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PSEUDORADIAL ORDER OF PSEUDORADIAL SPACES

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Abstract: Pseudoradial normal spaces of any order of pseudoradiality given by an ordinal number not greater than $\sigma_c(X)^+$ are constructed. Another construction with similar properties is given for compact T_1 spaces. Finally pseudoradial spaces of cardinality ω_α and pseudoradial order $\omega_{\alpha+1}$ are exhibited. The most important tools to perform such constructions are those contained in Lemma 2.7, Th. 2.8, Th. 4.2 and Th. 5.2.

1. Introduction

As is well known, a topological space X is called *pseudoradial* or *chain-net* (see [6], [1] or [4]) provided that for every non-closed subset A of X, there exist a point $x \in \overline{A} \setminus A$ and a transfinite sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ in A such that $x_\alpha \to x$ when $\alpha \to \lambda$.

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Definition 1.1. Let X be a topological space. Let $A \subseteq X$. Following [3], we define the *pseudoradial* or *chain-net closure of A in X* as the set $\widehat{A} = \{x \in X \mid \text{there is a transfinite sequence } \langle x_\alpha \rangle_{\alpha<\lambda} \text{ in } A \text{ converging to } x\}$. We can now use transfinite recursion to define the *pseudoradial* or *chain-net iterated closure* of A.

$$\begin{array}{lcll} \widehat{A}^{(0)} & = & A; & \\ \widehat{A}^{(\alpha+1)} & = & \widehat{\left(\widehat{A}^{(\alpha)}\right)} & \text{for every ordinal } \alpha; \\ \widehat{A}^{(\beta)} & = & \bigcup_{\alpha<\beta} A^{(\alpha)} & \text{if } \beta \text{ is a limit ordinal.} \end{array}$$

Remark 1.2. Let X be a topological space. Let $A \subseteq X$. Then

(i) $A \subseteq \widehat{A} \subseteq \overline{A}$;

(ii) for each pair of ordinals α, β, if $\alpha \leq \beta$, then $\widehat{A}^{(\alpha)} \subseteq \widehat{A}^{(\beta)}$;

(iii) for each ordinal α, $A \subseteq \widehat{A}^{(\alpha)} \subseteq \overline{A}$.

Definition 1.3. Let X be a pseudoradial space. The *pseudoradial order* of X is the least ordinal number α such that for each $A \subseteq X$,

$$\widehat{A}^{(\alpha)} = \overline{A}.$$

The pseudoradial order of a pseudoradial space X is denoted by $\mathrm{pro}(X)$.

The following proposition presents an upper bound on the number of times the pseudoradial closure has to be iterated in order to get the topological closure and therefore assures us that the pseudoradial order of a pseudoradial space always exists. We follow the proof presented in [5] or [8], where it is proved that the sequential order of a sequential space is not greater that ω_1.

We recall that the *chain character* of a pseudoradial space X is the least infinite cardinal $\sigma_c(X)$ such that for each non-closed subset A of X, there exist a point $x \in \overline{A} \setminus A$ and a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ in A of length $\lambda \leq \sigma_c(X)$ such that $x_\alpha \to x$ when $\alpha \to \lambda$.

We also recall that for each cardinal number λ the least cardinal that is strictly greater than λ is denoted by λ^+. It turns out that λ^+ is always a regular cardinal.

Finally, we will denote the cardinality of a set Y by $|Y|$. Now we can prove the following

Proposition 1.4. *Let X be a pseudoradial space, let $A \subseteq X$. Let $\sigma = \sigma_c(X)$. Then $\widehat{A}^{(\sigma^+)} = \overline{A}$.*

Proof. Assume that $\widehat{A}^{(\sigma^+)} \subsetneq \overline{A}$, i.e. $\widehat{A}^{(\sigma^+)}$ is not closed, and then there exist $x \in \overline{A} \setminus \widehat{A}^{(\sigma^+)}$ and a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ in $\widehat{A}^{(\sigma^+)}$ of length $\lambda \leq \sigma$

such that $x_\alpha \to x$ when $\alpha \to \lambda$. Since $\widehat{A}^{(\sigma^+)} = \bigcup_{\gamma<\sigma^+} \widehat{A}^{(\gamma)}$, for each $\alpha < \lambda$ there exists an ordinal $\gamma(\alpha) < \sigma^+$ such that $x_\alpha \in \widehat{A}^{(\gamma(\alpha))}$. Let $E = \{\gamma(\alpha) \mid \alpha < \lambda\}$. Let $\Gamma = \sup(E)$. Since $|E| \leq \lambda \leq \sigma < \sigma^+$ and σ^+ is a regular cardinal, $\Gamma < \sigma^+$. Thus for each $\alpha < \lambda$, $\gamma(\alpha) \leq \Gamma < \sigma^+$ and $x_\alpha \in \widehat{A}^{(\gamma(\alpha))} \subseteq \widehat{A}^{(\Gamma)}$. Therefore $\{x_\alpha\}_{\alpha<\lambda} \subseteq \widehat{A}^{(\Gamma)}$ and so $x \in \widehat{A}^{(\Gamma+1)}$. But $\Gamma + 1 < \sigma^+$ and so $x \in \widehat{A}^{(\sigma^+)}$, a contradiction. $\Diamond$

Corollary 1.5. *Let X be a pseudoradial space. Then there exists an ordinal α such that $\alpha = \mathrm{pro}(X)$. Furthermore* $\mathrm{pro}(X) \leq \sigma_c(X)^+$.

Remark 1.6. We will show later on in Sec. 3 that there are pseudoradial spaces X such that $\mathrm{pro}(X) = (\sigma_c(X))^+$. For example see the space M_{ω_2}, which has pseudoradial order ω_2 and chain character ω_1.

Let us now prove the following proposition, concerning the pseudoradial order of the topological sum of a collection of pseudoradial spaces.

Proposition 1.7. *Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of pseudoradial spaces and for each $\alpha \in I$, let $\sigma_\alpha = \mathrm{pro}(X_\alpha)$. Then the topological sum $\coprod_{\alpha\in I} X_\alpha$ is pseudoradial and*

$$\mathrm{pro}\Big(\coprod_{\alpha\in I} X_\alpha\Big) = \sup_{\alpha\in I} \sigma_\alpha.$$

Proof. We denote by i_α the canonical inclusion $i_\alpha : X_\alpha \longrightarrow \coprod_{\alpha\in I} X_\alpha$, $i_\alpha(x) = (x, \alpha)$. It is well known that the topological sum of a collection of pseudoradial spaces is pseudoradial (see [6]).

Let now $\sigma = \sup_{\alpha\in I} \sigma_\alpha$. In order to prove that the pseudoradial order of $\coprod_{\alpha\in I} X_\alpha$ is σ, it suffices to prove that:

(i) for each $A \subseteq \coprod_{\alpha\in I} X_\alpha$, $\widehat{A}^{(\sigma)} = \overline{A}$;

(ii) for each $\beta < \sigma$, there exists $A \subseteq \coprod_{\alpha\in I} X_\alpha$, such that $\widehat{A}^{(\beta)} \subsetneqq \overline{A}$.

Let $A \subseteq \coprod_{\alpha\in I} X_\alpha$; it is easy to prove that for each $\alpha \in I$, $\overline{i_\alpha^{-1}(A)} = i_\alpha^{-1}(\overline{A})$ and also $\widehat{i_\alpha^{-1}(A)}^{(\tau)} = i_\alpha^{-1}(\widehat{A}^{(\tau)})$, for each ordinal τ. For each $\alpha \in I$, let $A_\alpha = i_\alpha^{-1}(A)$. Since X_α is pseudoradial and $\mathrm{pro}(X_\alpha) = \sigma_\alpha$, there exists an ordinal $\tau_\alpha \leq \sigma_\alpha$ such that $\widehat{A_\alpha}^{(\tau_\alpha)} = \overline{A_\alpha}$. Since $\tau_\alpha \leq \sigma_\alpha \leq \sigma$, we have that $\widehat{A_\alpha}^{(\sigma)} = \overline{A_\alpha}$. Therefore $\widehat{A}^{(\sigma)} = \bigcup_{\alpha\in I} i_\alpha^{-1}(\widehat{A}^{(\sigma)}) \times \{\alpha\} = \bigcup_{\alpha\in I} \widehat{i_\alpha^{-1}(A)}^{(\sigma)} \times \{\alpha\} = \bigcup_{\alpha\in I} \overline{i_\alpha^{-1}(A)} \times \{\alpha\} = \bigcup_{\alpha\in I} i_\alpha^{-1}(\overline{A}) \times \{\alpha\} = \overline{A}$.

Let us now prove that for each $\beta < \sigma$, there exists $A \subseteq \coprod_{\alpha\in I} X_\alpha$ such that $\hat{A}^{(\beta)} \subsetneqq \overline{A}$. Let $\beta < \sigma = \sup_{\alpha\in I} \sigma_\alpha$. There exists $\alpha \in I$ such that $\beta < \sigma_\alpha = \mathrm{pro}(X_\alpha)$. Therefore there exists $A_\alpha \subseteq X_\alpha$ such that $\widehat{A_\alpha}^{(\beta)} \subsetneqq$

$\subsetneqq \overline{A_\alpha}$. Let $A = i_\alpha(A_\alpha)$. Since $\widehat{A_\alpha}^{(\beta)} \subsetneqq \overline{A_\alpha}$, there exists $y \in \overline{A_\alpha} \setminus \widehat{A_\alpha}^{(\beta)}$. Therefore $y \in \overline{A_\alpha} = \overline{i_\alpha^{-1}(A)} = i_\alpha^{-1}(\overline{A})$ and $y \notin \widehat{i_\alpha^{-1}(A)}^{(\beta)} = i_\alpha^{-1}(\widehat{A}^{(\beta)})$. Therefore $i_\alpha(y) \in \overline{A} \setminus \widehat{A}^{(\beta)}$ and so $\widehat{A}^{(\beta)} \subsetneqq \overline{A}$. $\Diamond$

We call a topological space *normal* provided that it is both T_4 and T_1. If X_α is a normal space for each $\alpha \in I$, then $\coprod_{\alpha \in I} X_\alpha$ is a normal space.

Finally we recall that the quotient of a pseudoradial space is pseudoradial (see [6]).

2. The pseudoradial sum and its order of pseudoradiality

In this section we introduce the pseudoradial sum of a family of topological spaces and prove some of its properties. Especially in Th. 2.8, through the result obtained in Lemma 2.7, we prove that under some conditions the pseudoradial order of the pseudoradial sum of a family of pseudoradial spaces is the sup of the pseudoradial orders of these spaces +1. This fact will allow us to use transfinite recursion in order to construct pseudoradial spaces of any order.

Definition 2.1. Let λ be a regular cardinal. Let $\{X_\alpha\}_{\alpha<\lambda}$ be topological spaces; for each $\alpha < \lambda$, let $0_\alpha \in X_\alpha$ be a point; let S be $\lambda + 1$ with the following topology:

(i) for each $\alpha < \lambda$, α is an isolated point;

(ii) the basic neighborhoods of the point λ are the intervals $(\alpha, \lambda]$, with $\alpha < \lambda$.

Let X be the topological sum of the family $\{X_\alpha\}_{\alpha<\lambda}$. Let $A = \{0_\alpha \mid \alpha < < \lambda\} \subseteq X$; let $f : A \longrightarrow S$, $f(0_\alpha) = \alpha$. Notice that f is injective. Therefore we can define

$$\sum_{\alpha<\lambda} (X_\alpha, 0_\alpha) = X \cup_f S$$

($X \cup_f S$ denotes the adjunction space of X to S along f). The space $\sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$ is called the *pseudoradial sum* of the family $\{X_\alpha\}_{\alpha<\lambda}$, with *base points* $0_\alpha \in X_\alpha$.

Let us introduce some notations that we will use throughout this section. Let $f_\alpha : X_\alpha \longrightarrow \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$, $g : S \longrightarrow \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$ be the canonical inclusions. By definition of adjunction space, if $V \subseteq \subseteq \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$, then

$$V \text{ is open} \Longleftrightarrow \begin{cases} f_\alpha^{-1}(V) \subseteq X_\alpha & \text{is open in } X_\alpha \text{ for each } \alpha < \lambda, \\ g^{-1}(V) \subseteq S & \text{is open in } S. \end{cases}$$

Finally, for the sake of convenience, rename the point $g(\lambda)$ as $\Lambda = g(\lambda)$. We call Λ the *end point* of the pseudoradial sum $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$.

Remark 2.2. $f_\alpha : X_\alpha \longrightarrow f_\alpha(X_\alpha)$ is a homeomorphism and furthermore $f_\alpha(X_\alpha)$ is both an open and closed subspace of $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$.

Remark 2.3. The above space S is pseudoradial. So if $\{X_\alpha\}_{\alpha<\lambda}$ are pseudoradial spaces, then $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$, the adjunction space, is the quotient of the topological sum of pseudoradial spaces and so it is pseudoradial.

Let us now prove the following

Proposition 2.4. *Let $\{X_\alpha\}_{\alpha<\lambda}$ be normal spaces, λ a regular cardinal; for each $\alpha < \lambda$, let $0_\alpha \in X_\alpha$; then $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ is normal.*

Proof. We have to prove that $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ is a T_1 space and a T_4 space. It is trivial to show that $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ is a T_1 space. We now prove that $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ is a T_4 space.

Let $E, F \subseteq \sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ be closed and disjoint. For each $\alpha < \lambda$, let $E_\alpha = f_\alpha^{-1}(E)$, $F_\alpha = f_\alpha^{-1}(F)$; E_α, F_α are closed in X_α and disjoint.

If $\Lambda \notin E \cup F$, the proof is trivial. Assume now that $\Lambda \in E \cup F$. We can suppose $\Lambda \in E$ (then $\Lambda \notin F$). So there exists $\overline{\alpha} < \lambda$ such that for each $\alpha \geq \overline{\alpha}$, $f_\alpha(0_\alpha) \notin F$ (if not, $\Lambda \in \overline{F} \setminus F$ and therefore F would not be closed). For each $\alpha < \lambda$, consider

$$E_\alpha^{(1)} = \begin{cases} E_\alpha & \text{if } \alpha < \overline{\alpha}, \\ E_\alpha \cup \{0_\alpha\} & \text{if } \alpha \geq \overline{\alpha}. \end{cases}$$

Since X_α is normal (and therefore it is a T_1 space), $E_\alpha^{(1)}$ is closed in X_α and furthermore $E_\alpha^{(1)} \cap F_\alpha = \emptyset$. Therefore there exist U_α, V_α, open in X_α and disjoint, such that $E_\alpha^{(1)} \subseteq U_\alpha$ and $F_\alpha \subseteq V_\alpha$. Let us now define $U = \bigcup_{\alpha<\lambda} f_\alpha(U_\alpha) \cup \{\Lambda\}$ and $V = \bigcup_{\alpha<\lambda} f_\alpha(V_\alpha)$. U, V are open, disjoint and $E \subseteq U$, $F \subseteq V$. $\Diamond$

Now we want to find the pseudoradial order of $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$. In order to do that, we need some lemmas.

Lemma 2.5. *Let $V \subseteq \sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ be a neighborhood of Λ; then there exists $\overline{\alpha} < \lambda$ and for each $\alpha \in (\overline{\alpha}, \lambda)$, there exists $U_\alpha \subseteq X_\alpha$, U_α a neighborhood of 0_α such that*

$$\Big(\bigcup_{\overline{\alpha}<\alpha<\lambda} f_\alpha(U_\alpha)\Big) \cup \{\Lambda\} \subseteq V.$$

Proof. Since V is a neighborhood of Λ in $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$, $g^{-1}(V)$ is a neighborhood of λ in S. Therefore there exists $\overline{\alpha} < \lambda$ such that for each $\alpha \in (\overline{\alpha}, \lambda]$, $\alpha \in g^{-1}(V)$, i.e. $g(\alpha) \in V$. But, by definitions of g and f_α, for each $\alpha < \lambda$, $g(\alpha) = f_\alpha(0_\alpha)$, so for each $\alpha \in (\overline{\alpha}, \lambda)$, $f_\alpha(0_\alpha) \in V$. Hence $f_\alpha^{-1}(V)$ is a neighborhood of 0_α in X_α. Thus for each $\alpha \in (\overline{\alpha}, \lambda)$, there exists an open set $U_\alpha \subseteq X_\alpha$ such that $0_\alpha \in U_\alpha$ and $U_\alpha \subseteq f_\alpha^{-1}(V)$, and so $f_\alpha(U_\alpha) \subseteq V$. As a result

$$\Big(\bigcup_{\overline{\alpha}<\alpha<\lambda} f_\alpha(U_\alpha)\Big) \cup \{\Lambda\} \subseteq V. \qquad \Diamond$$

Lemma 2.6. *Let $A \subseteq \sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$; assume that $\Lambda \in \overline{A} \setminus A$. Then for each $\alpha < \lambda$, there exists $\alpha' \geq \alpha$, $\alpha' < \lambda$ such that $g(\alpha') \in \overline{A}$.*

Proof. If not, there exists $\overline{\alpha} < \lambda$ such that for each $\alpha \in [\overline{\alpha}, \lambda)$, $g(\alpha) \notin \overline{A}$. Thus for each $\alpha \geq \overline{\alpha}$, $\alpha < \lambda$, there exists V_α, a neighborhood of $g(\alpha)$, such that $V_\alpha \cap A = \emptyset$ and furthermore $g(\lambda) = \Lambda \notin A$. In $V = \{g(\alpha) \,|\, \overline{\alpha} < < \alpha \leq \lambda\} \cup \Big(\bigcup_{\alpha\in(\overline{\alpha},\lambda)} V_\alpha\Big)$, a neighborhood of Λ, there are no points of A, a contradiction. $\Diamond$

The following lemma and theorem are very important for the construction of the spaces M_α.

Lemma 2.7. *Let λ be a regular cardinal. Let $\{X_\alpha\}_{\alpha<\lambda}$ be pseudoradial, T_1 spaces and let $0_\alpha \in X_\alpha$ for each $\alpha < \lambda$. Let $\langle x_\beta\rangle_{\beta<\kappa}$ be a transfinite sequence of length κ in $\sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$ and assume that $x_\beta \to \Lambda$. Then $\kappa \geq \lambda$ and there exists $\overline{\beta} < \kappa$ such that for each $\beta \in [\overline{\beta}, \kappa)$, $x_\beta \in g(S)$ (i.e. the sequence is eventually in $g(S)$).*

Proof. First, assume $\kappa < \lambda$. Let $E = \{\alpha < \lambda \mid f_\alpha^{-1}(\{x_\beta\}_{\beta<\kappa}) \neq \emptyset\}$. Let $\overline{\alpha} = \sup E$. Since $|E| < \lambda$ and λ is a regular cardinal, $\overline{\alpha} < \lambda$. So for each $\alpha > \overline{\alpha}$, the sequence $\langle x_\beta\rangle_{\beta<\kappa}$ never gets in $f_\alpha(X_\alpha)$. But so the set $U = g((\overline{\alpha}, \lambda]) \cup \bigcup_{\alpha\in(\overline{\alpha},\lambda)} f_\alpha(X_\alpha)$ is a neighborhood of Λ, in which the sequence $\langle x_\beta\rangle_{\beta<\kappa}$ never gets in, a contradiction. So κ cannot be less than λ.

Assume now $\kappa > \lambda$. For each $\delta < \lambda$, consider the set $U_\delta \subseteq \subseteq \sum_{\alpha<\lambda}(X_\alpha, 0_\alpha)$, defined by $U_\delta = g((\delta, \lambda]) \cup \bigcup_{\gamma\in(\delta,\lambda)} f_\gamma(X_\gamma)$. For each $\delta < \lambda$, U_δ is a neighborhood of Λ. Therefore for each $\delta < \lambda$, there exists $\underline{\beta}(\delta) < \kappa$ such that for each $\beta \geq \underline{\beta}(\delta)$, $x_\beta \in U_\delta$. Let $\overline{\beta} = \sup_{\delta<\lambda} \underline{\beta}(\delta)$; $\overline{\beta} < \kappa$, because $\lambda < \kappa$ are regular cardinals. Therefore for each $\beta \in (\overline{\beta}, \kappa)$, $x_\beta \in \bigcap_{\delta<\lambda} U_\delta = \{\Lambda\}$ i.e. $x_\beta = \Lambda$ and so the thesis is proved.

Finally, assume $\kappa = \lambda$. Assume that for each $\beta < \kappa$, there exists

$\beta' \geq \beta$ such that $x_{\beta'} \notin g(S)$; then, without loss of generality, we can suppose

$$(\star) \qquad \{x_\beta\}_{\beta<\kappa} \subseteq \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha) \setminus g(S).$$

We now want to find a subsequence of $\langle x_\beta \rangle_{\beta<\kappa}$, which does not converge to Λ. For each $\alpha<\lambda$, consider $F_\alpha = f_\alpha^{-1}\big(\{x_\beta\}_{\beta<\kappa}\big)$, $E_\alpha = \{\beta<\kappa \mid x_\beta \in F_\alpha\}$. For each $\alpha < \lambda$, let

$$\mu_\alpha = \begin{cases} \sup E_\alpha & \text{if } E_\alpha \neq \emptyset, \\ 0 & \text{if } E_\alpha = \emptyset. \end{cases}$$

If $\mu_\alpha = \kappa$ for some $\alpha < \lambda$, then we can find a subsequence of $\langle x_\beta \rangle_{\beta<\kappa}$, which is contained in $f_\alpha(X_\alpha)$, but such a subsequence cannot converge to Λ, a contradiction.

So for each $\alpha < \lambda$, $\mu_\alpha < \kappa$. Let $\mu = \sup_{\alpha<\lambda} \mu_\alpha$. If $\mu < \kappa$, then for each $\beta \in (\mu, \kappa)$, x_β cannot be in any F_α and therefore $x_\beta = \Lambda$ eventually, but this contradicts the assumption $(\star)$.

So $\mu = \sup_{\alpha<\lambda} \mu_\alpha = \kappa = \lambda$. Therefore in $\{\mu_\alpha \mid \alpha < \lambda\}$ there exists a strictly increasing sequence $(\mu_{\alpha_\gamma})_{\gamma<\lambda}$ of length λ (since λ is a regular cardinal), such that $\mu_{\alpha_\gamma} \to \lambda$ when $\gamma \to \lambda$. Now, for each $\gamma < \lambda$, consider $\mu_{\alpha_{\gamma+1}} = \sup(E_{\alpha_{\gamma+1}})$. Since the sequence $(\mu_{\alpha_\gamma})_{\gamma<\lambda}$ is strictly increasing, then for each $\gamma < \lambda$, $\mu_{\alpha_\gamma} < \mu_{\alpha_{\gamma+1}}$ and therefore there exists $\nu_{\alpha_\gamma} \in E_{\alpha_{\gamma+1}}$ such that $\mu_{\alpha_\gamma} < \nu_{\alpha_\gamma} \leq \mu_{\alpha_{\gamma+1}}$. The sequence $(\nu_{\alpha_\gamma})_{\gamma<\lambda}$ converges to λ when $\gamma \to \lambda$ and for each $\gamma < \lambda$, $\nu_{\alpha_\gamma} \in E_{\alpha_{\gamma+1}}$, i.e. $x_{\nu_{\alpha_\gamma}} \in f_{\alpha_{\gamma+1}}(X_{\alpha_{\gamma+1}})$.

Since $X_{\alpha_{\gamma+1}}$ is a T_1 space, there exists $U_{\alpha_{\gamma+1}} \subseteq X_{\alpha_{\gamma+1}}$, a neighborhood of $0_{\alpha_{\gamma+1}}$, such that $x_{\nu_{\alpha_\gamma}} \notin f_{\alpha_{\gamma+1}}(U_{\alpha_{\gamma+1}})$. Let us consider the neighborhood of Λ defined by $U = g(S) \cup \bigcup_{\beta<\lambda} f_\beta(U_\beta)$ where

$$U_\beta = \begin{cases} U_{\alpha_{\gamma+1}} & \text{if } \beta = \alpha_{\gamma+1} \text{ for some } \gamma, \\ X_\beta & \text{otherwise.} \end{cases}$$

Now we want to prove that for each $\alpha < \lambda = \kappa$, there exists $\alpha' > \alpha$ such that $x_{\alpha'} \notin U$ or, in other words, that there exists a subsequence of $\langle x_\beta \rangle_{\beta<\kappa}$ such that no term of this subsequence belongs to U, which is a neighborhood of Λ. For each $\overline{\alpha} < \lambda$, there exists $\gamma < \lambda$ such that $\nu_{\alpha_\gamma} > \overline{\alpha}$; furthermore:

(i) $x_{\nu_{\alpha_\gamma}} \in f_{\alpha_{\gamma+1}}(X_{\alpha_{\gamma+1}})$ and so $x_{\nu_{\alpha_\gamma}} \notin f_\beta(U_\beta)$, for each $\beta \neq \alpha_{\gamma+1}$;

(ii) $x_{\nu_{\alpha_\gamma}} \notin f_{\alpha_{\gamma+1}}(U_{\alpha_{\gamma+1}})$;

(iii) $x_{\nu_{\alpha_\gamma}} \notin g(S)$, because no term of the sequence $\langle x_\beta \rangle_{\beta<\kappa}$ belongs to $g(S)$,

and so $x_{\nu_{\alpha_\gamma}} \notin U$.

Therefore for each $\alpha < \lambda = \kappa$, there exists $\alpha' > \alpha$ (take $\alpha' = \nu_{\alpha_\gamma}$), such that $x_{\alpha'} \notin U$ and so the sequence $\langle x_\beta \rangle_{\beta<\kappa}$ cannot converge to Λ, a contradiction. $\Diamond$

Now we can state and prove the following

Theorem 2.8. *Let $\{X_\alpha\}_{\alpha<\lambda}$ be pseudoradial T_1 spaces, λ a regular cardinal; for each $\alpha < \lambda$, let $\sigma_\alpha = \mathrm{pro}(X_\alpha)$. Let us assume that:*

(i) *for each $\alpha < \lambda$, σ_α is a successor ordinal;*

(ii) *if $\alpha \leq \beta < \lambda$, then $\sigma_\alpha \leq \sigma_\beta$.*

Then for each $\alpha < \lambda$, there exists $0_\alpha \in X_\alpha$ such that $\sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$ is pseudoradial and

$$\mathrm{pro}\left(\sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)\right) = \left(\sup_{\alpha<\lambda} \sigma_\alpha\right) + 1.$$

Proof. Let $\sigma = \sup_{\alpha<\lambda} \sigma_\alpha$. First, for each $\alpha < \lambda$ let us choose $0_\alpha \in X_\alpha$. By hypothesis, we know that for each $\alpha < \lambda$, $\sigma_\alpha = \tau_\alpha + 1$. Therefore there exists $G_\alpha \subseteq X_\alpha$ such that $\widehat{G_\alpha}^{(\tau_\alpha)} \subsetneqq \overline{G_\alpha}$. Thus let us choose $0_\alpha \in \overline{G_\alpha} \setminus \widehat{G_\alpha}^{(\tau_\alpha)}$. $\sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$ is pseudoradial (Rem. 2.3).

In order to prove that $\mathrm{pro}\left(\sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)\right) = (\sup_{\alpha<\lambda} \sigma_\alpha) + 1$, it suffices to prove that:

(i) for each $A \subseteq \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$, $\widehat{A}^{(\sigma+1)} = \overline{A}$;

(ii) there exists $A \subseteq \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$, such that $\widehat{A}^{(\sigma)} \subsetneqq \overline{A}$.

Let us begin by proving the second claim. Let us define $A = \bigcup_{\alpha<\lambda} f_\alpha(G_\alpha)$ and let us prove that $\Lambda = g(\lambda) \in \overline{A} \setminus \widehat{A}^{(\sigma)}$. First let us prove that $\Lambda \in \overline{A}$. Notice that for each $\alpha < \lambda$, $g(\alpha) \in \overline{A}$. In fact $0_\alpha \in \overline{G_\alpha}$ and so

$$g(\alpha) = f_\alpha(0_\alpha) \in f_\alpha(\overline{G_\alpha}) = \overline{f_\alpha(G_\alpha)} \subseteq \overline{\bigcup_{\alpha<\lambda} f_\alpha(G_\alpha)} = \overline{A}.$$

Now let $V \subseteq \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$ be a neighborhood of Λ; therefore, according to Lemma 2.5, there exists $\overline{\alpha} < \lambda$ and for each $\alpha \in (\overline{\alpha}, \lambda)$ there exists $U_\alpha \subseteq X_\alpha$, a neighborhood of 0_α, such that $\left(\bigcup_{\overline{\alpha}<\alpha<\lambda} f_\alpha(U_\alpha)\right) \cup \{\Lambda\} \subseteq V$. Now let $\alpha \in (\overline{\alpha}, \lambda)$; $f_\alpha(U_\alpha)$ is a neighborhood of $f_\alpha(0_\alpha)$, but we have just seen that $f_\alpha(0_\alpha) \in \overline{A}$, so there exists $y \in A \cap f_\alpha(U_\alpha)$. So $y \in A \cap f_\alpha(U_\alpha) \subseteq A \cap V$. Therefore for each neighborhood V of Λ, there exists $y \in A \cap V$. Consequently $\Lambda \in \overline{A}$.

Let us now prove that $\Lambda \notin \widehat{A}^{(\sigma)}$. Assume that $\Lambda \in \widehat{A}^{(\sigma)}$. Let $E = \{\eta < \sigma \mid \Lambda \notin \widehat{A}^{(\eta)}\}$. $E \neq \emptyset$, because $0 \in E$. So let $\tau = \sup E$. If $\tau \notin E$, τ is a limit ordinal, $\Lambda \in \widehat{A}^{(\tau)}$, but $\Lambda \notin \widehat{A}^{(\eta)}$ for each $\eta < \tau$, a contradiction. Thus $\tau = \max E$. Therefore $\Lambda \in \widehat{A}^{(\tau+1)} \setminus \widehat{A}^{(\tau)}$. So there exists a sequence $\langle x_\beta \rangle_{\beta<\kappa}$ in $\widehat{A}^{(\tau)}$ such that $x_\beta \to \Lambda$ and $x_\beta \neq \Lambda$ for each β. But according to Lemma 2.7, such a sequence have to be eventually in $g(S)$, i.e. there exists $\beta_1 < \kappa$ such that for each $\beta \geq \beta_1$, $x_\beta \in g(S)$. So for each $\beta \geq \beta_1$, there exists $\alpha_\beta < \lambda$ such that $x_\beta = g(\alpha_\beta) = f_{\alpha_\beta}(0_{\alpha_\beta}) \in \widehat{A}^{(\tau)}$.

But $\tau < \sigma = \sup_{\alpha<\lambda} \sigma_\alpha$ and by hypothesis $(\sigma_\alpha)_{\alpha<\lambda}$ is an increasing sequence of ordinals. So there exists $\overline{\alpha} < \lambda$ such that for each $\alpha > \overline{\alpha}$, we have that $\tau < \sigma_\alpha$. Furthermore every σ_α is a successor ordinal, and so for each α, $\sigma_\alpha = \tau_\alpha + 1$. Therefore for each $\alpha > \overline{\alpha}$, we have that $\tau \leq \tau_\alpha$ and so $\widehat{A}^{(\tau)} \subseteq \widehat{A}^{(\tau_\alpha)}$. Since $x_\beta \to \Lambda$ when $\beta \to \kappa$, there is $\beta_2 < \kappa$ such that for each $\beta \geq \beta_2$, $x_\beta \in \bigcup_{\alpha\in(\overline{\alpha},\lambda)} f_\alpha(X_\alpha)$.

So for each $\beta \geq \max\{\beta_1, \beta_2\}$,

$$x_\beta = g(\alpha_\beta) \in \widehat{A}^{(\tau)} \subseteq \widehat{A}^{(\tau_{\alpha_\beta})}. \tag{1}$$

But at the same time, for each $\alpha < \lambda$, $0_\alpha \notin \widehat{G_\alpha}^{(\tau_\alpha)}$; therefore $f_\alpha(0_\alpha) \notin$ $\notin f_\alpha(\widehat{G_\alpha}^{(\tau_\alpha)}) = (\widehat{f_\alpha(G_\alpha)})^{(\tau_\alpha)}$. Consequently for each $\beta \geq \max\{\beta_1, \beta_2\}$, $x_\beta = f_{\alpha_\beta}(0_{\alpha_\beta}) \notin (\widehat{f_{\alpha_\beta}(G_{\alpha_\beta})})^{(\tau_{\alpha_\beta})}$ and so

$$x_\beta \notin \widehat{A}^{(\tau_{\alpha_\beta})} \tag{2}$$

$\big($because if $x_\beta \in \widehat{A}^{(\tau_{\alpha_\beta})}$, it would be in $\big(\widehat{f_{\alpha_\beta}(G_{\alpha_\beta})}\big)^{(\tau_{\alpha_\beta})}\big)$. But (1) and (2) contradict each other. So $\Lambda \notin \widehat{A}^{(\sigma)}$. Thus we have just proved that $\widehat{A}^{(\sigma)} \subsetneqq \overline{A}$.

We have still to prove that for each $A \subseteq \sum_{\alpha<\lambda} (X_\alpha, 0_\alpha)$, $\widehat{A}^{(\sigma+1)} =$ $= \overline{A}$; clearly it suffices to prove that $\overline{A} \subseteq \widehat{A}^{(\sigma+1)}$. So let $x \in \overline{A}$.

If there exists $x_\alpha \in X_\alpha$ such that $x = f_\alpha(x_\alpha)$, then $f_\alpha(x_\alpha) \in \overline{A}$, so $x_\alpha \in f_\alpha^{-1}(\overline{A}) = \overline{f_\alpha^{-1}(A)} \subseteq (\widehat{f_\alpha^{-1}(A)})^{(\sigma_\alpha)}$, where the last inclusion is justified by the fact that $\mathrm{pro}(X_\alpha) = \sigma_\alpha$. Therefore $x_\alpha \in (\widehat{f_\alpha^{-1}(A)})^{(\sigma_\alpha)} =$ $= f_\alpha^{-1}(\widehat{A}^{(\sigma_\alpha)})$ and so $x = f_\alpha(x_\alpha) \in \widehat{A}^{(\sigma_\alpha)} \subseteq \widehat{A}^{(\sigma+1)}$.

Let now $x = \Lambda \in \overline{A}$. We can assume that $\Lambda \notin A$; therefore, according to Lemma 2.6, for each $\alpha < \lambda$ there exists $\alpha' \geq \alpha$, $\alpha' < \lambda$, such that $g(\alpha') \in \overline{A}$. So it is possible to find a sequence $(x_{\alpha'(\alpha)})_{\alpha<\lambda}$ in $g(S)\setminus\{\Lambda\}$

such that $x_{\alpha'(\alpha)} \in \overline{A}$ for each $\alpha < \lambda$ and $x_{\alpha'(\alpha)} \to \Lambda$ when $\alpha \to \lambda$. But $x_{\alpha'} = g(\alpha') = f_{\alpha'}(0_{\alpha'})$ and $0_{\alpha'} \in X_{\alpha'}$. So $x_{\alpha'} \in \widehat{A}^{(\sigma_{\alpha'})} \subseteq \widehat{A}^{(\sigma)}$. Thus the sequence $(x_{\alpha'(\alpha)})_{\alpha<\lambda}$ is entirely included in $\widehat{A}^{(\sigma)}$ and consequently $\Lambda = \lim_{\alpha\to\lambda} x_{\alpha'(\alpha)} \in \widehat{A}^{(\sigma+1)}$.

We have proved that $\overline{A} \subseteq \widehat{A}^{(\sigma+1)}$; the thesis is proved. $\Diamond$

3. Construction of the spaces M_α

Theorem 3.1. *For each ordinal α there exists a pseudoradial and normal space M_α, such that* $\mathrm{pro}(M_\alpha) = \alpha$.

Proof. We will use transfinite recursion. Let us define $M_0 = \{0\}$ and $M_1 = \sum_{\alpha<\omega_0} (M_0, 0)$. Clearly M_0, M_1 are pseudoradial, normal spaces; furthermore $\mathrm{pro}(M_0) = 0$ and $\mathrm{pro}(M_1) = 1$.

Let us assume that M_β is defined for each $\beta < \alpha$, in such a way that M_β is pseudoradial, normal and $\mathrm{pro}(M_\beta) = \beta$ and let us define M_α. We distinguish three cases:

(i) $\alpha = \beta + 1$ and β is a successor ordinal; let us consider ω_0 copies of M_β[1]. We have that M_β is normal (and therefore it is T_1), $\mathrm{pro}(M_\beta) = \beta$ and β is a successor ordinal. Thus, according to Th. 2.8, for each $\gamma < \omega_0$, we can find a base point $0_\gamma \in M_\beta$ such that $\sum_{\gamma<\omega_0} (M_\beta, 0_\gamma)$ is pseudoradial and $\mathrm{pro}\left(\sum_{\gamma<\omega_0} (M_\beta, 0_\gamma)\right) = \left(\sup_{\gamma<\omega_0} (\mathrm{pro}(M_\beta))\right) + 1 = \beta + 1 = \alpha$. Furthermore $\sum_{\gamma<\omega_0} (M_\beta, 0_\gamma)$ is the pseudoradial sum of normal spaces and so according to Prop. 2.4 it is normal. So we define

$$M_\alpha = \sum_{\gamma<\omega_0} (M_\beta, 0_\gamma). \quad [2]$$

(ii) $\alpha = \beta + 1$, but β is a limit ordinal, $\beta \neq 0$; so let $\lambda = \mathrm{cf}(\beta)$; we can find an increasing sequence $(\beta_\nu)_{\nu<\lambda}$ of successor ordinals, each of which is less than β, such that the sequence $(\beta_\nu)_{\nu<\lambda}$ converges to β when $\nu \to \lambda$. So let us consider the collection

[1] The choice to take ω_0 copies of M_β, and not, for example, ω_1 copies of M_β, is totally arbitrary. We could construct M_α by taking λ copies of M_β, where λ is any regular cardinal. We choose ω_0 because it will be useful in Sec. 5.

[2] For each $n < \omega_0$, M_n is the space S_n constructed by Arhangel'skiĭ and Franklin in [2] and therefore it is a sequential space. If we want M_n to be a pseudoradial and not sequential space, we have to construct it by taking not less than ω_1 copies of M_{n-1}.

of spaces $\{M_{\beta_\nu}\}_{\nu<\lambda}$, where $\mathrm{pro}(M_{\beta_\nu}) = \beta_\nu$. This collection of spaces satisfies the hypotheses of Th. 2.8 and therefore for each $\nu < \lambda$ there exists $0_\nu \in M_{\beta_\nu}$ such that $\sum_{\nu<\lambda}(M_{\beta_\nu}, 0_\nu)$ is pseudoradial and $\mathrm{pro}\left(\sum_{\nu<\lambda}(M_{\beta_\nu}, 0_\nu)\right) = \left(\sup_{\nu<\lambda}(\mathrm{pro}(M_{\beta_\nu}))\right) + 1 = (\sup_{\nu<\lambda}\beta_\nu)+1 = \beta+1 = \alpha$. Furthermore, according to Prop. 2.4, $\sum_{\nu<\lambda}(M_{\beta_\nu}, 0_\nu)$ is normal. So we define

$$M_\alpha = \sum_{\nu<\lambda}(M_{\beta_\nu}, 0_\nu).$$

(iii) α is a limit ordinal. Let $\lambda = \mathrm{cf}(\alpha)$. We can find an increasing sequence $(\beta_\nu)_{\nu<\lambda}$ of ordinals, each of which is less than α, such that the sequence $(\beta_\nu)_{\nu<\lambda}$ converges to α when $\nu \to \lambda$. So let us consider the collection of spaces $\{M_{\beta_\nu}\}_{\nu<\lambda}$, where $\mathrm{pro}(M_{\beta_\nu}) = \beta_\nu$.

We define[3]

$$M_\alpha = \coprod_{\nu<\lambda} M_{\beta_\nu}.$$

M_α is normal (since it is the topological sum of normal spaces), and, according to Prop. 1.7, M_α is pseudoradial and $\mathrm{pro}(M_\alpha) = \mathrm{pro}(\coprod_{\nu<\lambda} M_{\beta_\nu}) = \sup_{\nu<\lambda}\beta_\nu = \alpha$. $\Diamond$

4. Construction of the spaces K_α

The spaces M_α constructed in Sec. 3 are normal, but they are not compact. In this section we construct for each ordinal $\alpha \geq 1$ a pseudoradial compact space K_α, such that $\mathrm{pro}(K_\alpha) = \alpha$. However these spaces K_α are not normal neither Hausdorff, but only T_1.

Let us recall that the *cofinite topology* on an underlying set X is the coarsest T_1 topology on this set. Closed subspaces in the cofinite topology are finite sets and the whole set X.

Let S be the Sierpiński space, i.e. the space $\{0, 1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space. Following the notation introduced in [7], let $(S^\mu)_1$ be the product of μ copies of S with the topology which is the join of the product topology and the cofinite topology on the set $\{0, 1\}^\alpha$. Clearly basic neighborhoods of $(S^\mu)_1$

[3]It seems easier to define $M_\alpha = \coprod_{\beta<\alpha} M_\beta$, but the construction we present in this section will be useful in Sec. 5.

are of the form $\left(\prod_{\eta<\mu} U_\eta\right) \setminus F$, where U_η is an open subspace of S, $U_\eta = \{0,1\}$ for all but a finite number of indices and F is a finite set.

For each $x \in S^\mu$, let $\mathrm{Z}(x) = \{\eta < \mu \mid x(\eta) = 0\}$. For each $\eta < \mu$, let $\pi_\eta : S^\mu \longrightarrow S$ be the canonical projection onto the η-th coordinate.

Definition 4.1. For each ordinal $\alpha \geq 1$ we define

$$K_\alpha = \begin{cases} S^{\omega_\beta} & \text{if } \alpha = \beta + 1, \\ \{x \in S^{\omega_\alpha} \mid |\mathrm{Z}(x)| < \omega_\alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

(in the case that α is a limit ordinal, the topology is the induced subspace topology of $(S^{\omega_\alpha})_1$).

Theorem 4.2. *For each ordinal $\alpha \geq 1$, K_α is a T_1 compact pseudoradial space and* $\mathrm{pro}(K_\alpha) = \alpha$.

First we prove the following lemmas.

Lemma 4.3. *Let $A \subseteq K_\alpha$, let $x \in \overline{A}$. For each ordinal $\beta \geq 1$, if $|\mathrm{Z}(x)| < \omega_\beta$, then $x \in \widehat{A}^{(\beta)}$.*

Proof. First we prove the lemma for all spaces $K_{\alpha+1} = S^{\omega_\alpha}$. We use transfinite induction.

Let us prove the lemma for $\beta = 1$. Let $x \in \overline{A}$, $|\mathrm{Z}(x)| < \omega_1$, i.e. $|\mathrm{Z}(x)| \leq \omega_0$. If $|\mathrm{Z}(x)| = j < \omega_0$, we can suppose without restriction $\mathrm{Z}(x) = \{1, \ldots, j\}$. Consider the following neighborhood of x: $U = \prod_{\eta<\omega_\alpha} U_\eta$ where

$$U_\eta = \begin{cases} \{0\} & \text{if } \eta \in \{1, \ldots, j\}, \\ \{0,1\} & \text{if } \eta \notin \{1, \ldots, j\}. \end{cases}$$

For each $k < \omega_0$, let $x_k \in (U \cap A) \setminus \{x_1, \ldots, x_{k-1}\}$. It is easy to prove that the sequence $\langle x_k \rangle_{k<\omega_0}$ converges to x.

Assume now $|\mathrm{Z}(x)| = \omega_0$. We can suppose without restriction $\mathrm{Z}(x) = \omega_0$. For each $k < \omega_0$ consider the following neighborhood of x: $U^{(k)} = \prod_{\eta<\omega_\alpha} U_\eta^{(k)}$ where

$$U_\eta^{(k)} = \begin{cases} \{0\} & \text{if } \eta \leq k, \\ \{0,1\} & \text{if } \eta > k. \end{cases}$$

For each $k < \omega_0$ let $x_k \in (U^{(k)} \cap A) \setminus \{x_1, \ldots, x_{k-1}\}$. We now prove that $x_k \to x$. Let $U \setminus F$ be a neighborhood of x, $U = \prod_{\eta<\omega_\alpha} U_\eta$, F a finite set. Let $\overline{\eta} = \max\{\eta \mid U_\eta \neq \{0,1\}\}$. Clearly $\overline{\eta} < \omega_0$. Let $\overline{k} = \max\{k < \omega_0 \mid x_k \in F\}$. For each $k > \max\{\overline{\eta}, \overline{k}\}$, we have that $x_k \in U \setminus F$ and so $x_k \to x$.

Assume that the lemma is proved for some ordinal β and let us prove it for the ordinal $\beta+1$. Let $x \in \overline{A}$, $|\mathrm{Z}(x)| < \omega_{\beta+1}$, i.e. $|\mathrm{Z}(x)| \leq \omega_\beta$.

If $x \in A$, the lemma is proved. Assume $x \in \overline{A} \setminus A$. If $|\mathrm{Z}(x)| < \omega_\beta$, by inductive assumption, $x \in \widehat{A}^{(\beta)} \subseteq \widehat{A}^{(\beta+1)}$. If $|\mathrm{Z}(x)| = \omega_\beta$, we can suppose without restriction $\mathrm{Z}(x) = \omega_\beta$. For each $\gamma < \omega_\beta$, let x_γ such that

$$x_\gamma(\eta) = \begin{cases} 0 & \text{if } \eta < \gamma, \\ 1 & \text{if } \eta \geq \gamma. \end{cases}$$

First we prove that $x_\gamma \in \overline{A}$ for each $\gamma < \omega_\beta$. Let $U \setminus F$ be a neighborhood of x_γ, $U = \prod_{\eta<\omega_\alpha} U_\eta$, F a finite set. Then $(U \setminus F) \cup \{x\}$ is a neighborhood of x. Thus in $(U \setminus F) \cup \{x\}$ there are points of A, but $x \notin A$ and so in $U \setminus F$ there are points of A. Then $x_\gamma \in \overline{A}$.

Furthermore $|\mathrm{Z}(x_\gamma)| < \omega_\beta$, so, by inductive assumption, $x_\gamma \in \widehat{A}^{(\beta)}$.

We now prove that $x_\gamma \to x$ when $\gamma \to \omega_\beta$. Let $U \setminus F$ be a neighborhood of x, $U = \prod_{\eta<\omega_\alpha} U_\eta$, F a finite set. Let $\overline{\eta} = \max\big\{\eta \mid U_\eta \neq \{0,1\}\big\}$. Since $\mathrm{Z}(x) = \omega_\beta$, $\overline{\eta} < \omega_\beta$. Let $\overline{\gamma} = \max\{\gamma \mid x_\gamma \in F\}$. For each $\gamma > \max\{\overline{\eta}, \overline{\gamma}\}$, $x_\gamma \in U \setminus F$. Therefore $x_\gamma \to x$ and so $x \in \widehat{A}^{(\beta+1)}$.

Assume now that the lemma is proved for each $\gamma < \beta$, β a limit ordinal and let us prove it for β. Let $x \in \overline{A}$, $|\mathrm{Z}(x)| < \omega_\beta$. Since β is a limit ordinal, there exists $\gamma < \beta$ such that $|\mathrm{Z}(x)| = \omega_\gamma$. So $x \in \widehat{A}^{(\gamma)} \subseteq \widehat{A}^{(\beta)}$.

In a similar way the lemma can be proved for all spaces K_α with α a limit ordinal. $\Diamond$

Lemma 4.4. *Let $A = \{x \in K_\alpha \mid |\mathrm{Z}(x)| < \omega_0\}$, let $x \in \overline{A}$. For each ordinal β, if $x \in \widehat{A}^{(\beta)}$, then $|\mathrm{Z}(x)| < \omega_\beta$.*

Proof. We prove the lemma by transfinite induction. For $\beta = 0$ the proof is trivial.

Assume that the lemma is proved for some ordinal β and let us prove it for the ordinal $\beta + 1$. Let $x \in \widehat{A}^{(\beta+1)}$. Therefore there exists a sequence $\langle x_\gamma \rangle_{\gamma<\lambda}$ in $\widehat{A}^{(\beta)}$ converging to x. By inductive assumption for each $\gamma < \lambda$, $|\mathrm{Z}(x_\gamma)| < \omega_\beta$. We have to prove that $|\mathrm{Z}(x)| < \omega_{\beta+1}$. Assume $|\mathrm{Z}(x)| \geq \omega_{\beta+1}$.

If $\lambda \leq \omega_\beta$, we have $\big|\bigcup_{\gamma<\lambda} \mathrm{Z}(x_\gamma)\big| \leq \omega_\beta$. Therefore there exists $\eta \in \mathrm{Z}(x) \setminus \cup_{\gamma<\lambda}\mathrm{Z}(x_\gamma)$. Thus $x(\eta) = 0$, but for each $\gamma < \lambda$, $x_\gamma(\eta) = 1$ and so the sequence $\langle x_\gamma \rangle_{\gamma<\lambda}$ cannot converge to x.

If $\lambda > \omega_\beta$, i.e. $\lambda \geq \omega_{\beta+1}$, let E be a subset of $\mathrm{Z}(x)$ of cardinality ω_β. For each $\eta \in E$, let $U_\eta = \pi_\eta^{-1}(0)$ be a neighborhood of x (π_η is the projection onto the η-th coordinate). So for each $\eta \in E$, there exists $\gamma(\eta) < \lambda$, such that for each $\gamma \geq \gamma(\eta)$, $x_\gamma(\eta) = 0$. Let $\overline{\gamma} = \sup\{\gamma(\eta) \mid \eta \in E\}$. Since $|E| = \omega_\beta$, λ is a regular cardinal and $\omega_\beta < \lambda$, we have that $\overline{\gamma} < \lambda$.

So for each $\gamma \in (\overline{\gamma}, \lambda)$ and for each $\eta \in E$, $x_\gamma(\eta) = 0$, i.e. $|\mathrm{Z}(x_\gamma)| \geq \omega_\beta$, a contradiction.

Assume now that the lemma is proved for each ordinal $\gamma < \beta$, with β a limit ordinal and let us prove it for β. Let $x \in \widehat{A}^{(\beta)}$. Therefore there exists an ordinal $\gamma < \beta$, such that $x \in \widehat{A}^{(\gamma)}$. By inductive assumption, $|\mathrm{Z}(x)| < \omega_\gamma < \omega_\beta$. $\Diamond$

Proof of Theorem 4.2. Let $\alpha \geq 1$ be an ordinal. Clearly K_α is a T_1 space.

We now prove the compactness. Let $\{V_i\}_{i \in I}$ be an open cover of K_α. Let $x \in K_\alpha$ such that $x(\eta) = 1$ for each $\eta < \omega_\alpha$. Let $i \in I$ such that $x \in V_i$. It is easy to see that V_i has the form $K_\alpha \setminus F$, F a finite set. Thus $\{V_i\}_{i \in I}$ has a finite subcover and so K_α is a compact space.

In order to prove that K_α is a pseudoradial space and that $\mathrm{pro}(K_\alpha) = \alpha$ it suffices to prove that:

(i) for each $A \subseteq K_\alpha$, $\widehat{A}^{(\alpha)} = \overline{A}$;

(ii) there exists $A \subseteq K_\alpha$ such that for each $\beta < \alpha$, $\widehat{A}^{(\beta)} \subsetneq \overline{A}$.

Let us prove the first claim. Let $A \subseteq K_\alpha$. Let $x \in \overline{A}$. According to the definition of K_α we have that $|\mathrm{Z}(x)| < \omega_\alpha$. Therefore, by Lemma 4.3, $x \in \widehat{A}^{(\alpha)}$ and so $\widehat{A}^{(\alpha)} = \overline{A}$.

Let us now prove the second claim. Let $A = \{x \in K_\alpha \mid |\mathrm{Z}(x)| < \omega_0\}$. Let $\beta < \alpha$. Let $x \in K_\alpha$ such that $x(\eta) = 0$ for each $\eta < \omega_\beta$. We have that $x \in \overline{A}$, but by Lemma 4.4, since $|\mathrm{Z}(x)| = \omega_\beta$, $x \notin \widehat{A}^{(\beta)}$. $\Diamond$

5. Construction of the spaces F_α

In [2] Arhangel'skiĭ and Franklin constructed a sequential space S_ω having sequential order ω_1 and cardinality ω_0. Let us observe that the spaces $M_{\omega_{\alpha+1}}$ constructed in Sec. 3 have pseudoradial order $\omega_{\alpha+1}$ and cardinality $\omega_{\alpha+1}$. In this section we construct for each ordinal α a pseudoradial space F_α such that $\mathrm{pro}(F_\alpha) = \omega_{\alpha+1}$ but $|F_\alpha| = \omega_\alpha$.

Let ω_α be a cardinal. For each regular cardinal $\lambda \leq \omega_\alpha$, let $S^{(\lambda)}$ be $\lambda + 1$ with the same topology of S as in Def. 2.1. Consider the topological sum of all the spaces $S^{(\lambda)}$ for each regular cardinal $\lambda \leq \omega_\alpha$ with all the non-isolated points identified. The fan space obtained in such a way is called $S(\omega_\alpha)$. For each regular cardinal $\lambda \leq \omega_\alpha$, let $g^{(\lambda)} : S^{(\lambda)} \longrightarrow S(\omega_\alpha)$ be the canonical inclusion and $\xi = g^{(\lambda)}(\lambda)$. Clearly $S(\omega_\alpha)$ is pseudoradial.

By recursion, following [2], we now construct for each $n < \omega_0$ a pseudoradial space G_n of cardinality not greater than ω_α and at the same time we define the level $\mathcal{L}_n(x)$ for all points $x \in G_n$. Let $G_0 = \{\Omega\}$ be an one-point space and let $\mathcal{L}_0(\Omega) = 0$. Assume now already defined the space G_n and the level $\mathcal{L}_n(x)$ for all points $x \in G_n$. For each point x of level n in G_n, take a copy $S_x(\omega_\alpha)$ of $S(\omega_\alpha)$ and let X be the topological sum of the spaces $S_x(\omega_\alpha)$. Let $A = \{\xi_x \in S_x(\omega_\alpha) \mid x \in G_n \text{ such that } \mathcal{L}_n(x) = = n\} \subseteq X$, and define $f : A \longrightarrow G_n$ by $f(\xi_x) = x$. Define the space G_{n+1} as the adjunction space $X \cup_f G_n$. Now choose $x \in G_{n+1}$. If $x \in G_n$, let $\mathcal{L}_{n+1}(x) = \mathcal{L}_n(x)$. If not, let $\mathcal{L}_{n+1}(x) = n + 1$. It is easy to see that G_n is pseudoradial and $|G_n| \leq \omega_\alpha$ for each $n < \omega_0$.

Let us observe that there is a natural embedding $\phi_n : G_n \longrightarrow G_{n+1}$. We define by recursion a partial order $\leq_n$ on G_n. Suppose that we have defined a partial order $\leq_n$ on G_n, with Ω as maximal element. Then let $\leq_{n+1}$ be the partial order on G_{n+1} generated by $\leq_n \cup \{\langle y, x\rangle \mid y \in \in S_x(\omega_\alpha)\}$.

Now, using the maps ϕ_n, we define for each pair $m < n < \omega_0$ a map $\phi^n_m : G_m \longrightarrow G_n$ by $\phi^n_m = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_m$ and obtain a direct system $\langle G_n, \phi^n_m\rangle$ of spaces and maps. Denote by F_α the direct limit of this system[4]. We denote by $\psi_n : G_n \longrightarrow F_\alpha$ the canonical embedding.

Lemma 5.1. *Let $\leq$ be the partial order on F_α defined by the rule that $x \leq y$ if and only if there exists a triple n, a, b such that $a \in \psi_n^{-1}(x)$, $b \in \psi_n^{-1}(y)$ and $a \leq_n b$. Then for each $x \in F_\alpha$, $I(x) = \{y \in F_\alpha \mid y \leq x\}$ is homeomorphic to F_α.*

Proof. Noting that $\mathcal{L}_n(x) = k$ implies that $\mathcal{L}_{n+1}(\phi_n(x)) = k$, we may unambiguously define the level $\mathcal{L}(x)$ of a point $x \in F_\alpha$ by choosing some n and $a \in \psi_n^{-1}(x)$ and setting $\mathcal{L}(x) = \mathcal{L}_n(a)$. It is easy to verify that $x \leq y$ implies $\mathcal{L}(x) \geq \mathcal{L}(y)$.

For each $x \in F_\alpha$, let $I(x) = \{y \in F_\alpha \mid y \leq x\}$. We will show by an induction on the level of x that each $I(x)$ is homeomorphic to F_α. For $\mathcal{L}(x) = 0$ the assertion is trivial. Suppose $\mathcal{L}(x) = 1$ and let $T_n = \psi_n^{-1}(I(x))$ for each $n < \omega_0$. Then $T_0 = \emptyset$ and for each $n > 0$, T_n is homeomorphic to G_{n-1}. But clearly $I(x)$ is the inductive limit of the system $\langle T_n, \phi^n_m \restriction T_m\rangle$ and hence it is homeomorphic to F_α.

Now suppose our assertion is true for all points at level $n-1$ and that $\mathcal{L}(x) = n$. Then there exists exactly one $y \in F_\alpha$ with $\mathcal{L}(y) = n - 1$

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[4]We remind the reader we began our construction by choosing a cardinal ω_α in order to construct a space F_α such that $|F_\alpha| = \omega_\alpha$, but $\mathrm{pro}(F_\alpha) = \omega_{\alpha+1}$. We also observe that F_0 is the space S_ω constructed by Arhangel'skiĭ and Franklin in [2].

and $x < y$. The point x is at level 1 with respect to $I(y)$ which is homeomorphic to F_α by inductive assumption, and hence $I(x)$ is homeomorphic to F_α by the level-one argument. $\Diamond$

Theorem 5.2. *F_α is a pseudoradial space with* $\mathrm{pro}(F_\alpha) = \omega_{\alpha+1}$*, but $|F_\alpha| = \omega_\alpha$, and it contains a copy of M_β for each $\beta < \omega_{\alpha+1}$.*

Proof. It is easy to see that $|F_\alpha| = \omega_\alpha$. Since F_α is the direct limit of a collection of pseudoradial spaces, F_α is pseudoradial. It is also easy to see that $\sigma_c(F_\alpha) = \omega_\alpha$. Therefore, by Cor. 1.5, $\mathrm{pro}(F_\alpha) \leq \omega_{\alpha+1}$. The opposite inequality will follow from the fact that M_β is a closed subspace of F_α for each $\beta < \omega_{\alpha+1}$.

For each regular cardinal $\lambda \leq \omega_\alpha$, let $\Sigma^{(\lambda)}$ be the pseudoradial sum of λ copies of F_α choosing the level-zero point of each F_α as base point. Note we can embed each $\Sigma^{(\lambda)}$ in F_α by observing that $\Sigma^{(\lambda)}$ is homeomorphic to $\{\Omega\} \cup \bigcup_{x \in S^{(\lambda)} \setminus \{0\}} I(x)$.

We will now recursively prove that for each $\beta < \omega_{\alpha+1}$, M_β is a closed subspace of F_α and the end point of M_β is the level-zero point of F_α whenever β is not a limit ordinal.

Clearly M_0 is a subspace of F_α. It is also easy to see that M_1 is homemorphic to the sequence $S^{(\omega_0)}$ which is a subspace of $S(\omega_\alpha)$ which is a subspace of F_α. So M_1 is homeomorphic to a subspace of F_α. Furthermore we can suppose that the end point of M_1 is the level-zero point of F_α.

Now let $\beta < \omega_{\alpha+1}$. Assume that $\beta = \gamma + 1$ and that γ is also a successor ordinal. By inductive assumption, we know that M_γ is a subspace of F_α and the end point of M_γ is the level-zero point of F_α. We know that M_β is the pseudoradial sum of ω_0 copies of M_γ choosing the end point of each M_γ as base point (see Sec. 3). So M_β is embedded in the pseudoradial sum of ω_0 copies of F_α choosing the level-zero point of each F_α as base point. Therefore M_β is embedded in $\Sigma^{(\omega_0)}$ which is a subspace of F_α.

Assume now $\beta = \gamma + 1$, but γ is a limit ordinal, $\gamma \neq 0$. Let $\lambda = \mathrm{cf}(\gamma) \leq \omega_\alpha$ and let $\langle \gamma_\nu \rangle_{\nu < \lambda}$ be an increasing sequence of successor ordinals which converges to γ. By inductive assumption, we know that for each $\nu < \lambda$, M_{γ_ν} is a subspace of F_α and the end point of M_{γ_ν} is the level-zero point of F_α. We know that M_β is the pseudoradial sum of the family $\{M_{\gamma_\nu}\}_{\nu < \lambda}$ choosing the end point of each M_{γ_ν} as base point. So M_β is embedded in the pseudoradial sum of λ copies of F_α choosing the level-zero point of each F_α as base point. Therefore M_β is embedded in

$\Sigma^{(\lambda)}$ which is a subspace of F_α.

Assume now β is a limit ordinal. Let $\lambda = \mathrm{cf}(\beta) \leq \omega_\alpha$ and let $\langle \beta_\nu \rangle_{\nu<\lambda}$ be an increasing sequence of ordinals which converges to β. By inductive assumption, we know that for each $\nu < \lambda$, M_{β_ν} is a subspace of F_α. We know that M_β is the topological sum of the family $\{M_{\beta_\nu}\}_{\nu<\lambda}$. So M_β is embedded in the topological sum of λ copies of F_α. Therefore M_β is embedded in $\Sigma^{(\lambda)} \setminus \{\Omega\}$ which is a subspace of F_α.

So for each $\beta < \omega_{\alpha+1}$, M_β is a subspace of F_α. It is also easy to see that for each $\beta < \omega_{\alpha+1}$, M_β is a closed subspace of F_α. Therefore $\mathrm{pro}(F_\alpha) \geq \omega_{\alpha+1}$. We have already proved the opposite inequality and so $\mathrm{pro}(F_\alpha) = \omega_{\alpha+1}$. $\Diamond$

References

[1] ARHANGEL'SKIĬ, A. V.: The structure and classification of topological spaces and cardinal invariants, *Uspekhi Mat. Nauk* **33** (1978), 29–84.

[2] ARHANGEL'SKIĬ, A. V. and FRANKLIN, S. P.: Ordinal invariants for topological spaces, *Michigan Math. J.* **15** (3) (1968), 313–320.

[3] ARHANGEL'SKIĬ, A. V., ISLER, R. and TIRONI, G.: On pseudo-radial spaces, *Comment. Math. Univ. Carolinae* **27** (1986), 137–156.

[4] BELLA, A. and TIRONI, G.: d-4 Pseudoradial spaces, in: *Encyclopedia of general topology*, Elsevier, 2003, 165–168.

[5] DOLCHER, M.: Topologie e strutture di convergenza *Ann. Scuola Norm. Sup. Pisa* **14** (1960), 63–92.

[6] HERRLICH, H.: Quotienten geordneter Räume und Folgenkonvergenz, *Fund. Math.* **61** (1967), 79–81.

[7] SLEZIAK, M.: Subspaces of pseudoradial spaces, *Math. Slovaca* **53** (5) (2003), 505–514.

[8] VAIDYANATHASWAMY, R.: *Set topology*, Chelsea Publishing Company, New York, second edition, 1960.