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UNIQUE RANGE SETS OF MEROMORPHIC FUNCTIONS WITH FINITE WEIGHT

Abhijit **Banerjee**

Department of Mathematics, West Bengal State University, Barasat, North 24 Prgs., West Bengal, Kolkata-700126, India

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Abstract: We employ the notion of weighted sharing of sets to deal with the problem of Unique Range Sets for meromorphic functions and obtain a result which improve and supplement all the results obtained earlier in this aspect.

1. Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We shall use the standard notations of value distribution theory:

$$T(r,f), \quad m(r,f), \quad N(r,\infty;f), \quad \overline{N}(r,\infty;f), \dots$$

(see [6]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $(r\longrightarrow\infty, r\notin E)$.

For any constant a, we define

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

E-mail addresses: abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

As a simple application of his own value distribution theory, Nevanlinna proved that a non-constant meromorphic function is uniquely determined by the inverse image of 5 distinct values (including the infinity), IM. Thus, the study of the relationship between two meromorphic functions via the preimage sets of several distinct values in the range has a long history. Inspired by the Nevanlinna's four and five value theorems, in 1970s F. Gross and C. C. Yang started to study the similar but more general questions of two functions that share sets of distinct elements instead of values. For instance, they proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$. In 1977 F. Gross proposed the following question in [6]:

"Is there a finite set S so that an entire function is determined uniquely by the pre-image of the set S, CM?"

We recall that a set S is called a unique range set for meromorphic functions (URSM) if for any pair of non-constant meromorphic functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$. Similarly a set S is called a unique range set for entire functions (URSE) if for any pair of non-constant entire functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$. We will call any set $S \subset \mathbb{C}$ a unique range set for meromorphic functions ignoring multiplicities(URSM-IM) for which $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$ for any pair of non-constant meromorphic functions.

In the last couple of years the concept of URSE, URSM and URSM-IM have gradually increased among the researchers. The study is focused mainly on two problems: finding different URSM with smallest cardinality and at the same time characterizing the URSM (see [1]–[4], [10]–[19]).

A recent advent in the uniqueness literature is the notion of weighted sharing instead of sharing IM/CM which implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001 in [8, 9]. Below we are giving the definition.

Definition 1.1 [8, 9]**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 [8]**.** Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. Let $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

In 2003 Y. Xu [16] proved the following theorem which is the improvement of Fang and Hua [2].

Theorem A [16]**.** *If f and g are two non-constant meromorphic functions and $\Theta(\infty; f) > \frac{3}{4}$, $\Theta(\infty; g) > \frac{3}{4}$, then there exists a set with seven elements such that $E_f(S, \infty) = E_g(S, \infty)$ implies $f \equiv g$.*

Dealing with the question of Yi raised in [19] Lahiri and Banerjee exhibited a unique range set S with higher cardinalities than Y. Xu [16] but significantly weaken the condition over the ramification indexes on f and g. They obtained the following result.

Theorem B [10]**.** *Let*

$$S = \{z : z^n + az^{n-1} + b = 0\},$$

where n (≥ 9) be an integer and a, b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If $E_f(S, 2) = E_g(S, 2)$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n-1}$ then $f \equiv g$.

In [1] and [3] Bartels and Fang-Guo both independently proved the existence of a URSM-IM with 17 elements.

In this paper we shall continue the investigations and provide better results from those mentioned earlier and at the same time supplement them.

The following theorem is the main result of the paper.

Theorem 1.1. *Let*
$$S=\left\{z:\frac{(n-1)(n-2)}{4}z^{n}-\frac{n(n-2)}{2}z^{n-1}+\frac{n(n-1)}{4}z^{n-2}-1=0\right\},$$
where n (≥ 6) is an integer. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,m)=E_g(S,m)$. If

(i) $m\geq 2$ *and* $\Theta_f+\Theta_g+\min\{\Theta(1;f),\Theta(1;g)>10-n$,

(ii) *or if* $m=1$ *and* $\Theta_f+\Theta_g+\min\{\Theta(1;f),\Theta(1;g)\}+\frac{1}{2}\min\{\Theta(0;f)+$ $+\Theta(\infty;f),\Theta(0;g)+\Theta(\infty;g)\}>11-n$,

(iii) *or if* $m=0$ *and* $\Theta_f+\Theta_g+\Theta(0;f)+\Theta(\infty;f)+\Theta(0;g)+\Theta(\infty;g)+$ $+\min\{\Theta(0;f)+\Theta(1;f)+\Theta(\infty;f),\Theta(0;g)+\Theta(1;g)+\Theta(\infty;g)>$ $>16-n$,

then $f\equiv g$, where $\Theta_f=2\Theta(0;f)+2\Theta(\infty;f)+\Theta(1;f)$ and Θ_g can be similarly defined.

Corollary 1.1. *In Th. 1.1 when $m=2$ and $n\geq 7$ and $n\geq 9$ it is the improvements of the results of Y. Xu* [16] *and Lahiri–Banerjee* [10] *respectively. On the other hand when $m=0$ and $n\geq 17$ it is the improvement of the results of Bartel's* [1] *as well as Fang-Guo* [3].

We have already assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [6]. We are still going to explain some notations as these are used in the paper.

Definition 1.3 [7]**.** For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $N(r,a;f\mid=1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r,a;f\mid\leq m)(N(r,a;f\mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq m)$ $(\overline{N}(r,a;f\mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$, $\overline{N}(r,a;f\mid<m)$ and $\overline{N}(r,a;f\mid>m)$ are defined analogously.

Definition 1.4 [9]**.** We denote by $N_2(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)$.

Definition 1.5. Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the reduced counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where

$p = q = 1$, by $\overline{N}_E^{(2}(r, a; f)$ the reduced counting function of those a-points of f and g where $p = q \geq 2$. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$. In a similar manner we can define $\overline{N}_L(r, a; f)$ and $\overline{N}_L(r, a; g)$ for $a \in \mathbb{C} \cup \{\infty\}$. When f and g share (a, m), $m \geq 1$ then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 1.6 [8, 9]**.** Let f, g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly

$\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f and g be two non-constant meromorphic function and for an integer $n \geq 3$

$$F = \frac{(n-1)(n-2)}{4} f^n - \frac{n(n-2)}{2} f^{n-1} + \frac{n(n-1)}{4} f^{n-2}, \tag{2.1}$$

$$G = \frac{(n-1)(n-2)}{4} g^n - \frac{n(n-2)}{2} g^{n-1} + \frac{n(n-1)}{4} g^{n-2}. \tag{2.2}$$

Henceforth we shall denote by H the following functions

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 2.1 [20]**.** If F, G are two non-constant meromorphic functions such that they share (1,0) and $H \not\equiv 0$ then

$$N_E^{1)}(r, 1; F \mid= 1) = N_E^{1)}(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2. *Let F, G be given by* (2.1) *and* (2.2)*. If $H \not\equiv 0$ then*

$$N(r, H) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}_*(r, 1; F, G) + \overline{N}(r, \infty; f) + \\ + \overline{N}(r, \infty; g) + \overline{N}(r, 1; f) + \overline{N}(r, 1; g) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g'),$$

where $\overline{N}_0(r, 0; f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $F-1$, $\overline{N}_0(r, 0; g')$ is similarly defined.

Proof. First we note that $F' = n(n-1)(n-2)f^{n-3}(f-1)^2 f'/4$ and $G' = n(n-1)(n-2)g^{n-3}(g-1)^2 g'/4$. We can easily verify that possible poles of H occur at (i) zeros (1-points) of f and g, (ii) poles of f and g, (iii) those 1-points of F and G whose multiplicities are distinct from the multiplicities of the corresponding 1-points of G and F respectively, (iv) zeros of f' which are not the zeros of $f(f-1)$ and $F-1$, (v) zeros of which are not the zeros of $g(g-1)$ and $G-1$.

Since H has only simple poles, clearly the lemma follows from above explanations. $\Diamond$

Lemma 2.3 [11]**.** *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f)+ \\ + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f).$$

Lemma 2.4 [15]**.** *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r,f) + O(1)$.*

Lemma 2.5. *Let f, g be two non-constant meromorphic functions and suppose α_1 and α_2 are the roots of the equation $\frac{(n-1)(n-2)}{4}z^2 - \frac{n(n-2)}{2}z + \frac{n(n-1)}{4} = 0$. Then*

$$(n-1)^2(n-2)^2 f^{n-2}(f-\alpha_1)(f-\alpha_2)g^{n-2}(g-\alpha_1)(g-\alpha_2) \not\equiv 16$$

and n (≥ 5) is an integer.

Proof. If possible, let us suppose

$$(n-1)^2(n-2)^2 f^{n-2}(f-\alpha_1)(f-\alpha_2)g^{n-2}(g-\alpha_1)(g-\alpha_2) \equiv 16. \tag{2.3}$$

Let z_0 be a zero of f with multiplicity p. Then z_0 is a pole of g with multiplicity q such that

$$(n-2)p = (n-2)q + 2q = nq. \tag{2.4}$$

From (2.4) we see that $2q = (n-2)(p-q) \geq n-2$ and so $p = \frac{n}{n-2}q \geq \frac{n}{2}$.

Let z_0 be a zero of $f-\alpha_i$ $i = 1,2$ with multiplicity p. Then z_0 is a pole of g with multiplicity q such that $p = (n-2)q + 2q = nq \geq n$.

Since the poles of f are the zeros of g and $g-\alpha_i$ $i = 1,2$, we get

$$\begin{aligned} \overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) \leq \\ &\leq \frac{2}{n}N(r,0;g) + \frac{1}{n}\,N(r,\alpha_1;g) + \frac{1}{n}\,N(r,\alpha_2;g) \leq \\ &\leq \frac{4}{n}T(r,g). \end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned}2T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)+\overline{N}(r,\infty;f)+S(r,f) \le \\ &\le \frac{2}{n}N(r,0;f)+\frac{1}{n}\,N(r,\alpha_1;f)+\frac{1}{n}\,N(r,\alpha_2;f)+ \\ &\quad +\frac{4}{n}T(r,g)+S(r,f)\le \\ &\le \frac{4}{n}T(r,f)+\frac{4}{n}T(r,g)+S(r,f).\end{aligned}$$

i.e.,

$$\left(2-\frac{4}{n}\right)T(r,f)\le \frac{4}{n}\,T(r,g)+S(r,f). \tag{2.5}$$

Similarly

$$\left(2-\frac{4}{n}\right)T(r,g)\le \frac{4}{n}\,T(r,f)+S(r,g) \tag{2.6}$$

Adding (2.5) and (2.6) we get

$$\left(2-\frac{8}{n}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

a contradiction for $n \ge 5$. This proves the lemma. $\Diamond$

Lemma 2.6 [3]**.** *Let f, g be two non-constant meromorphic functions and suppose n (≥ 6) is an integer. If*

$$\begin{aligned}&\frac{(n-1)(n-2)}{2}f^n-n(n-2)f^{n-1}+\frac{n(n-1)}{2}f^{n-2}\equiv \\ &\equiv \frac{(n-1)(n-2)}{2}g^n-n(n-2)g^{n-1}+\frac{n(n-1)}{2}g^{n-2},\end{aligned}$$

then $f\equiv g$.

Lemma 2.7. *Let F, G be given by (2.1), where $n \ge 7$ is an integer. Also let S be given as in Th. 1.1. If $E_f(S,0)=E_g(S,0)$ then $S(r,f)=$ $=S(r,g)$.*

Proof. Since $E_f(S,0)=E_g(S,0)$, it follows that F and G share $(1,0)$. We first note that the polynomial

$$p(z)=\frac{(n-1)(n-2)}{4}z^n-\frac{n(n-2)}{2}z^{n-1}+\frac{n(n-1)}{4}z^{n-2}-1$$

has only simple zeros. In fact

$$p'(z)=\frac{n(n-1)(n-2)}{4}z^{n-3}(z-1)^2.$$

Also we note that $p(0), p(1) \neq 0$. Thus all the zeros of $p(z)$ are simple and we denote them by w_j, $j = 1, 2, \ldots n$. Since F, G share $(1, 0)$ from the second fundamental theorem we have

$$\begin{aligned}(n-2)T(r,g) &\leq \sum_{j=1}^{n} \overline{N}\left(r, w_j; g\right) + S(r,g) = \\ &= \sum_{j=1}^{n} \overline{N}\left(r, w_j; f\right) + S(r,g) \leq \\ &\leq nT(r,f) + S(r,g).\end{aligned}$$

Similarly we can deduce

$$(n-2)T(r,f) \leq nT(r,g) + S(r,f).$$

The last inequalities imply $T(r,f) = O\left(T(r,g)\right)$ and $T(r,g) = O\left(T(r,f)\right)$ and so we have $S(r,f) = S(r,g)$. $\Diamond$

Lemma 2.8. *Under the condition of Th. 1.1, $H \equiv 0$.*

Proof. Let F, G be given by (2.1) and (2.2). Since $E_f(S,m) = E_g(S,m)$ it follows that F, G share $(1, m)$. If possible let us suppose that $H \not\equiv 0$.

Case 1. $m \geq 1$. While $m \geq 2$, using Lemma 2.3 we note that

$$\begin{aligned}(2.7)\quad & \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \leq \\ &\leq \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3) \leq \\ &\leq \overline{N}_0(r,0;g') + \sum_{j=1}^{n}\left\{\overline{N}(r,\omega_j;g \mid= 2) + 2\overline{N}(r,\omega_j;g \mid\geq 3)\right\} \leq \\ &\leq N(r,0;g' \mid g \neq 0) + S(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g).\end{aligned}$$

Hence using (2.7), Lemmas 2.1 and 2.2 we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned}(2.8)\quad & (n+1)\,T(r,f) \leq \\ &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + N(r,1;F \mid= 1) + \\ &\quad + \overline{N}(r,1;F \mid\geq 2) - N_0(r,0;f') + S(r,f) \leq \\ &\leq 2\left\{\overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f)\right\} + \overline{N}(r,0;g) + \\ &\quad + \overline{N}(r,1;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;G \mid\geq 2) + \\ &\quad + N_*(r,1;F,G) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g) \leq \\ &\leq 2\left\{\overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g)\right\} + \\ &\quad + \overline{N}(r,1;g) + S(r,f) + S(r,g)\end{aligned}$$

$$\leq \big(11 - 2\Theta(0;f) - 2\Theta(0;g) - \\ - 2\Theta(\infty;f) - 2\Theta(\infty;g) - 2\Theta(1;f) - \Theta(1;g) + \varepsilon\big)T(r) + S(r).$$

In a similar way we can obtain

$$(2.9)\quad (n+1)\,T(r,g) \leq \big(11-2\Theta(0;f)-2\Theta(0;g)-2\Theta(\infty;f)- \\ -2\Theta(\infty;g)-\Theta(1;f)-2\Theta(1;g)+\varepsilon\big)T(r)+S(r).$$

Combining (2.8) and (2.9) we see that

$$(2.10)\quad \big(n-10+2\Theta(0;f)+2\Theta(\infty;f)+\Theta(1;f)+2\Theta(0;g)+ \\ + 2\Theta(\infty;g)+\Theta(1;g)+\min\{\Theta(1;f),\Theta(1;g)\}-\varepsilon\big)\,T(r)\leq S(r).$$

Since $\varepsilon > 0$, (2.10) leads to a contradiction.

While $m = 1$, using Lemma 2.3, (2.7) changes to

$$(2.11)\quad \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \leq \\ \leq \overline{N}_0(r,0;g')+\overline{N}(r,1;G \mid\geq 2)+\overline{N}_L(r,1;G)+\overline{N}(r,1;F \mid\geq 3)\leq \\ \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\sum_{j=1}^{n}\{N(r,\omega_j;f) - \overline{N}(r,\omega_j;f)\} \leq \\ \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\big\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\big\}+ \\ + S(r,f) + S(r,g).$$

So using (2.11), Lemmas 2.1 and 2.2 and proceeding as in (2.8) we get from second fundamental theorem for $\varepsilon > 0$ that

$$(2.12)\quad (n+1)\,T(r,f) \leq \\ \leq 2\,\big\{\overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\big\} + \\ + \overline{N}(r,1;g) + \frac{1}{2}\big\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\big\} + S(r,f) + S(r,g) \leq \\ \leq \left\{\frac{5}{2}\overline{N}(r,0;f)+2\overline{N}(r,1;f)+\frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,0;g)+2\overline{N}(r,\infty;g)\right\}+ \\ + \overline{N}(r,1;g) + S(r,f) + S(r,g) \leq \\ \leq \left(12-\frac{5}{2}\Theta(0;f)-2\Theta(0;g)-\frac{5}{2}\Theta(\infty;f)- \right. \\ \left. - 2\Theta(\infty;g)-2\Theta(1;f)-\Theta(1;g)+\varepsilon\right)T(r) + S(r).$$

Similarly we can obtain

(2.13)
$$(n+1)\,T(r,g) \le \left(12 - 2\Theta(0;f) - \frac{5}{2}\Theta(0;g) - 2\Theta(\infty;f) - \right.$$
$$\left. - \frac{5}{2}\Theta(\infty;g) - \Theta(1;f) - 2\Theta(1;g) + \varepsilon\right)T(r) + S(r).$$

Combining (2.12) and (2.13) we see that

(2.14)
$$\Big(n - 11 + 2\Theta(0;f) + 2\Theta(\infty;f) + \Theta(1;f) + 2\Theta(0;g) +$$
$$+ 2\Theta(\infty;g) + \Theta(1;g) + \min\{\Theta(1;f), \Theta(1;g)\} +$$
$$+ \frac{1}{2}\min\big\{\Theta(0;f) + \Theta(\infty;f), \Theta(0;g) + \Theta(\infty;g)\big\} - \varepsilon\Big)\, T(r) \le$$
$$\le S(r).$$

Since $\varepsilon > 0$, (2.14) leads to a contradiction.

Case 2. $m = 0$. Using Lemma 2.3 we note that

(2.15)
$$\overline{N}_0(r,0;g') + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \le$$
$$\le \overline{N}_0(r,0;g') + \overline{N}_E^{(2}(r,1;G) + \overline{N}_L(r,1;G) + \overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \le$$
$$\le \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_L(r,1;G) + 2\overline{N}_L(r,1;F) \le$$
$$\le N(r,0;g' \mid g \ne 0) + \overline{N}(r,1;G \mid\ge 2) + 2\overline{N}(r,1;F \mid\ge 2) \le$$
$$\le 2\big\{\overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + \overline{N}(r,\infty;f)\big\} + S(r,f) + S(r,g).$$

Hence using (2.15), Lemmas 2.1 and 2.2 we get from second fundamental theorem for $\varepsilon > 0$ that

(2.16)
$$(n+1)\,T(r,f) \le$$
$$\le \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + N_E^{1)}(r,1;F) + \overline{N}_L(r,1;F) +$$
$$+ \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) - N_0(r,0;f') + S(r,f) \le$$
$$\le 2\left\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;f)\right\} + \overline{N}(r,0;g) +$$
$$+ \overline{N}(r,1;g) + \overline{N}(r,\infty;g) + \overline{N}_E^{(2}(r,1;F) + 2\overline{N}_L(r,1;G) +$$
$$+ 2\overline{N}_L(r,1;F) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g) \le$$

$$\leq 4\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}+3\left\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right\}+$$
$$+2\overline{N}(r,1;f)+\overline{N}(r,1;g)+S(r,f)+S(r,g)\leq$$
$$\leq \big(17-4\Theta(0;f)-4\Theta(\infty;f)-3\Theta(0;g)-3\Theta(\infty;g)-$$
$$-2\Theta(1;f)-\Theta(1;g)+\varepsilon\big)T(r)+S(r).$$

In a similar manner we can obtain

$$(n+1)\,T(r,g)\leq \big(17-3\Theta(0;f)-3\Theta(\infty;f)-4\Theta(0;g)- \\ -4\Theta(\infty;g)-\Theta(1;f)-2\Theta(1;g)+\varepsilon\big)T(r)+S(r). \tag{2.17}$$

Combining (2.16) and (2.17) we see that

$$\big(n-16+3\Theta(0;f)+3\Theta(\infty;f)+\Theta(1;f)+ \\ +3\Theta(0;g)+3\Theta(\infty;g)+\Theta(1;g)+ \\ +\min\big\{\Theta(0;f)+\Theta(1;f)+\Theta(\infty;f),\Theta(0;g)+ \\ +\Theta(1;g)+\Theta(\infty;g)\big\}-\varepsilon\big)T(r)\leq S(r). \tag{2.18}$$

Since $\varepsilon>0$, (2.18) leads to a contradiction. $\Diamond$

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1) and (2.2). Since $E_f(S,m)=E_g(S,m)$ it follows that F, G share $(1,m)$. By a simple computation it can be easily seen that 1 is a root with multiplicity 3 of $F-\frac{1}{2}$ and hence $F-\frac{1}{2}=(f-1)^3\ Q_{n-3}(f)$, where $Q_{n-3}(f)$ is a polynomial in f of degree $n-3$ and thus $N_2\left(r,\frac{1}{2};F\right)\leq 2\overline{N}(r,1;f)+$ $+N\left(r,0;Q_{n-3}(f)\right)\leq 2\overline{N}(r,1;f)+(n-3)T(r,f)+S(r,f)$.

From Lemma 2.8 we have $H\equiv 0$. So

$$F\equiv\frac{aG+b}{cG+d}, \tag{3.1}$$

where a, b, c, d are constants such that $ad-bc\neq 0$. Also

$$T(r,F)=T(r,G)+O(1). \tag{3.2}$$

We now consider the following cases.

Case 1. Let $ac \neq 0$. From (3.1) we get

$$\overline{N}(r,\infty;G) = \overline{N}\left(r,\frac{a}{c};F\right). \tag{3.3}$$

So in view of (3.2), (3.3) and the second fundamental theorem we get

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r,\frac{a}{c};F\right) + S(r,F) = \\ &= \overline{N}(r,0;f) + 2T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) \leq \\ &\leq 5T(r,f) + S(r,f),\end{aligned}$$

which in view of by Lemma 2.4 gives a contradiction for $n \geq 6$.

Case 2. Let $a \neq 0$ and $c = 0$. Then $F = \alpha G + \beta$, where $\alpha = \frac{a}{d}$ and $\beta = \frac{b}{d}$.

If F has no 1-point, by the second fundamental theorem we get

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + S(r,f) \leq \\ &\leq 3T(r,f) + \overline{N}(r,\infty;f) + S(r,f),\end{aligned}$$

which implies a contradiction in view of Lemma 2.4.

If F and G have some 1-points then $\alpha + \beta = 1$ and so

$$F \equiv \alpha G + 1 - \alpha. \tag{3.4}$$

Suppose $\alpha \neq 1$. If $1 - \alpha \neq \frac{1}{2}$ then in view of (3.2) and the second fundamental theorem we get

$$\begin{aligned}2T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,1-\alpha;F) + \overline{N}\left(r,\frac{1}{2};F\right) + \\ &\quad + \overline{N}(r,\infty;F) + S(r,F) \leq \\ &\leq 3T(r,f) + \overline{N}(r,0;G) + (n-2)T(r,f) + \overline{N}(r,\infty;f) + S(r,f) \leq \\ &\leq (n+5)T(r,f) + S(r,f),\end{aligned}$$

which implies a contradiction in view of Lemma 2.4 and $n \geq 6$. If $\alpha = \frac{1}{2}$, then we have from (3.4)

$$F \equiv \frac{1}{2}(G+1).$$

So by the second fundamental theorem we can obtain using (3.2) that

$$\begin{aligned}2T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{2};G\right) + \overline{N}(r,-1;G) + \overline{N}(r,\infty;G) + S(r,G) \leq \\ &\leq 3T(r,g) + (n-2)T(r,g) + \overline{N}(r,0;F) + \overline{N}(r,\infty;g) + S(r,g) \leq \\ &\leq (n+5)T(r,g) + S(r,g),\end{aligned}$$

which implies a contradiction in view of Lemma 2.4 and $n \geq 6$.

So $\alpha = 1$ and hence $F \equiv G$. So by Lemma 2.6 we get $f \equiv g$.

Case 3. Let $a = 0$ and $c \neq 0$. Then $F \equiv \frac{1}{\gamma G+\delta}$, where $\gamma = \frac{c}{b}$ and $\delta = \frac{d}{b}$.

If F has no 1-point then as in Subcase 2.2 we can deduce a contradiction.

If F and G have some 1-points then $\gamma + \delta = 1$ and so

$$F \equiv \frac{1}{\gamma G + 1 - \gamma}. \tag{3.5}$$

Suppose $\gamma \neq 1$ If $\gamma \neq -1$, then by the second fundamental theorem we get

$$\begin{aligned}2\,T(r,F) \leq \overline{N}(r,0;F) + \overline{N}\left(r,\frac{1}{1-\gamma};F\right) + \overline{N}\left(r,\frac{1}{2};F\right) + \\ + \overline{N}(r,\infty;f) + S(r,f) \leq \\ \leq 3T(r,f) + \overline{N}(r,0;G) + (n-2)T(r,f) + \overline{N}(r,\infty;f) + S(r,f) \leq \\ \leq (n+5)T(r,f) + S(r,f),\end{aligned}$$

which gives a contradiction in view of Lemma 2.4 and $n \geq 6$. If $\gamma = -1$ from (3.5) we have

$$F \equiv \frac{1}{-G+2}.$$

Now the second fundamental theorem with the help of (3.2) yields

$$\begin{aligned}&2T(r,G) \leq \\ &\leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{2};G\right) + \overline{N}(r,2;G) + \overline{N}(r,\infty;G) + S(r,G) \leq \\ &\leq 3T(r,g) + (n-2)T(r,g) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,g) \leq \\ &\leq (n+3)T(r,g) + S(r,g),\end{aligned}$$

which implies a contradiction in view of Lemma 2.4 and $n \geq 6$.

So we must have $\gamma = 1$ then $FG \equiv 1$, which is impossible by Lemma 2.5. This completes the proof of the theorem. $\diamondsuit$

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