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ON SPECIAL WEAKLY RICCI SYMMETRIC, RICCI BI-SYMMETRIC AND R-HARMONIC QUASI-EINSTEIN MANIFOLDS

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Abstract: In this paper, we have studied some geometric properties of special weakly Ricci symmetric quasi-Einstein manifold, special weakly Ricci bi-symmetric quasi-Einstein manifold and R-harmonic quasi-Einstein manifold.

1. Introduction

A non-flat Riemannian manifold (M^n, g) $(n \geqslant 3)$ is called quasi-Einstein manifold [5] if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{1.1}$$

where a, b are scalars of which $b \neq 0$ and η is a nonzero 1-form such that

$$g(X,\xi) = \eta(X) \quad \forall X, \tag{1.2}$$

and ξ is a unit vector field. In such a case a, b are called the associated scalars, η is called the associated 1-form and ξ is called the generator of

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the manifold. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$.

In [1], [6], [7] and [4], the authors studied quasi-Einstein manifolds and gave some examples of quasi-Einstein manifold. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance the Robertson–Walker spacetime are quasi-Einstein manifolds [9].

As a generalization of Chaki's pseudosymmetric and pseudo Ricci symmetric manifolds (see [2] and [3]), the notion of weakly symmetric and weakly Ricci symmetric manifolds were introduced by L. Tamássy and T. Q. Binh (see [15] and [16]). These type of manifolds were studied with different structures by many authors (see [8], [11] and [12]). The notion of special weekly Ricci symmetric manifold was introduced and studied by Singh and Khan in [13].

An n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ is called a special weakly Ricci-symmetric manifold $(SWRS)_n$ (see [13]) if the Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X), \tag{1.3}$$

for any vector fields X, Y, Z on M^n, where α is 1-form and is defined by

$$\alpha(X) = g(X, \rho), \tag{1.4}$$

where ρ is a associated vector field and ∇ is the Levi-Civita connection of M^n.

Also the notion of special weakly Ricci bi-symmetric manifold was introduced by Singh and Sinha [14]. An n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ is said to be a special weakly Ricci bi-symmetric manifold $(SWRBS)_n$ (see [14]) if it satisfies the condition

$$(\nabla_W \nabla_X S)(Y,Z) = 2\beta(W,X)S(Y,Z) + \beta(W,Y)S(X,Z) + \beta(W,Z)S(Y,X), \tag{1.5}$$

where β is a 2-form and is defined as

$$\beta(W,X) = g((W,X),T), \tag{1.6}$$

where T is a vector field.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be a R-harmonic manifold (see [10]) if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = (\nabla_Z S)(X, Y), \tag{1.7}$$

for all vector fields X, Y, Z on M^n.

Motivated by the above studies, in this study we consider special weakly Ricci symmetric quasi-Einstein manifold, special weakly Ricci bi-symmetric quasi-Einstein manifold and R-harmonic quasi-Einstein manifold. The paper is organized as follows: First, it is shown that if a special weakly Ricci symmetric quasi-Einstein manifold admits a cyclic parallel Ricci tensor with $a + b \neq 0$ then the 1-form α must vanish. Next, it is proved that if in such a manifold the generator is a parallel vector field then the scalar function a of such a manifold is constant along the generator. Also, the condition under which the generator of such a manifold is parallel is enquired. Moreover a special weakly Ricci bi-symmetric quasi-Einstein manifold has been studied. Further some interesting properties regarding R-harmonic quasi-Einstein manifold are obtained.

2. Preliminaries

We consider a $(QE)_n$ with associated scalars a, b, associated 1-form η and generator ξ. Since ξ is a unit vector field,

$$g(\xi, \xi) = 1 \quad \text{i.e.} \quad \eta(\xi) = 1. \tag{2.1}$$

Contracting (1.1) over X and Y we get

$$r = na + b, \tag{2.2}$$

where r denotes the scalar curvature of the manifold. Putting $Y = \xi$ in (1.1) we have

$$S(X, \xi) = (a + b)\eta(X). \tag{2.3}$$

Putting $X = Y = \xi$ in (1.1), we have

$$S(\xi, \xi) = (a + b). \tag{2.4}$$

Let L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$g(LX, Y) = S(X, Y) \quad \forall\, X,\, Y. \tag{2.5}$$

Also,

$$L\xi = (a+b)\xi. \tag{2.6}$$

These results will be used in the sequel.

3. On special weakly Ricci symmetric quasi-Einstein manifolds

In this section we consider a special weakly Ricci symmetric quasi-Einstein manifold M^n, i.e. equations (1.1), (1.2), (1.3) and (1.4) are satisfied in M^n. Taking cyclic sum in (1.3), we get

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(Y,X) \\ = 4\big[\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(X,Y)\big].\end{aligned} \tag{3.1}$$

Let M^n admit a cyclic parallel Ricci tensor. Then (3.1) reduces to

$$\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(X,Y) = 0. \tag{3.2}$$

Taking $Z = \xi$ in (3.2) and using (2.3), (1.4) and (1.2) we have

$$(a+b)\alpha(X)\eta(Y) + (a+b)\alpha(Y)\eta(X) + \eta(\rho)S(Y,X) = 0. \tag{3.3}$$

Now putting $Y = \xi$ in (3.3) and using (2.1), (2.3) and (1.4) we get

$$(a+b)\alpha(X) + (a+b)\eta(\rho)\eta(X) + (a+b)\eta(\rho)\eta(X) = 0. \tag{3.4}$$

Taking $X = \xi$ in (3.4) and using (2.1) and (1.4) we obtain

$$(a+b)\eta(\rho) = 0, \tag{3.5}$$

which implies $\eta(\rho) = 0$ provided $(a+b) \neq 0$. Using $\eta(\rho) = 0$ in (3.4) we have

$$\alpha(X) = 0, \quad \text{since} \quad (a+b) \neq 0 \tag{3.6}$$

for any vector fields X on M^n. Hence we can state the following theorem:

Theorem 1. *If a special weakly Ricci symmetric quasi-Einstein manifold admits a cyclic parallel Ricci tensor with $a + b \neq 0$ then the 1-form α must vanish.*

Next we suppose that the vector field ξ is parallel in M^n. Then $\nabla_X\xi = 0$, which implies $R(X,Y)\xi = 0$. Hence contracting this equation with respect to Y we obtain $S(X,\xi) = 0$. So from (2.3) we have $a+b = 0$, which implies $a = -b$.

Then equation (1.1) becomes

$$S(X,Y) = a[g(X,Y) - \eta(X)\eta(Y)], \tag{3.7}$$

which implies that

$$(\nabla_X S)(Y,Z) = X[a][g(Y,Z) - \eta(Y)\eta(Z)] - a[(\nabla_X\eta)(Y)\eta(Z) + \eta(Y)(\nabla_X\eta)(Z)], \tag{3.8}$$

where $X[a]$ denotes the derivative of a with respect to the vector field X. Since ξ is a parallel vector field, $(\nabla_X\eta)(\mathrm{Y}) = 0 \ \ \forall\ X,Y,Z$. Therefore equation (3.8) becomes

$$(\nabla_X S)(Y,Z) = X[a][g(Y,Z) - \eta(Y)\eta(Z)]. \tag{3.9}$$

Since M^n is special weakly Ricci symmetric, by the use of (1.3) and (3.9), we can write

$$X[a][g(Y,Z)-\eta(Y)\eta(Z)] = 2\alpha(X)S(Y,Z)+\alpha(Y)S(X,Z)+\alpha(Z)S(Y,X). \tag{3.10}$$

Putting $X = \xi$ in (3.10) and using (3.7), we have

$$\xi[a] = 2a\alpha(\xi). \tag{3.11}$$

Taking $Z = \xi$ in (3.10) and using (3.7), we get

$$\alpha(\xi) = 0. \tag{3.12}$$

So, in view of (3.11) and (3.12) we have $\xi[a] = 0$, which implies a is constant along the vector field ξ. Hence we can state the following theorem:

Theorem 2. *Let M^n be a special weakly Ricci symmetric quasi-Einstein manifold under the condition that ξ is a parallel vector field. Then the scalar function a is constant along the vector field ξ.*

Now we assume that the associated scalars a and b are constants in M^n. Then for a $(QE)_n$, we have from (1.1),

$$(\nabla_X S)(Y,Z) = b[(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y)]. \tag{3.13}$$

From (1.3) and (3.13) we have,

$$2\alpha(X)S(Y,Z)+\alpha(Y)S(X,Z)+\alpha(Z)S(Y,X) = b[(\nabla_X\eta)(Y)\eta(Z)+(\nabla_X\eta)(Z)\eta(Y)]. \tag{3.14}$$

Putting $Z=\xi$ in (3.14) and using (2.3) we get,

$$2(a+b)\alpha(X)\eta(Y)+(a+b)\alpha(Y)\eta(X)+\alpha(\xi)S(Y,X)=b(\nabla_X\eta)(Y). \tag{3.15}$$

Taking $Y=\xi$ in (3.15) and using (2.3) we obtain,

$$(a+b)\alpha(X)+(a+b)\alpha(\xi)\eta(X)=0. \tag{3.16}$$

Putting $X=\xi$ in (3.15) we have

$$2(a+b)\alpha(\xi)\eta(Y)+(a+b)\alpha(Y)+(a+b)\alpha(\xi)\eta(Y)=b(\nabla_\xi\eta)(Y). \tag{3.17}$$

Replacing Y with X in (3.17), we have

$$3(a+b)\alpha(\xi)\eta(X)+(a+b)\alpha(X)=b(\nabla_\xi\eta)(X). \tag{3.18}$$

Adding (3.16) and (3.18) we obtain,

$$2(a+b)\alpha(X)+4(a+b)\alpha(\xi)\eta(X)=b(\nabla_\xi\eta)(X). \tag{3.19}$$

Again, taking $X=\xi$ in (3.14) we have

$$2\alpha(\xi)S(Y,Z)+(a+b)\alpha(Y)\eta(Z)+(a+b)\alpha(Z)\eta(Y) = b\big[(\nabla_\xi\eta)(Y)\eta(Z)+(\nabla_\xi\eta)(Z)\eta(Y)\big]. \tag{3.20}$$

Now putting $Y=Z=\xi$ in (3.20), we get

$$(a+b)\alpha(\xi)=0,$$

which implies

$$\alpha(\xi)=0, \quad \text{provided} \quad (a+b)\neq 0. \tag{3.21}$$

Again $Y=\xi$ in (3.20) implies,

$$3(a+b)\alpha(\xi)\eta(Z)+(a+b)\alpha(Z)=b(\nabla_\xi\eta)(Z). \tag{3.22}$$

Replacing Z by X in (3.22) implies

$$3(a+b)\alpha(\xi)\eta(X)+(a+b)\alpha(X)=b(\nabla_\xi\eta)(X). \tag{3.23}$$

Now, from (3.19) and (3.23), we have

$$(a+b)\alpha(X) + (a+b)\alpha(\xi)\eta(X) = 0. \tag{3.24}$$

So, in view of (3.21) we get from (3.24)

$$\alpha(X) = 0, \quad \forall X \quad [\because (a+b) \neq 0]. \tag{3.25}$$

Now, by virtue of (3.21) and (3.25) we obtain from (3.15)

$$(\nabla_X \eta)(Y) = 0, \quad [\because b \neq 0]. \tag{3.26}$$

We can write (3.26) as follows:

$$g(\nabla_X \xi, Y) = 0, \quad \forall X,\, Y. \tag{3.27}$$

From (3.27) it follows that

$$\nabla_X \xi = 0,$$

which implies that the vector field ξ is parallel. Hence we can state the following theorem:

Theorem 3. *Let M^n be a special weakly Ricci symmetric quasi-Einstein manifold with constants associated scalars a, b and $(a+b) \neq 0$. Then the generator of such a manifold is parallel.*

4. On special weakly Ricci bi-symmetric quasi-Einstein manifolds

Let us consider a special weakly Ricci bi-symmetic quasi-Einstein manifold M^n $(n > 3)$, which is conformally flat. It is known [17] (p. 40) that for a conformally flat (M^n, g), the Riemann curvature tensor has the following form:

$$\begin{aligned} {}'R(X,Y,Z,W) = \frac{1}{n-2}\big[&S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\big] \\ &+ \frac{r}{(n-1)(n-2)}\big[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big], \end{aligned} \tag{4.1}$$

where $'R(X,Y,Z,W) = g(R(X,Y,Z),W)$. Taking bi covariant derivative of (4.1) with respect to X and W, respectively, we get

$$(\nabla_W\nabla_X R)(Y,Z,V) = \frac{1}{n-2}\left[(\nabla_W\nabla_X S)(Z,V)Y - (\nabla_W\nabla_X S)(Y,V)Z\right]. \tag{4.2}$$

Permuting twice the vectors X, Y, Z in equation (4.2) and using Bianchi's second identity, we get

$$\begin{aligned}(\nabla_W\nabla_X S)(Z,V)Y &- (\nabla_W\nabla_X S)(Y,V)Z\\ + (\nabla_W\nabla_Y S)(X,V)Z &- (\nabla_W\nabla_Y S)(Z,V)X\\ + (\nabla_W\nabla_Z S)(Y,V)X &- (\nabla_W\nabla_Z S)(X,V)Y = 0.\end{aligned} \tag{4.3}$$

Using (1.5) in (4.3) and also using the symmetric properties of Ricci tensor, we have

$$\begin{aligned}&\beta(W,X)S(Z,V)Y - \beta(W,X)S(Y,V)Z + \beta(W,Y)S(X,V)Z\\ &-\beta(W,Y)S(Z,V)X + \beta(W,Z)S(Y,V)X - \beta(W,Z)S(X,V)Y = 0.\end{aligned} \tag{4.4}$$

Contracting (4.4) with respect to X, we have

$$\beta(W,Z)S(Y,V) - \beta(W,Y)S(Z,V) = 0. \tag{4.5}$$

By factoring off V in (4.5), we get

$$\beta(W,Z)L(Y) - \beta(W,Y)L(Z) = 0. \tag{4.6}$$

Contracting (4.6) with respect to Y, we have

$$\beta(W,Z)r - \beta(W,L(Z)) = 0. \tag{4.7}$$

Putting $Z = \xi$ in (4.7), we get

$$\begin{aligned}&\beta(W,\xi)r = (a+b)\beta(W,\xi), \quad \text{[using (2.6)]}\\ \text{or,}\quad &\{r-(a+b)\}\beta(W,\xi) = 0,\\ \text{or,}\quad &\beta(W,\xi) = 0, \quad [\because\ r \neq a+b],\end{aligned} \tag{4.8}$$

i.e. the 2-form β is zero for all W and the vector field ξ.

Theorem 4. *In a special weakly Ricci bi-symmetric quasi-Einstein manifolds M^n, the 2-form β is zero for all vector fields X and the vector field ξ, i.e. $\beta(X,\xi) = 0, \forall X$.*

5. On R-harmonic quasi-Einstein manifolds

Next we assume that M^n is an R-harmonic quasi-Einstein manifold. If ξ is a parallel vector field then from (1.7) and (3.9) we have

$$(5.1)\quad (\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = X[a][g(Y,Z) - \eta(Y)\eta(Z)] - Z[a][g(X,Y) - \eta(X)\eta(Y)] = 0.$$

Now taking X $= \xi$ in (5.1) we get

$$\xi[a] = 0,$$

which implies that a is constant along the vector field ξ. This leads to the following theorem:

Theorem 5. *Let M^n be an R-harmonic quasi-Einstein manifold under the condition that ξ is a parallel vector field. Then the scalar function a is constant along the vector field ξ.*

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