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ON NEARLY QUASI-EINSTEIN MANIFOLDS

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Abstract: The notion of nearly quasi-Einstein manifold have been introduced by U. C. De and A. K. Gazi [7]. In the present paper we study some properties of a nearly quasi-Einstein manifold.

1. Introduction

In 2000 M. C. Chaki and R. K. Maity introduced the notion of quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g) $(n>2)$ is said to be quasi-Einstein manifold ([2, 5, 6, 9, 11]) if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1}$$

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where a and b are scalars such that $b \neq 0$ and A is a non-zero 1-form defined by $g(X, U) = A(X)$ for all vector fields X; U being a unit vector field, called the generator of the manifold. An n-dimensional manifold of this kind is denoted by $(QE)_n$. If $b = 0$, the manifold reduces to an Einstein manifold.

Einstein manifolds play an important role in the Riemannian geometry, as well as in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds due to a curvature condition imposed on their Ricci tensor ([1], pp. 432–433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing relation (1).

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson–Walker spacetime are quasi-Einstein manifolds [10]. Considering this aspect we are motivated to study such a manifold.

In the present paper we consider the nearly quasi-Einstein manifold, which is a weaker class of a quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ whose Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bE(X, Y) \tag{2}$$

where a and b are non-zero scalars and E is a non-zero (0, 2) tensor. Such a manifold shall be called as nearly quasi-Einstein manifold. This notion has been introduced by U. C. De and A. K. Gazi [7].

It is noted ([8], p. 39) that the outer product of 2 covariant vectors is a covariant tensor of type (0, 2) but the converse is not true, in general. Hence the manifolds which are quasi-Einstein are also nearly quasi-Einstein, but the converse is not true, in general.

An n-dimensional nearly quasi-Einstein manifold will be denoted by $N(QE)_n$. We shall call E the associated tensor and a and b as associated scalars.

A concrete example of a nearly quasi-Einstein manifold was also given in [7] by the following example:

Example 1.1. Let $(\mathbf{R}^4, g)$ be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2,$$

$(i, j = 1, 2, 3, 4)$. Then $(\mathbf{R}^4, g)$ is a $N(QE)_4$ with non-zero and non-constant scalar curvature which is not a quasi-Einstein manifold.

The paper is organized as follows: In Sect. 2, we give preliminaries and known results for a nearly quasi-Einstein manifold. Sect. 3 is devoted to the study of conformally flat $N(QE)_n$ and introduced the notion of nearly quasi-constant curvature. In Sect. 4 we study $N(QE)_n$ with cyclic associated tensor. The last section gives the example of a manifold of nearly quasi-constant curvature.

2. Preliminaries and known results

Let Q and L be two symmetric endomorphisms of the tangent space at each point of the manifold corresponding to the Ricci tensor S and to the associated tensor E, respectively. Then

$$g(QX, Y) = S(X, Y), \qquad g(LX, Y) = E(X, Y). \tag{3}$$

Also, let $\widetilde{e}$ be the scalar corresponding to E, that is, $\widetilde{e} = \sum_{i=1}^{n} E(e_i, e_i)$, where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at each point of the manifold.

Now, putting $X = Y = e_i$ in (2) we get

$$r = na + b\widetilde{e} \tag{4}$$

where r is the scalar curvature.

Further, let s^2 and e^2 denote the squares of the length of the Ricci tensor S and the associated tensor E respectively. Then $s^2 = \sum_{i=1}^{n} S(Qe_i, e_i)$ and $e^2 = \sum_{i=1}^{n} E(Le_i, e_i)$. Now from (2) we get

$$\sum_{i=1}^{n} S(Qe_i, e_i) = na^2 + nb\widetilde{e} + b\sum_{i=1}^{n} S(Le_i, e_i). \tag{5}$$

Also from (2) we obtain

$$\sum_{i=1}^{n} S(Le_i, e_i) = a\widetilde{e} + be^2. \tag{6}$$

Hence from (5) and (6) it follows that

$$s^2 = na^2 + (n + a)\widetilde{e}b + b^2e^2. \tag{7}$$

From (7) it follows that $b > \frac{s}{e}$ (respectively, $<$, $=$) according as $(na^2 + (n+a)\widetilde{e}b) < 0$ (resp., $>$, $=$). Hence, we can state the following:

Theorem 2.1. *In an $N(QE)_n$ $(n > 2)$ the associated scalar b is less than or equal to or greater than the ratio which the length of the Ricci tensor S bears to the length of the associated tensor E according as $(na^2 + (n+a)\widetilde{e}b) < 0$ or, $= 0$, < 0 respectively.*

3. Conformally flat $N(QE)_n$ $(n > 3)$

The Weyl conformal curvature tensor C of type (1,3) of an n-dimensional Riemannian manifold (M^n, g) $(n > 3)$ is defined by [3]

$$(8)\qquad \begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\big[g(Y,Z)X - g(X,Z)Y\big] + \\ + g(X,W)Y - g(Y,W)X] + \\ + \frac{r}{(n-1)(n-2)}\big[g(Y,Z)X - g(X,Z)Y\big], \end{aligned}$$

where r is the scalar curvature of the manifold.

Let R be the curvature tensor of type (0, 4) of a conformally flat $N(QE)_n$. From (8) we have

$$(9)\quad \begin{aligned} R(X,Y,Z,W) = \frac{1}{n-2}\big[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + \\ + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)\big] - \\ - \frac{r}{(n-1)(n-2)}\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big]. \end{aligned}$$

Using (4) and (3) in (9), we obtain

$$(10)\qquad \begin{aligned} R(X,Y,Z,W) = \\ = \left[\frac{-a - b\widetilde{e}}{(n-1)(n-2)}\right]\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + \\ + \frac{b}{n-2}\big[E(X,W)g(Y,Z) - E(X,Z)g(Y,W) + \\ + E(Y,Z)g(X,W) - E(Y,W)g(X,Z)\big]. \end{aligned}$$

According to Chen and Yano [3], a Riemannian manifold (M^n, g) $(n > 3)$ is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) has the form

$$(11)\quad \begin{aligned} R(X,Y,Z,W) = a_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + \\ + a_2\big[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) + \\ + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\big], \end{aligned}$$

where A is a 1-form and $a_1,\ a_2$ are scalars of which $a_2 \neq 0$. Generalizing this notion we introduce the following definition.

A Riemannian manifold (M^n, g) $(n > 3)$ is said to be of nearly quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) satisfies the condition

$$(12)\quad \begin{aligned} R(X,Y,Z,W) = \alpha_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + \\ + \alpha_2\big[g(Y,Z)E(X,W) - g(X,Z)E(Y,W) + \\ + g(X,W)E(Y,Z) - g(Y,W)E(X,Z)\big] \end{aligned}$$

where α_1 and α_2 are non-zero scalars and E is a symmetric tensor of type (0, 2). Now the relation (10) can be written as

$$(13)\quad \begin{aligned} R(X,Y,Z,W) = \beta_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + \\ + \beta_2\big[g(Y,Z)E(X,W) - g(X,Z)E(Y,W) + \\ + g(X,W)E(Y,Z) - g(Y,W)E(X,Z)\big], \end{aligned}$$

where $\beta_1 = \frac{-a-b\widetilde{e}}{(n-1)(n-2)}$ and $\beta_2 = \frac{b}{n-2}$ are non-zero scalars. Comparing (12) and (13), it follows that the manifold is of nearly quasi-constant curvature. This leads to the following:

Theorem 3.1. *A conformally flat* $N(QE)_n$ $(n > 3)$ *is a manifold of nearly quasi-constant curvature.*

Let us consider a manifold of nearly quasi-constant curvature. Then from (12) it follows that

$$(14)\quad S(Y,Z) = \widetilde{\alpha} g(Y,Z) + \widetilde{\beta} E(Y,Z),$$

where $\widetilde{\alpha} = (n-1)\alpha_1 + \alpha_2\widetilde{e}$ and $\widetilde{\beta} = (n-2)\alpha_2$ are non-zero scalars. Thus we have the following:

Theorem 3.2. *A manifold* (M^n, g) $(n > 2)$ *of nearly quasi-constant curvature is* $N(QE)_n$.

Now $N(QE)_n$ is not a manifold of nearly quasi-constant curvature in general. However, since a 3-dimensional Riemannian manifold is conformally flat, it follows by virtue of Th. 3.1 that $N(QE)_3$ is a manifold of nearly quasi-constant curvature. This leads to the following:

Corollary 3.3. *A $N(QE)_3$ is a manifold of nearly quasi-constant curvature.*

4. $N(QE)_n$ with cyclic associated tensor

In this section we assume that the associated scalars of an $N(QE)_n$ are constants, that is, a and b are constants. Now, if an $N(QE)_n$ satisfies cyclic Ricci tensor, then we have

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{15}$$

Taking covariant differentiation to both sides of (2), we get

$$(\nabla_X S)(Y,Z) = b(\nabla_X E)(Y,Z). \tag{16}$$

Since $b \neq 0$, equations (15) and (16) together imply that the associated tensor E is of cyclic associated type. That is,

$$(\nabla_X E)(Y,Z) + (\nabla_Y E)(Z,X) + (\nabla_Z E)(X,Y) = 0. \tag{17}$$

This leads to the following:

Theorem 4.1. *An $N(QE)_n$ with associated scalars as constants satisfies the cyclic Ricci tensor if and only if its associated tensor is of cyclic type.*

Now we consider a $N(QE)_n$ with an cyclic associated tensor. Putting $Y = Z = e_i$ in (17) and taking summation over i, $1 \leq i \leq n$ we have

$$(\nabla_X E)(e_i,e_i) + 2(\nabla_{e_i} E)(e_i,X) = 0. \tag{18}$$

Now

$$(\nabla_X E)(e_i,e_i) = \nabla_X E(e_i,e_i) - 2E(\nabla_X e_i, e_i). \tag{19}$$

In local coordinates $\nabla_X e_i = X^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, $\nabla_X e_i = 0$. Hence from (19) it follows that

$$(\nabla_X E)(e_i,e_i) = \nabla_X E(e_i,e_i) = d\widetilde{e}(X). \tag{20}$$

We know that the associated operator L defined by $g(LX, Y) = E(X, Y)$ is the (1, 1) associated tensor. Then

$$(\nabla_Z E)(X, Y) = g\big((\nabla_Z L)(X), Y\big). \tag{21}$$

Taking $Y = Z = \{e_i\}$ in (21)and taking summation over i, $1 \leq i \leq n$, we have $(\nabla_{e_i} E)(X, e_i) = g\big((\nabla_{e_i} L)(X), e_i\big)$. But it is known that $(\operatorname{div} L)(X) = = tr\big(Z \to (\nabla_Z L)(X)\big) = \sum_i g\big((\nabla_{e_i} L)(X), e_i\big)$, $g\big((\nabla_{e_i} Q)(X), e_i\big) = = bg\big((\nabla_{e_i} L)(X), e_i\big)$ ([12]), and $(\operatorname{div} Q)(X) = \frac{1}{2} dr(X)$. This implies

$$(\nabla_{e_i} E)(X, e_i) = \frac{1}{2} d\widetilde{e}(X). \tag{22}$$

Now using (20) and (22) in (18) we obtain

$$d\widetilde{e}(X) = 0 \quad \text{for all} \quad X, \tag{23}$$

which implies that $\widetilde{e}$ is constant.

Thus we state the following:

Theorem 4.2. *If a $N(QE)_n$ with associated constant scalars satisfies the cyclic associated tensor condition* (17)*, then the scalar curvature $\widetilde{e}$ corresponding to E is zero.*

From Th. 4.1 and Th. 4.2 we conclude that

Corollary 4.3. *If a $N(QE)_n$ with associated constant scalars satisfies the cyclic Ricci tensor condition* (15)*, then the scalar curvature r corresponding to S is zero.*

5. Example of a $N(QE)_n$

Let $(M^{n-1}, \widetilde{g})$ be a hypersurface of the Euclidean space (M^n, g). If A is the (1, 1) tensor corresponding to the normal valued second fundamental tensor H, then we have [4]

$$\widetilde{g}\big(A_\xi(X), Y\big) = g\big(H(X, Y), \xi\big) \tag{24}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields.

Let H_ξ be the symmetric (0, 2) tensor associated with A_ξ defined by

$$\widetilde{g}\big(A_\xi(X), Y\big) = H_\xi(X, Y). \tag{25}$$

A hypersurface of a Riemannian manifold (M^n, g) is called quasi-umbilical [6] if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y) \tag{26}$$

where ω is a 1-form. The vector field corresponding to the 1-form ω is a unit vector field, and α, β are scalars. If $\alpha = 0$ (resp. $\beta = 0$ or $\alpha = \beta = 0$) holds, then M^n is called cylindrical (resp. umbilical or geodesic).

In this section, we define nearly quasi-umbilical hypersurface of a Riemannian manifold.

Definition 5.1. A hypersurface of a Riemannian manifold (M^n, g) is called nearly quasi-umbilical if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha g(X,Y) + D(X,Y) \tag{27}$$

where D is a symmetric (0, 2) tensor and α is a scalar. If $\alpha = 0$ (resp. $D = 0$ or $\alpha = D = 0$) holds, then M^n is called nearly cylindrical (resp. umbilical or geodesic).

Now from (24), (25) and (27) we obtain

$$g\big(H(X,Y),\xi\big) = \alpha g(X,Y)g(\xi,\xi) + D(X,Y)g(\xi,\xi) \tag{28}$$

which implies that

$$H(X,Y) = \alpha g(X,Y)\xi + D(X,Y)\xi, \tag{29}$$

since ξ is the only unit normal vector field.

The Gauss equation on M^n in E^{n+1} can be written as

$$\widetilde{g}\big(\widetilde{R}(X,Y)Z,W\big) = \widetilde{g}\big(H(X,W),H(Y,Z)\big) - \widetilde{g}\big(H(Y,W),H(X,Z)\big) \tag{30}$$

where $\widetilde{R}$ is the curvature tensor of M^n.

Let us assume that the hypersurface is nearly quasi-umbilical, then from (29) and (30) it follows that

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y,Z,W)) = {} & \alpha^2\big[g(Y,Z)g(X,W) - g(Y,W)g(X,Z)\big] + \\ & + \alpha\big[D(Y,Z)g(X,W) - D(X,Z)g(Y,W) + \\ & + D(X,W)g(Y,Z) - D(Y,W)g(X,Z)\big]\end{aligned}$$

where $\widetilde{g}\big(\widetilde{R}(X,Y)Z,W\big) = \widetilde{R}(X,Y,Z,W)$. Contracting the above equation with $X = W = e_i$ and taking summation over i, $1 \leq i \leq n$, we obtain

$$\widetilde{S} = ag(Y,Z) + bD(Y,Z)$$

where $\widetilde{d} = \sum_{i=1}^{n} D(e_i,e_i)$, $a = \big[(n-1)\alpha^2 + \alpha\widetilde{d}\,\big]$, $b = (n-2)\alpha$.

Hence $(M^n,\widetilde{g})$ is a nearly quasi-Einstein manifold.

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