.8 of [10]. $\Diamond$

It is worthwhile noticing that with Cor. 3.4 of [10] at hand there exists a dsc-group G such that $p^{\omega+n}G$ is countable but G is not $\omega + n$-totally $p^{\omega+n}$-projective. Nevertheless, the following is valid:

Theorem 2.3 (CH)**.** *Any n-Σ-group is $\omega + n$-totally $p^{\omega+n}$-projective if and only if it is a dsc-group.*

Proof. By Cor. 2.2, such a group G cannot be a proper $\omega + n$-totally $p^{\omega+n}$-projective. So, owing to the notions of [10], it is either $p^{\omega+n}$-projective or a dsc-group. Furthermore, we employ Prop. D to get that G is a dsc-group, indeed. $\Diamond$

Conjecture. We conjecture that the last assertion is true in ZFC without the truthfulness of the CH. If yes, note that this will extend the fact we already used above that $p^{\omega+n}$-projective n-Σ-groups are dsc-groups (see, e.g., [2]).

We continue with the promised characterization of n-Σ-groups.

Proposition 2.4. *The following are equivalent:*

(i) *G is an n-Σ-group;*
(ii) *each $p^{\omega+n-1}$-high subgroup of G is a dsc-group;*
(iii) *each $p^{\omega+n-1}$-high subgroup of G is an n-Σ-group;*

(iv) *each* $p^{\omega+n}$*-high subgroup of* G *is an* n*-*Σ*-group;*

(v) *each* p^{α}*-high subgroup of* G *is an* n*-*Σ*-group for an arbitrary ordinal* α*.*

Proof. The equivalence (i) $\iff$ (ii), as aforementioned in Prop. C, was established in [7] and [4]. The implication (ii) $\Rightarrow$ (iii) is clear and, besides, the implication (iii) $\Rightarrow$ (ii) follows from the second part of Th. 2.1.

On the other hand, as for points (iv) and (v), we shall demonstrate that they are both equivalent to (i). In fact, for any ordinal α, p^{α}-high subgroups are themselves isotype in the whole group. However, we know with the aid of [2] that pure subgroups of n-Σ-groups are again n-Σ-groups. So (iv) and (v) hold provided that (i) is valid. Conversely, let (v) hold and assume that H is an arbitrary p^{α}-high subgroup of a group G for some arbitrary but a fixed ordinal α, whence H is an n-Σ-group. Moreover, suppose K is some $p^{\omega+n-1}$-high subgroup of G. In view of (iii), we may assume that $\alpha \geq \omega+n$. Since $K\cap p^{\alpha}G \subseteq K\cap p^{\omega+n-1}G = 0$, we may embed $K \subseteq H$ for some H described as above. But K is isotype in G and hence in H. We therefore deduce as previously observed that K is an n-Σ-group, so that (iii) follows and, by what we have already shown, (i) follows as well. The relationship (iv) yields (i) follows in the same manner. $\diamondsuit$

We proceed by proving a Nunke-like property for n-Σ-groups.

Theorem 2.5. *Suppose* $\omega \leq \lambda \leq \omega_1$ *is an ordinal.*

(a) *G is an n-Σ-group if and only if $G/p^{\lambda+n}G$ is an n-Σ-group.*

(b) *If $G/p^{\lambda}G$ is an n-Σ-group, then G is an n-Σ-group whereas the converse may fail.*

Proof. (a) First, we indicate the following useful facts:

(1) If a group A has one $p^{\omega+n-1}$-high subgroup which is a dsc-group, then all its $p^{\omega+n-1}$-high subgroups are dsc-groups.

This is a well-known classical fact.

(2) If $\gamma < \omega + n - 1$ and A has a $p^{\omega+n-1}$-high subgroup that is a dsc-group, then it has a p^{γ}-high subgroup that is a dsc-group.

This follows easily since any p^{γ}-high subgroup of G can be embedded as an isotype subgroup in some $p^{\omega+n-1}$-high subgroup of G and by a classical result of Hill (see, e.g., [11]) isotype subgroups of countable length of a dsc-group are again a dsc-group.

(3) If H is a $p^{\omega+n}$-high subgroup of G, then under the canonical map $G \to G/p^{\omega+n}G$, H maps to a $p^{\omega+n}$-high subgroup of $G/p^{\omega+n}G$.

From these three statements it plainly follows that G is a $C_{\omega+n}$-group if and only if $G/p^{\lambda+n}G$ is a $C_{\omega+n}$-group.

(b) Suppose H is a $p^{\omega+n}$-high subgroup of G. Then H is isotype in G, and it is long known that $H/p^\lambda H \cong (H + p^\lambda G)/p^\lambda G$ is isotype in $G/p^\lambda G$. Again utilizing [2], $H/p^\lambda H$ is an n-Σ-group. If $\lambda \geq \omega + n$, we are done because $p^\lambda H = 0$. But if $\lambda < \omega + n$, then $\lambda \leq \omega + n - 1$ and so Th. 2.1 allows us to infer that $H/p^\lambda H$ is a dsc-group. And since $p^\lambda H$ is bounded, it follows from Nunke's criterion in [17] that H is a dsc-group as well, and hence it is an n-Σ-group. Finally, Prop. 2.4 applies to get that G is an n-Σ-group.

In order to finish the proof, we appeal to [16] in which a Σ-group (= an 1-Σ-group) was constructed such that $p^\omega G$ is uncountable and $G/p^\omega G$ is an unbounded torsion-complete group. Thus this factor-group cannot be a Σ-cyclic group, i.e., not a Σ-group. $\Diamond$

Remark. If $\lambda \geq \omega$, G being an n-Σ-group does not imply that so is $p^\lambda G$ (see, e.g., [16] or [2]); in fact $p^\omega G$ can be unbounded torsion-complete.

Moreover, clause (b) was proved in ([2], Prop. 12) but using another more direct method which was based on the original definition (compare with the initial Def. 1 alluded to above).

Our next result, which is closely related to Th. 3, is to describe those n-Σ-groups G whose quotient group $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective.

Proposition 2.6. *If G is an n-Σ-group such that $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective, then G is pillared, i.e., $G/p^\omega G$ is a Σ-cyclic group.*

Proof. By virtue of Th. 2.5(a), $G/p^{\omega+n}G$ is also an n-Σ-group. Furthermore, we apply Prop. D to deduce that $G/p^{\omega+n}G$ is a dsc-group. This is just tantamount to $G/p^\omega G$ is Σ-cyclic; indeed one can sees that the sequence of isomorphisms

$$G/p^\omega G \cong G/p^{\omega+n}G/p^\omega G/p^{\omega+n}G \cong G/p^{\omega+n}G/p^\omega(G/p^{\omega+n}G)$$

holds, where $p^\omega(G/p^{\omega+n}G)$ is bounded by p^n, and henceforth Nunke's criterion from [17] may be applied. $\Diamond$

Imitating [15], a group G is called *strongly n-totally projective* if, for each limit ordinal λ, $G/p^{\lambda+n}G$ is $p^{\lambda+n}$-projective.

Corollary 2.7. *Suppose G is a group of length $< \omega\cdot 2$. If G is a strongly n-totally projective n-Σ-group, then G is a dsc-group and vice versa.*

Proof. By definition, $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective, so in virtue of the Prop. 2.6, $G/p^\omega G$ is Σ-cyclic which processing as above is, actually, equivalent to $G/p^{\omega+k}G$ is a dsc-group for any $k < \omega$. But length$(G) = \omega+t$ for some $t<\omega$. That is why, $G/p^{\omega+t}G \cong G$ is a dsc-group too, as required. $\Diamond$

Remark. Notice that for lengths strictly less than $\omega \cdot 2$, n-Σ-groups coincide with n-summable groups (e.g., cf. [1]). Recall that by [8] n-summable groups are those groups G such that $G[p^n]$ is the valuated direct sum of valuated countable subgroups. Thereby, the last corollary can also be found in [15].

In some instances n-summable groups and n-Σ-groups have satisfactory interrelations – notice that n-summable groups are n-Σ-groups while the converse manifestly fails. Nevertheless, the next assertion somewhat answers the Question on p. 161 of [2].

Proposition 2.8. *If G is a group such that $p^{\omega \cdot 2}G = 0$, then G is n-summable if and only if it is a summable n-Σ-group. In particular, if length$(G) < \omega \cdot 2$, G is n-summable if and only if it is an n-Σ-group.*

Proof. Clearly, as noted above, any n-summable group is a summable n-Σ-group. Conversely, observe that $p^\omega(p^\omega G) = 0$ and thus $p^\omega G$ is separable summable, whence a Σ-cyclic group, and hence it is n-summable. Finally, [8] applies to imply the claim.

As for the other statement, we observe that whenever $p^\omega G$ is bounded, it will be n-summable. $\Diamond$

Proposition 2.9. *Suppose L is a subgroup of G. If either L is large in G or G/L is bounded, then G is n-summable or strongly n-summable if and only if L is n-summable or strongly n-summable, respectively.*

Proof. According to [2], G is an n-Σ-group if and only if L is an n-Σ-group. Since $p^\omega L = p^\omega G$, the result follows from [8]. $\Diamond$

We next give one more new concept.

Definition 3. A group G is called *a strongly n-Σ-group* if its $p^{\omega+n-1}$-high subgroups are Σ-cyclic.

It follows easily that G is a strongly n-Σ-group if and only if it is an n-Σ-group and the Ulm functions $f_G(\alpha)$ of G satisfy the following equality $\sum_{\omega \leq \alpha < \omega+n-1} f_G(\alpha) = 0$.

The following statement sheds some light on the structure of strongly n-Σ-groups. Recall that in view of [8] strongly n-summable groups are those groups whose p^n-socle is the valuated direct sum of valuated cyclic subgroups.

Proposition 2.10. *If G is a group, then the following hold:*

(a) *G is strongly n-summable if and only if it is a strongly n-Σ-*

group and $p^{\omega}G$ is strongly n-summable.

(b) *If $p^{\omega\cdot 2}G = 0$, then G is strongly n-summable if and only if it is a summable strongly n-Σ-group.*

(c) *If length$(G) < \omega^2$, then G is strongly n-summable if and only if for each $m < \omega$, $p^{\omega\cdot m}G$ is a strongly n-Σ-group.*

(d) *If length$(G) = \omega^2$, then G is strongly n-summable if and only if it is summable and, for each $m < \omega$, $p^{\omega\cdot m}G$ is a strongly n-Σ-group.*

Proof. Under the above specifications of the definition of a strongly n-Σ-group, it follows in the same manner as the corresponding statement from [8]. $\Diamond$

We close the work with a number of interesting and non-trivial questions.

3. Open problems

There are several important classes of groups that merit investigation because they have some quite interesting properties.

Problem 1. Does it follow that n-Σ-groups of countable length are strongly n-totally projective if and only if they are dsc-groups?

Problem 2. Describe those groups G such that all high ($=\omega$-high) subgroups are $p^{\omega+n}$-projective.

It is clear that Σ-groups, ω-totally $p^{\omega+n}$-projective groups and n-totally projective groups lie in this group class.

Problem 3. Describe those groups G such that all $p^{\omega+n}$-high subgroups are $p^{\omega+n}$-projective.

It is plain that n-Σ-groups, $\omega + n$-totally $p^{\omega+n}$-projective groups and strongly n-totally projective groups belong to this group class.

Moreover, the latter group class is contained in the first one.

Problem 4. If G is an $\omega+n$-totally $p^{\omega+n}$-projective group, does it follow that $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective?

We conjecture that the answer is negative whenever G is a proper $\omega + n$-totally $p^{\omega+n}$-projective group. The reason is that, because $p^{\omega}G$ is countable by ([10], Cor. 3.7), if $n = 1$ and if $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective then, referring to Cor. 2.11(i) from [10], G will be the direct sum of a countable group and a separable $p^{\omega+1}$-projective group provided that Continuum Hypothesis holds. However, perhaps there is an $\omega + 1$-totally $p^{\omega+1}$-projective group which is not such a direct sum.

We next come to the generalized version of the Dieudonné criterion for strongly n-Σ-groups. In fact, the following was proved in [5]: Suppose B is a balanced subgroup of a group A. If B and A/B are both n-Σ-group, then so is A. Note that for $n = 1$, i.e. for Σ-groups, the reader can see [1].

So, we state the following.

Problem 5. Let B be a balanced subgroup of a group A. If both B and A/B are strongly n-Σ-groups, is A a strongly n-Σ-group?

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