Mathematica Pannonica
21/2 (2010), 251–263

ON QUASI-GENERALIZED RECURRENT MANIFOLDS

Absos Ali **Shaikh**

Department of Mathematics, University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India

Indranil **Roy**

Department of Mathematics, University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India

Received: February 2010

MSC 2000: 53 B 05, 53 B 50, 53 C 15, 53 C 25

Keywords: Quasi-generalized recurrent manifold, recurrent manifold, generalized recurrent manifold, Ricci recurrent manifold, generalized Ricci recurrent manifold, quasi-generalized Ricci recurrent manifold, quasi-generalized 2-Ricci recurrent manifold, projectively recurrent manifold, conformally recurrent manifold, scalar curvature.

Abstract: The object of the present paper is to introduce a kind of non-flat semi-Riemannian manifolds called *quasi-generalized recurrent manifolds*, to study a lot of their several geometric properties and to furnish also a proper example.

1. Introduction

Let (M^n, g) be an n-dimensional connected semi-Riemannian manifold with Levi-Civita connection ∇. Then M is said to be locally symmetric due to Cartan if its curvature tensor R satisfies $\nabla R = 0$. The notion of locally symmetric manifolds has been weakened by many au-

E-mail address: aask2003@yahoo.co.in

thors in several ways to a different extent such as recurrent manifolds by A. G. Walker ([15]), 2-recurrent manifolds by A. Lichnerowicz ([9]), concircular recurrent manifolds by T. Miyazawa ([10]), weakly symmetric manifolds by L. Tamássy and T. Q. Binh ([13]), conformally recurrent manifolds ([1]), projectively recurrent manifolds ([2]), generalized recurrent manifolds ([3], [4]).

Again M^n is said to be Ricci symmetric if its Ricci tensor S of type (0, 2) satisfies $\nabla S = 0$. The notion of Ricci symmetry has also been weakened by many authors such as Ricci recurrent manifolds by E. M. Patterson ([11]), weakly Ricci symmetric manifolds by L. Tamássy and T. Q. Binh ([14]), generalized Ricci recurrent manifolds ([5]).

We denote by $\nabla^i T$ the covariant differential of the ith order, $i \geq 1$, of a $(0, k)$ tensor field T, $k \geq 1$, defined on a semi-Riemannian manifold (M^n, g) with Levi-Civita connection ∇. The tensor field T is said to be recurrent, respectively, 2-recurrent ([12]), if the following condition holds on M

$$(\nabla T)(X_1, \ldots, X_k; X)T(Y_1, \ldots, Y_k) = \\ = (\nabla T)(Y_1, \ldots, Y_k; X)T(X_1, \ldots, X_k), \tag{1.1}$$

respectively,

$$(\nabla^2 T)(X_1, \ldots, X_k; X, Y)T(Y_1, \ldots, Y_k) = \\ = (\nabla T^2)(Y_1, \ldots, Y_k; X, Y)T(X_1, \ldots, X_k), \tag{1.2}$$

where X, Y, X_1, Y_1, $\ldots$, X_k, Y_k are vector fields on M. From (1.1), respectively (1.2), it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique 1-form ϕ, respectively, a (0, 2)-tensor ψ, defined on a neighbourhood U of x, such that

$$\nabla T = T \otimes \phi, \quad \phi = d\big(\log \|T\|\big),$$

respectively,

$$\nabla^2 T = T \otimes \psi,$$

holds on U, where $\|T\|$ denotes the norm of T, $\|T\|^2 = g(T, T)$.

A non-flat connected semi-Riemannian manifold (M^n, g) $(n \geq 2)$, is said to be recurrent ([15]) if its curvature tensor R of type (0, 4) satisfies the condition $\nabla R = A \otimes R$ where A is a non-zero 1-form. Such a manifold is denoted by K_n. Let $U_R = \{x \in M : \nabla R \neq A \otimes R \text{ at } x\}$. Then the manifold (M^n, g) is said to be generalized recurrent ([4]) if on $U_R \subset M$, we have $\nabla R = A \otimes R + B \otimes G$, where B is an 1-form on U_R and G is a tensor of type (0, 4) given by

$$G(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U) \tag{1.3}$$

for all $X, Y, Z, U \in \chi(M)$, $\chi(M)$ being the Lie algebra of smooth vector fields on M. Such a manifold is denoted by GK_n. It is clear that the 1-form B is non-zero at every point on U_R. It is also clear that every K_n is GK_n but not conversely.

The object of the present paper is to introduce a generalized class of recurrent manifolds called *quasi-generalized recurrent manifolds*.

A non-flat semi-Riemannian manifold (M^n, g) $(n > 2)$ [this condition is assumed throughout the paper] is said to be *quasi-generalized recurrent manifold* if on $U_R \subset M$ the condition

$$\nabla R = A \otimes R + B \otimes [G + g \wedge H] \tag{1.4}$$

holds, where A, B are two non-zero 1-forms such that $A(.) = g(., \alpha)$, $B(.) = g(., \beta)$, $H = \eta \otimes \eta$, η being a non-zero 1-form defined by $\eta(.) = = g(., \rho)$ such that $g(\rho, \rho) = \epsilon$; and the Kulkarni–Nomizu product $E \wedge F$ of two (0, 2) tensors E and F is defined by

$$\begin{aligned}&(E \wedge F)(X_1, X_2, X_3, X_4) =\\ &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4) -\\ &\quad - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3),\end{aligned}$$

$X_i \in \chi(M)$, $i = 1, 2, 3, 4$. Such a manifold is denoted by QGK_n.

A semi-Riemannian manifold (M^n, g), $(n > 2)$, is said to be Ricci recurrent ([11]) if its Ricci tensor is not identically zero and satisfies $\nabla S = A \otimes S$, where A is a non-zero 1-form. Such a kind of manifold is denoted by RK_n. Let $U_S = \{x \in M : \nabla S \neq A \otimes S \text{ at } x\}$. Then the manifold (M^n, g) is said to be generalized Ricci recurrent ([5]) if on $U_S \subset M$, the condition $\nabla S = A \otimes S + B \otimes g$ holds where B is an 1-form on U_S. It is clear that the 1-form B is non-zero at every point of U_S. Such a manifold is denoted by GRK_n. Extending the notion of GRK_n, we introduce the notion of quasi-generalized Ricci recurrent manifolds.

A semi-Riemannian manifold (M^n, g) is said to be *quasi-generalized Ricci recurrent* (briefly, $QGRK_n$) if on $U_S \subset M$, the condition

$$\nabla S = A \otimes S + B \otimes [g + \eta \otimes \eta] \tag{1.5}$$

holds, where B and η are two 1-forms on U_S. It is clear that the 1-forms B and η are non-zero at every point of U_S.

A semi-Riemannian manifold (M^n, g) $(n \geq 3)$, is said to be *quasi-generalized* 2-*Ricci recurrent* if its Ricci tensor S is not identically zero and satisfies the following:

$$(\nabla\nabla S) = K \otimes S + N_1 \otimes g + I \otimes H, \tag{1.6}$$

where K, N_1, I are tensors of type (0, 2) and $H = \eta \otimes \eta$.

A projective transformation on a semi-Riemannian manifold is a transformation under which geodesics transform into geodesics. The projective curvature tensor P of type (0, 4) on a semi-Riemannian manifold (M^n, g) is defined by ([6], [16])

$$P = R - \frac{1}{n-1} D, \tag{1.7}$$

where D is a tensor of type (0, 4) and is given by

$$D(X, Y, Z, U) = g(X, U)S(Y, Z) - g(Y, U)S(X, Z)$$

$\forall$ X, Y, Z, $U \in \chi(M)$.

A semi-Riemannian manifold (M^n, g) $(n > 2)$, is said to be projectively recurrent ([2]) if its projective curvature tensor P is not identically zero and satisfies

$$\nabla P = A \otimes P,$$

where A is a non-zero 1-form.

As a special subgroup of the conformal transformation group, Y. Ishii ([8]) introduced the notion of the conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor $\overline{C}$ of type (0, 4) on a Riemannian manifold (M^n, g) $(n > 3)$ (this condition is assumed as for $n = 3$ the Weyl conformal tensor vanishes) is given by

$$\overline{C} = R - \frac{1}{n-2} g \wedge S. \tag{1.8}$$

A semi-Riemannian manifold (M^n, g) $(n > 3)$, is called a generalized conharmonically recurrent if its conharmonic curvature tensor $\overline{C}$ satisfies the following

$$\nabla\overline{C} = A \otimes \overline{C} + B \otimes G,$$

where A and B are non-zero 1-forms, and is denoted by $G\overline{C}K_n$.

A semi-Riemannian manifold (M^n, g) $(n > 3)$, is said to be conformally recurrent ([1]) if its conformal curvature tensor C is non-vanishing and satisfies the following:

$$\nabla C = A \otimes C, \tag{1.9}$$

where A is a non-zero 1-form.

Section 2 deals with some geometric properties of QGK_n. An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(+, +, +, \cdots, +, -)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A spacetime is a connected 4-dimensional Lorentzian manifold. The existence of QGK_4 is ensured by a proper example and it is shown that a Lorentzian QGK_4 is a Gödel cosmological model.

2. Some geometric properties of QGK_n

Theorem 2.1. *In a semi-Riemannian manifold (M^n, g) $(n > 2)$, the following results hold:*

(i) *A QGK_n is a $QGRK_n$.*

(ii) *In a QGK_n the relation*

$$rA + (n-1)(n + 2\epsilon)B = 2\big[A(Q.) + (n-1+\epsilon)B + (n-2)B(\rho)\eta\big] \tag{2.1}$$

holds, where r is the scalar curvature and Q being the symmetric endomorphism corresponding to the Ricci tensor S.

(iii) *In a QGK_n with non-zero constant scalar curvature the associated 1-forms A and B are related by $rA + (n-1)(n + 2\epsilon)B = 0$, and the relation $A(QX) + (n-1+\epsilon)B(X) + (n-2)B(\rho)\eta(X) = 0$ holds for all X.*

(iv) *If in a QGK_n with non-zero constant scalar curvature, the vector field ρ is parallel and the vector fields α and β associated to the 1-forms A and B respectively are codirectional, then A and B are closed.*

(v) *If a QGK_n is projectively recurrent, then it is a GK_n.*

(vi) *A QGK_n is a $G\overline{C}K_n$.*

(vii) *A $G\overline{C}K_n$ satisfying the condition*

$$\nabla S = A \otimes S + (n-2) B \otimes H, \;\; H = \eta \otimes \eta, \quad \textit{is a } QGK_n. \tag{2.2}$$

(viii) *A QGK_n $(n > 3)$ is a conformally recurent manifold.*

(ix) *If in a QGK_n, the vector field ρ is parallel and the vector fields α, β are codirectional, then it is a quasi-generalized 2-Ricci recurrent.*

Proof of (i). Taking an orthogonal frame field and contracting suitably, (1.4) yields

$$\nabla S = A \otimes S + B_1 \otimes g + B_2 \eta \otimes \eta, \tag{2.3}$$

where B_1 and B_2 are 1-forms given by $B_1 = (n-1+\epsilon)B$ and $B_2 = (n-2)B$ of which B_1 and B_2 are non-zero as B is non-zero. This proves (i).

Proof of (ii). From (2.3) it follows that

$$dr(X) = rA(X) + (n-1)(n+2\epsilon)B(X), \quad \text{and} \tag{2.4}$$

$$dr(X) = 2\big[A(QX) + (n-1+\epsilon)B(X) + (n-2)B(\rho)\eta(X)\big], \tag{2.5}$$

r being the scalar curvature of the manifold. By virtue of (2.4) and (2.5), we get (2.1). This proves (ii).

Proof of (iii). If r is a non-zero constant, then (2.4) and (2.5) implies that

$$rA(X) + (n-1)(n+2\epsilon)B(X) = 0, \quad \text{and} \tag{2.6}$$

$$A(QX) + (n-1+\epsilon)B(X) + (n-2)B(\rho)\eta(X) = 0, \tag{2.7}$$

which proves (iii).

Proof of (iv). Differentiating (1.4) covariantly, and then using (1.4) we obtain

(2.8)

$$\begin{aligned}
&(\nabla_Y \nabla_X R)(Z,W,U,V) = \\
&= \big[(\nabla_Y A)(X) + A(X)A(Y)\big] R(Z,W,U,V) + \\
&\quad + A(X)B(Y)\big[g(Z,V)g(W,U) - g(Z,U)g(W,V) + g(Z,V)\eta(W)\eta(U) + \\
&\quad + g(W,U)\eta(Z)\eta(V) - g(Z,U)\eta(W)\eta(V) - g(W,V)\eta(Z)\eta(U)\big] + \\
&\quad + (\nabla_Y B)(X)\big[g(Z,V)g(W,U) - g(Z,U)g(W,V) + g(Z,V)\eta(W)\eta(U) +
\end{aligned}$$

$$\begin{aligned}
&+ g(W,U)\eta(Z)\eta(V) - g(Z,U)\eta(W)\eta(V) - g(W,V)\eta(Z)\eta(U)\big]+\\
&+ B(X)\big[g(Z,V)(\nabla_Y\eta)(W)\eta(U) + g(Z,V)(\nabla_Y\eta)(U)\eta(W)+\\
&+ g(W,U)(\nabla_Y\eta)(Z)\eta(V) + g(W,U)\eta(Z)(\nabla_Y\eta)(V)-\\
&- \{g(Z,U)(\nabla_Y\eta)(W)\eta(V) + g(Z,U)(\nabla_Y\eta)(V)\eta(W)+\\
&+ g(W,V)(\nabla_Y\eta)(Z)\eta(U) + g(W,V)(\nabla_Y\eta)(U)\eta(Z)\}\big].
\end{aligned}$$

Interchanging X and Y and then subtracting the result, we obtain

$$\begin{aligned}
(2.9)\quad &(\nabla_Y\nabla_X R)(Z,W,U,V) - (\nabla_X\nabla_Y R)(Z,W,U,V) =\\
&= [(\nabla_Y A)(X) - (\nabla_X A)(Y)]R(Z,W,U,V)+\\
&\quad + [(\nabla_Y B)(X) - (\nabla_X B)(Y)][G + g\wedge H](Z,W,U,V)+\\
&\quad + [B(Y)A(X) - A(Y)B(X)][G + g\wedge H](Z,W,U,V)+\\
&\quad + B(X)\big[g(Z,V)(\nabla_Y\eta)(W)\eta(U) + g(Z,V)(\nabla_Y\eta)(U)\eta(W)+\\
&\quad + g(W,U)(\nabla_Y\eta)(Z)\eta(V) + g(W,U)(\nabla_Y\eta)(V)\eta(Z)-\\
&\quad - \{g(Z,U)(\nabla_Y\eta)(W)\eta(V) + g(Z,U)(\nabla_Y\eta)(V)\eta(W)+\\
&\quad + g(W,V)(\nabla_Y\eta)(Z)\eta(U) + g(W,V)(\nabla_Y\eta)(U)\eta(Z)\}\big]-\\
&\quad - B(Y)\big[g(Z,V)(\nabla_X\eta)(W)\eta(U) + g(Z,V)(\nabla_X\eta)(U)\eta(W)+\\
&\quad + g(W,U)(\nabla_X\eta)(Z)\eta(V) + g(W,U)(\nabla_X\eta)(V)\eta(Z)-\\
&\quad - \{g(Z,U)(\nabla_X\eta)(W)\eta(V) + g(Z,U)(\nabla_X\eta)(V)\eta(W)+\\
&\quad + g(W,V)(\nabla_X\eta)(Z)\eta(U) + g(W,V)(\nabla_X\eta)(U)\eta(Z)\}\big],
\end{aligned}$$

where $H = \eta\otimes\eta$.

We suppose that α, β are codirectional and ρ is a parallel vector field. Then $B(X)A(Y) - A(X)B(Y) = 0$ and $(\nabla_X\eta)(Z) = 0$ for all X, Y. Hence (2.9) takes the form

$$\begin{aligned}
(2.10)\qquad &(\nabla_Y\nabla_X R)(Z,W,U,V) - (\nabla_X\nabla_Y R)(Z,W,U,V) =\\
&= \big[(\nabla_Y A)(X) - (\nabla_X A)(Y)\big]R(Z,W,U,V)+\\
&\quad + \big[(\nabla_Y B)(X) - (\nabla_X B)(Y)\big][G + g\wedge H](Z,W,U,V).
\end{aligned}$$

From Walker's lemma $\big($[15], equation (26)$\big)$ we have

$$\begin{aligned}
(2.11)\qquad &(\nabla_X\nabla_Y R)(Z,W,U,V) - (\nabla_Y\nabla_X R)(Z,W,U,V)+\\
&\quad + (\nabla_Z\nabla_W R)(X,Y,U,V) - (\nabla_W\nabla_Z R)(X,Y,U,V)+\\
&\quad + (\nabla_U\nabla_V R)(Z,W,X,Y) - (\nabla_V\nabla_U R)(Z,W,X,Y) = 0.
\end{aligned}$$

By virtue of (2.10), (2.11) yields

$$(2.12)\quad \begin{aligned} &M(X,Y)R(Z,W,U,V)+L(X,Y)(G+g\wedge H)(Z,W,U,V)+\\ &+M(Z,W)R(X,Y,U,V)+L(Z,W)(G+g\wedge H)(X,Y,U,V)+\\ &+M(U,V)R(Z,W,X,Y)+L(U,V)(G+g\wedge H)(Z,W,X,Y)=0, \end{aligned}$$

where $M(X,Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X)$, and $L(X,Y) = (\nabla_X B)(Y) - -(\nabla_Y B)(X)$.

If the scalar curvature is a non-zero constant, then we have the relation (2.6). Using (2.6) in (2.12) we obtain

$$(2.13)\quad \begin{aligned} M(X,Y)N(Z,W,U,V) + M(Z,W)N(X,Y,U,V)+&\\ + M(U,V)N(Z,W,X,Y) = 0,& \end{aligned}$$

where $N = R - \frac{r}{(n-1)(n+2\epsilon)}(G + g\wedge H)$, from which it follows that N is a symmetric (0, 4) tensor with respect to the first pair of two indices and the last pair of two indices. Consequently by virtue of Walker's lemma ([15], eq. (27)), we obtain

$$M(X,Y) = L(X,Y) = 0,$$

for all X, Y. And hence

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0,$$
$$(\nabla_X B)(Y) - (\nabla_Y B)(X) = 0.$$

Therefore $dA(X,Y) = 0$, $dB(X,Y) = 0$. This proves (iv).

Remark 2.1. If ρ is a concurrent vector field, then the result is also true.

Proof of (v). From (2.3), (1.4) and (1.7) we obtain

$$(2.14)\quad \begin{aligned} &(\nabla_W P)(X,Y,Z,U) =\\ &= A(W)P(X,Y,Z,U) - \frac{B(W)\epsilon}{n-1}\big[g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\big]+\\ &+\frac{B(W)}{n-1}\big[g(X,U)\eta(Y)\eta(Z) - g(Y,U)\eta(X)\eta(Z)\big]+\\ &+B(W)\big[g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)\big]. \end{aligned}$$

Suppose that the manifold under consideration is projectively recurrent. Then (2.14) yields

$$(2.15)\qquad -\frac{\epsilon}{n-1}\big[g(X,U)g(Y,Z) - g(X,Z)g(Y,U)\big] +$$

$$+\frac{1}{n-1}\big[g(X,U)\eta(Y)\eta(Z)-g(Y,U)\eta(X)\eta(Z)\big]+$$
$$+\big[g(Y,Z)\eta(X)\eta(U)-g(X,Z)\eta(Y)\eta(U)\big]=0.$$

Taking contraction over Y and Z, we obtain

$$\epsilon g(X,U)=n\eta(X)\eta(Y). \tag{2.16}$$

Using (2.16) in (1.4) we get

$$\nabla R=A\otimes R+B_1\otimes G,$$

where $B_1=\frac{n+2\epsilon}{n}B$ is an 1-form. This proves (v).

Proof of (vi). From (2.3), (1.4) and (1.8) we obtain

$$\begin{aligned}(\nabla_W\overline{C})(X,Y,Z,U)&=\\ &=A(W)\overline{C}(X,Y,Z,U)+\\ &\quad+B_1(W)\big[g(X,U)g(Y,Z)-g(X,Z)g(Y,U)\big],\end{aligned} \tag{2.17}$$

where $B_1=-\frac{n+2\epsilon}{n-2}B$ is an 1-form. This proves (vi).

Proof of (vii). If the manifold is $G\overline{C}K_n$, then we have

$$\nabla\overline{C}=A\otimes\overline{C}+B\otimes G,$$

which yields, by virtue of (1.8), that

$$\nabla R-\frac{1}{n-2}\big(g\wedge(\nabla S)\big)=A\otimes\left(R-\frac{1}{n-2}g\wedge S\right)+B\otimes G. \tag{2.18}$$

By virtue of (2.2), (2.18) takes the form

$$\nabla R=A\otimes R+B\otimes[G+g\wedge H].$$

This proves (vii).

Proof of (viii). The conformal curvature tensor C of type (0, 4) of a semi-Riemannian manifold (M^n,g) $(n>3)$, is given by

$$C=R-\frac{1}{n-2}(g\wedge S)+\frac{r}{(n-1)(n-2)}G, \tag{2.19}$$

where r is the scalar curvature of the manifold and G is defined in (1.3).

From (2.3), (2.4), (1.4) and (2.19) we obtain (1.9), which proves (viii).

Proof of (ix). From (2.3), it follows that

$$
\begin{aligned}
&(\nabla_Y\nabla_X S)(Z,W) = \\
&= \big[(\nabla_Y A)(X) + A(X)A(Y)\big]S(Z,W) + \\
&\quad + A(X)B(Y)\big[(n-1+\epsilon)g(Z,W) + (n-2)\eta(Z)\eta(W)\big] + \\
&\quad + (\nabla_Y B)(X)\big[(n-1+\epsilon)g(Z,W) + (n-2)\eta(Z)\eta(W)\big] + \\
&\quad + (n-2)B(X)\big[(\nabla_Y\eta)(Z)\eta(W) + (\nabla_Y\eta)(W)\eta(Z)\big].
\end{aligned} \tag{2.20}
$$

Interchanging X, Y and subtracting the result, we obtain

$$
\begin{aligned}
&(\nabla_X\nabla_Y S)(Z,W) - (\nabla_Y\nabla_X S)(Z,W) = \\
&= M(X,Y)S(Z,W) + \big[A(Y)B(X) - A(X)B(Y) + \\
&\quad + L(X,Y)\big]\big[(n-1+\epsilon)g(Z,W) + (n-2)\eta(Z)\eta(W)\big] + \\
&\quad + (n-2)\big[B(Y)\{(\nabla_X\eta)(Z)\eta(W) + (\nabla_X\eta)(W)\eta(Z)\} - \\
&\quad - B(X)\{(\nabla_Y\eta)(Z)\eta(W) + (\nabla_Y\eta)(W)\eta(Z)\}\big].
\end{aligned} \tag{2.21}
$$

If $A(Y)B(X) - A(X)B(Y) = 0$ and ρ is a parallel vector field, then (2.21) takes the form

$$
\begin{aligned}
&(\nabla_X\nabla_Y S)(Z,W) - (\nabla_Y\nabla_X S)(Z,W) = \\
&= M(X,Y)S(Z,W) + (n-1+\epsilon)L(X,Y)g(Z,W) + \\
&\quad + (n-2)L(X,Y)\eta(Z)\eta(W).
\end{aligned} \tag{2.22}
$$

In view of (2.22) and (2.3) we obtain

$$
\begin{aligned}
(R(X,Y).S)(Z,W) = {} & \gamma(X,Y)g(Z,W) + \\
& + \delta(X,Y)S(Z,W) + \sigma(X,Y)\eta(Z)\eta(W),
\end{aligned} \tag{2.23}
$$

where

$$
\begin{aligned}
\gamma(X,Y) &= (n-1+\epsilon)\left[XB(Y) - YB(X) - 2B([X,Y])\right], \\
\delta(X,Y) &= XA(Y) - YA(X) - 2A([X,Y]) \quad \text{and} \\
\sigma(X,Y) &= (n-2)\left[XB(Y) - YB(X) - 2B([X,Y])\right].
\end{aligned}
$$

The relation (2.23) implies that the manifold is a quasi-generalized 2-Ricci recurrent. This proves (ix).

3. A proper example of QGK_4

In this section the existence of QGK_4 is ensured by a proper example.

Example 3.1. Let M be an open connected subset of $\mathbb{R}^4$. We consider the semi-Riemannian manifold M equipped with the Gödel metric ([7]) given by

$$ds^2 = g_{ij}dx^i dx^j = a^2\Big[-(dx^1)^2 + \frac{1}{2}e^{2x^1}(dx^2)^2 - (dx^3)^2 + \\ + (dx^4)^2 + 2e^{x^1}dx^2dx^4\Big] \quad (i, j = 1, 2, 3, 4), \tag{3.1}$$

where a is a positive number and $x^1, \dots, x^4$ are the standard coordinates of $\mathbb{R}^4$.

The only non-vanishing components of the Christoffel symbols of second kind, the curvature tensor and their covariant derivatives are

$$\Gamma^1_{22} = \frac{1}{2}e^{2x^1}, \quad \Gamma^4_{12} = \Gamma^1_{42} = \frac{1}{2}e^{x^1}, \quad \Gamma^4_{41} = 1, \quad \Gamma^2_{41} = -e^{-x^1},$$

$$\begin{cases} R_{4141} = -\frac{1}{2}a^2, & R_{4112} = \frac{1}{2}a^2e^{x^1}, \quad R_{4242} = -\frac{1}{4}a^2e^{2x^1}, \\ R_{1212} = -\frac{3}{4}a^2e^{2x^1}, & R_{1221,\,1} = a^2e^{2x^1}, \quad R_{4112,\,1} = \frac{1}{2}a^2e^{x^1}, \end{cases} \tag{3.2}$$

and the components which can be obtained from these by the symmetry property of R. The scalar curvature of the manifold is $\frac{1}{a^2}$ which is non-vanishing and constant. We shall now show that M is a QGK_4, i.e., it satisfies the defining condition (1.4). In terms of local coordinates, we consider the components of the associated 1-forms as follows:

$$\begin{cases} A_i(\partial_i) = A_i = \begin{cases} \frac{3}{2} & \text{for} \quad i = 1, \\ 0 & \text{otherwise,} \end{cases} \\ B_i(\partial_i) = B_i = \begin{cases} \frac{1}{a^2} & \text{for} \quad i = 1, \\ 0 & \text{otherwise,} \end{cases} \\ \eta_i(\partial_i) = \eta_i = \begin{cases} \frac{\sqrt{3}}{2}a & \text{for} \quad i = 1, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{3.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ at any point $x \in M$.

In terms of local coordinates, the defining equation (1.4) of QGK_n can be written as

$$(3.4)\quad R_{hijk,l} = A_l R_{hijk} + B_l \big[g_{hk} g_{ij} - g_{hj} g_{ik} + g_{hk} \eta_i \eta_j + g_{ij} \eta_h \eta_k - \\ - g_{hj} \eta_i \eta_k - g_{ik} \eta_h \eta_j \big], \quad i,j,h,k,l = 1,2,\ldots,n.$$

By virtue of (3.2) and (3.3), it follows that (3.4) holds for all $i,j,h,k,l = = 1,2,3,4$. Therefore, M is a QGK_4. Thus we can state the following:

Theorem 3.1. *Let* (M^4, g) *be a semi-Riemannian manifold equipped with the metric given by* (3.1). *Then* (M^4, g) *is a* QGK_4 *with non-vanishing scalar curvature which is neither* K_4 *nor* GK_4.

Acknowledgement. The authors wish to express their sincere thanks to the referee for his valuable comments towards the improvement of the paper. The authors gratefully acknowledge the financial support of CSIR, New Delhi, India [Project F. No. 25(0171)/09/EMR-II].

References

[1] ADATI, T. and MIYAZAWA, T.: On Riemannian space with recurrent conformal curvature, *Tensor (N.S.)* **18** (1967), 348–354.

[2] ADATI, T. and MIYAZAWA, T.: On projective transformations of projective recurrent spaces, *Tensor (N.S.)* **31** (1977), 49–54.

[3] ARSLAN, K., DE, U. C., MURATHAN, C. and YILDIZ, A.: On generalized recurrent Riemannian manifolds, *Acta Math. Hung.* **123** (2009), 27–39.

[4] DE, U. C. and GUHA, N.: On generalized recurrent manifolds, *J. Nat. Acad. of Math. India* **9** (1991), 85–92.

[5] DE, U. C., GUHA, N. and KAMILYA, D.: On generalized Ricci recurrent manifolds, *Tensor (N.S)* **56** (1995), 312–317.

[6] DE, U. C. and SHAIKH, A. A.: *Differential geometry of manifolds*, Narosa Publ. Pvt. Ltd., New Delhi, 2007.

[7] GÖDEL, K.: An example of a new type of cosmological solutions of Einstein's field equations of gravitations, *Rev. Mod. Phys.* **21** (1949), 447.

[8] ISHII, I.: On conharmonic transformations, *Tensor (N.S.)* **11** (1957), 73–80.

[9] LICHNEROWICZ, A.: Courbure, nombres de Betti, et espaces symmetriques, *Proc. of the Int. Cong. of Math.* **2** (1952), 216–223.

[10] MIYAZAWA, T.: On Riemannian space admitting some recurrent tensors, *TRU Math. J.* **2** (1996), 11–18.

[11] PATTERSON, E. M.: Some theorems on Ricci recurrent spaces, *J. London. Math. Soc.* **27** (1952), 287–295.

[12] RUSE, H. S., WALKER, A. G. and WILLMORE, T. J.: *Harmonic spaces*, Edizioni Cremonese, Roma, 1961.

[13] TAMÁSSY, L. and BINH, T. Q.: On weakly symmetric and weakly projective symmetric Riemannian manifolds, *Coll. Math. Soc. J. Bolyai* **56** (1989), 663–670.

[14] TAMÁSSY, L. and BINH, T. Q.: On weak symmetries of Einstein and Sasakian manifolds, *Tensor (N. S.)* **53** (1993), 140–148.

[15] WALKAR, A. G. On Ruse's spaces of recurrent curvature, *Proc. London Math. Soc.* **52** (1950), 36–64.

[16] YANO, K. and KON, M.: *Structure on manifolds*, World Sci. Publ., Singapore, 1989.