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THE MARTINGALE HARDY TYPE INEQUALITY FOR THE MAXIMAL OPERATOR OF THE (C,α) MEANS OF CUBIC PARTIAL SUMS OF THE d-DIMENSIONAL CONJUGATE WALSH–FOURIER SERIES

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Abstract: The main aim of this paper is to prove that for any $0 < p \leq d/ / (d+\alpha)$ there exists a martingale $f \in H_p$ such that the maximal operators of (C,α) means of cubic partial sums of d-dimensional conjugate Walsh–Fourier series do not belong to the space L_p.

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1. Introduction

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [4] has proved that for $f \in L\log L([0,2\pi]^2)$ the means

$$\sigma_n^1 f = \frac{1}{n}\sum_{j=1}^{n} S_{j,j}(f)$$

converge a.e. to f as $n \to \infty$. Zhizhiashvili [9] improved this result and proved that for $f \in L_1([0,2\pi]^2)$ the (C,α)-means

$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1} S_{j,j}(f), \quad \alpha > 0$$

converge a.e. to f as $n \to \infty$.

For the Marcinkiewicz–Fejér means of the two-dimensional Walsh–Fourier series Weisz [8] proved that the following is true

Theorem A (Weisz). *Let $p > 2/3$. Then the maximal operators σ_*^1 and $\widetilde{\sigma}_*^{1,(t)}$ are bounded from the Hardy space $H_p(G \times G)$ to the space $L_p(G \times G)$.*

The second author [1] generalized the theorem of Weisz for the d-dimensional Walsh–Fourier series and proved that the maximal operator σ_*^1 is bounded from the d-dimensional dyadic martingale Hardy space $H_p(G \times \cdots \times G)$ to the space $L_p(G \times \cdots \times G)$ for $p > d/(d+1)$ and is of weak type (1,1). We also proved [2] that for the boundedness of the maximal operator σ_*^1 from the Hardy space $H_p(G \times \cdots \times G)$ to the space $L_p(G \times \cdots \times G)$ the assumption $p > d/(1+d)$ is essential.

In [3] it is proved that the maximal operators σ_*^α $(0 < \alpha < 1)$ of the (C,α) means of cubical partial sums of the d-dimensional Walsh–Fourier series is bounded from the d-dimensional dyadic martingale Hardy space $H_p(G \times \cdots \times G)$ to the space $L_p(G \times \cdots \times G)$, when $p > d/(d+\alpha)$ and for the boundedness of the maximal operator σ_*^α from the Hardy space $H_p(G \times \cdots \times G)$ to the space $L_p(G \times \cdots \times G)$ the assumption $p > d/(\alpha+d)$ is essential. It is easy to show that (see Weisz [8]) the conjugate maximal operators $\widetilde{\sigma}_*^{\alpha,(t)}$ $(0 < \alpha \leq 1)$ of the (C,α) means of cubical partial sums of the d-dimensional Walsh–Fourier series is bounded from the d-dimensional dyadic martingale Hardy space $H_p(G \times \cdots \times G)$ to the space $L_p(G \times \cdots \times G)$, when $p > d/(d+\alpha)$.

In this paper we prove that for every $0 < p \leq d/(d+\alpha), 0 < \alpha \leq 1$ there exists a martingale $f \in H_p(G \times \cdots \times G)$ such that

$$\left\|\widetilde{\sigma}_*^{\alpha,(t)} f\right\|_p = +\infty.$$

We note that in case $\alpha = 1$ and $d = 2$ above mentioned result contains answer to the question of Weisz [8].

2. Dyadic Hardy spaces and conjugate transforms

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0,1\}$ $(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G, \ \ I_n(x) := I_n(x_0, \dots, x_{n-1}) :=$$
$$:= \{y \in G : \ y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\} \qquad (x \in G, n \in \mathbf{N}).$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)$ $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the k-th Rademacher function.

The dyadic d-dimensional rectangles are of the form

$$I_n(x_1, \dots, x_d) := I_n(x_1) \times \dots \times I_n(x_d).$$

The σ-algebra generated by the dyadic rectangles

$$\{I_n(x_1, \dots, x_d) : (x_1, \dots, x_d) \in G \times \dots \times G\}$$

is denoted by F_n.

The norm (or quasinorm) of the space $L_p(G \times \dots \times G)$ is defined by

$$\|f\|_p := \left(\int\limits_{G \times \dots \times G} |f(x_1, \dots, x_d)|^p \, d\mu(x_1, \dots, x_d) \right)^{1/p} \qquad (0 < p < +\infty).$$

Denote by $f = \left(f^{(n)}, n \in N\right)$ one parameter martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e.g. [6, 7]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbf{N}} \left|f^{(n)}\right|.$$

In case $f \in L_1(G\times\cdots\times G)$, the maximal function can also be given by

$$f^*(x_1,\dots,x_d) =$$
$$= \sup_{n\in\mathbf{N}} \frac{1}{\mu(I_n(x_1,\dots,x_d))}\left|\int\limits_{I_n(x_1,\dots,x_d)} f(u_1,\dots,u_d)\, d\mu(u_1,\dots,u_d v)\right|,$$
$$(x_1,\dots,x_d)\in G\times\cdots\times G.$$

For $0<p<\infty$ the Hardy martingale space $H_p(G\times\cdots\times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

For a martingale

$$f \sim \sum_{n=0}^{\infty}\left(f^{(n)} - f^{(n-1)}\right)$$

the conjugate transforms are defined by the martingale

$$\widetilde{f}^{(t)} \sim \sum_{n=1}^{\infty} r_n(t)\left(f^{(n)} - f^{(n-1)}\right),$$

where $t\in G$ is fixed. Note that $\widetilde{f}^{(0)} = f$. As is well known, if f is an integrable function, then conjugate transforms $\widetilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general.

3. Walsh system and (C,α) means

Let $n\in\mathbf{N}$, then $n = \sum\limits_{i=0}^{\infty} n_i 2^i$, where $n_i\in\{0,1\}$ $(i\in\mathbf{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j\in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|}\le n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty}(r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \quad (x\in G, n\in\mathbf{P}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that ([5])

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{if } x \in G \backslash I_n. \end{cases} \tag{1}$$

The rectangular partial sums of the d-dimensional Walsh–Fourier series are defined as follows:

$$S_{M_1,\dots,M_d} f(x_1,\dots,x_d) := \sum_{i_1=0}^{M_1-1} \cdots \sum_{i_d=0}^{M_d-1} \widehat{f}(i_1,\cdots,i_d) \prod_{j=1}^{d} w_{i_j}(x_j),$$

where the number

$$\widehat{f}(i_1,\cdots,i_d) = \int\limits_{G\times\cdots\times G} f(x_1,\dots,x_d) \prod_{j=1}^{d} w_{i_j}(x_j)\, d\mu(x_1,\dots,x_d)$$

is said to be the $(i_1,\cdots,i_d)$th Walsh–Fourier coefficient of the function f.

If $f \in L_1(G\times\cdots\times G)$ then it is easy to show that the sequence $(S_{2^n,\dots,2^n}(f) : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n)} : n \in \mathbf{N})$ then the Walsh–Fourier coefficients must be defined in a little bit different way:

(2)

$$\widehat{f}(i_1,\cdots,i_d) = \lim_{k\to\infty} \int\limits_{G\times\cdots\times G} f^{(k)}(x_1,\dots,x_d) \prod_{j=1}^{d} w_{i_j}(x_j)\, d\mu(x_1,\dots,x_d).$$

The Walsh–Fourier coefficients of $f \in L_1(G\times\cdots\times G)$ are the same as the ones of the martingale $(S_{2^n,\dots,2^n}(f) : n \in \mathbf{N})$ obtained from f.

For $n = 1,2,\dots$ and martingale f the (C,α)-mean of order n of the d-dimensional Walsh–Fourier series of f is given by

$$\sigma_n^\alpha f(x_1,\dots,x_d) = \frac{1}{A_{n-1}^\alpha} \sum_{j=1}^{n} A_{n-j}^{\alpha-1} S_{j,\dots,j} f(x_1,\dots,x_d),$$

where

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \quad (n \in N,\ \alpha \neq -1,-2,\dots).$$

It is known that (see Zygmund [10])

$$A_n^\alpha \sim n^\alpha \quad (n \in N). \tag{3}$$

It is evident that

$$\sigma_n^\alpha f(x_1,\dots,x_d)= = \int\limits_{G\times\cdots\times G} f(u_1,\dots,u_d)\,K_n^\alpha(x_1+u_1,\dots,x_d+u_d)\,d\mu(u_1,\dots,u_d),$$

where

$$K_n^\alpha(x_1,\dots,x_d)=\frac{1}{A_{n-1}^\alpha}\sum_{j=1}^n A_{n-j}^{\alpha-1}\prod_{i=1}^d D_j(x_i).$$

Let

$$\rho_{0,\dots,0}=r_0,\quad \rho_{i_1,\dots,i_d}=r_j$$

if $i_k\in\{0,1,\dots,2^j-1\}$ and at least one $i_l\in\{2^{j-1},\dots,2^j-1\}$.

Then $(M_1,\dots,M_d)$th partial sums of the conjugate transforms is given by

$$\widetilde{S}^{(t)}_{M_1,\dots,M_d}f(x_1,\dots,x_d):=\sum_{i_1=0}^{M_1-1}\cdots\sum_{i_d=0}^{M_d-1}\rho_{i_1,\dots,i_d}(t)\,\widehat{f}(i_1,\dots,i_d)\prod_{j=1}^d w_{i_j}(x_j).$$

The conjugate (C,α)-means of a martingale f are introduced by

$$\widetilde{\sigma}_n^{\alpha,(t)}f(x_1,\dots,x_d)=\frac{1}{A_{n-1}^\alpha}\sum_{j=1}^n A_{n-j}^{\alpha-1}\widetilde{S}^{(t)}_{j,\dots,j}f(x_1,\dots,x_d).$$

It is evident that $\widetilde{\sigma}_n^{\alpha,(0)}f=\sigma_n^\alpha f$.

The maximal operator and the conjugate maximal operator are defined by

$$\sigma_*^\alpha f=\sup_n|\sigma_n^\alpha f|,\ \ \widetilde{\sigma}_*^{\alpha,(t)}f=\sup_n|\widetilde{\sigma}_n^{\alpha,(t)}f|.$$

A bounded measurable function a is a p-atom, if there exists a dyadic d-dimensional cube $I\times\cdots\times I$, such that

a) $\int\limits_{I\times\cdots\times I} a\,d\mu=0$;

b) $\|a\|_\infty\le\mu(I\times\cdots\times I)^{-1/p}$;

c) supp $a\subset I\times\cdots\times I$.

The basic result of atomic decomposition is the following one.

Theorem A (Weisz [7]). *A martingale $f=\left(f^{(n)}:n\in\mathbf{N}\right)$ is in H_p $(0<p\le1)$ if and only if there exists a sequence $(a_k,k\in\mathbf{N})$ of p-atoms and a sequence $(\mu_k,k\in\mathbf{N})$ of real numbers such that for every $n\in\mathbf{N}$,*

$$\sum_{k=0}^\infty \mu_k S_{2^n,\dots,2^n}a_k=f^{(n)}, \tag{4}$$

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty.$$

Moreover,

$$\|f\|_{H_p}\sim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}.$$

4. Main result

Theorem 1. *Let $0<p\le d/(d+\alpha)$. Then there exists a martingale $f\in H_p(G\times\cdots\times G)$ such that*

$$\left\|\widetilde{\sigma}_*^{\alpha,(t)}f\right\|_p=+\infty.$$

Corollary 1. *Let $0<p\le d/(d+\alpha)$. Then there exists a martingale $f\in H_p(G\times\cdots\times G)$ such that*

$$\|\sigma_*^{\alpha}f\|_p=+\infty$$

5. Proof of main result

Proof of Theorem 1. Let $\{m_k : k\in\mathbf{P}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=1}^{\infty}\frac{1}{m_k^p}<\infty, \tag{5}$$

$$\sum_{l=0}^{k-1}\frac{2^{2m_l d/p}}{m_l}<\frac{2^{2m_k d/p}}{m_k}, \tag{6}$$

$$\frac{2^{2dm_{k-1}/p}}{m_{k-1}}<\frac{2^{m_k}}{m_k}. \tag{7}$$

Let

$$f^{(A)}(x_1,\ldots,x_d):=\sum_{\{k:2m_k<A\}}\lambda_k a_k(x_1,\ldots,x_d),$$

where $\lambda_k:=\dfrac{2^d}{m_k}$ and

$$a_k\left(x_1,\dots,x_d\right):=2^{2d(1/p-1)m_k-d}\prod_{j=1}^{d}\left(D_{2^{2m_k+1}}\left(x_j\right)-D_{2^{2m_k}}\left(x_j\right)\right).$$

It is easy to show that the martingale $f:=\left(f^{(0)},f^{(1)},\dots,f^{(A)},\dots\right)\in$ $\in H_p\left(G\times\dots\times G\right)$. Indeed, since

$$\|a_k\|_\infty=2^{2d(1/p-1)m_k-d}2^{2m_kd+d}=2^{2m_kd/p}=\left(\operatorname{supp}\left(a_k\right)\right)^{-1/p},$$

$$S_{2^A,\dots,2^A}a_k\left(x_1,\dots,x_d\right)=\begin{cases}0, & A\le 2m_k\\ a_k, & A>2m_k\end{cases},$$

$$f^{(A)}\left(x_1,\dots,x_d\right)=\sum_{\{k:2m_k<A\}}\lambda_ka_k\left(x_1,\dots,x_d\right)=$$

$$=\sum_{k=0}^{\infty}\lambda_kS_{2^A,\dots,2^A}a_k\left(x_1,\dots,x_d\right)$$

from (5) and Th. A we conclude that $f\in H_p\left(G\times\dots\times G\right)$.

Let $q_{A,s}=2^{2A}+2^{2s}$, $A>s$. We write $(s<m_k)$

$$\widetilde{\sigma}^{\alpha,(t)}_{q_{m_k,s}}f(x_1,\dots,x_d)=\frac{1}{A^{\alpha}_{q_{m_k,s}-1}}\sum_{j=1}^{2^{2m_k}-1}A^{\alpha-1}_{q_{m_k,s}-j}\widetilde{S}^{(t)}_{j,\dots,j}f\left(x_1,\dots,x_d\right)+$$
$$+\frac{1}{A^{\alpha}_{q_{m_k,s}-1}}\sum_{j=2^{2m_k}}^{q_{m_k,s}}A^{\alpha-1}_{q_{m_k,s}-j}\widetilde{S}^{(t)}_{j,\dots,j}f(x_1,\dots,x_d)=$$
$$=I+II. \tag{8}$$

Let $(j_1,\dots,j_d)\in\{2^{2m_k},\dots,2^{2m_k+1}-1\}\times\dots\times\{2^{2m_k},\dots,2^{2m_k+1}-1\}$ for some $k\in\mathbf{P}$. Then

$$\widehat{f}\left(j_1,\dots,j_d\right)=\lim_{A\to\infty}\widehat{f^{(A)}}\left(j_1,\dots,j_d\right)=\frac{2^{2d(1/p-1)m_k}}{m_k} \tag{9}$$

and

$$\widehat{f}\left(j_1,\dots,j_d\right)=0 \tag{10}$$

if $(j_1,\dots,j_d)\notin\{2^{2m_k},\dots,2^{2m_k+1}-1\}\times\dots\times\{2^{2m_k},\dots,2^{2m_k+1}-1\}$, $k\in P$.

Let $j<2^{2m_k}$. Then from (6), (9) and (10) we have

$$\left|\widetilde{S}^{(t)}_{j,\dots,j} f\left(x_1,\dots,x_d\right)\right| =$$
$$= \left|\sum_{l=0}^{k-1} r_{2m_l}\left(t\right) \sum_{v_1=2^{2m_l}}^{2^{2m_l+1}-1} \cdots \sum_{v_d=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}\left(v_1,\dots,v_d\right) \prod_{j=1}^{d} w_{v_j}\left(x_j\right)\right| \le$$
$$\le \sum_{l=0}^{k-1} \sum_{v_1=2^{2m_l}}^{2^{2m_l+1}-1} \cdots \sum_{v_d=2^{2m_l}}^{2^{2m_l+1}-1} \left|\widehat{f}\left(v_1,\dots,v_d\right)\right| \le$$
$$\le \sum_{l=0}^{k-1} \sum_{v_1=2^{2m_l}}^{2^{2m_l+1}-1} \cdots \sum_{v_d=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2d(1/p-1)m_l}}{m_l} =$$
$$= \sum_{l=0}^{k-1} \frac{2^{2d(1/p-1)m_l}}{m_l} 2^{2m_l d} =$$
$$= \sum_{l=0}^{k-1} \frac{2^{2dm_l/p}}{m_l} < 2\frac{2^{2m_{k-1}d/p}}{m_{k-1}}.$$

Consequently

$$I \le \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \sum_{j=1}^{2^{2m_k}-1} A^{\alpha-1}_{q_{m_k,s}-j} \frac{2^{2m_{k-1}d/p+1}}{m_{k-1}} \le c\left(\alpha\right) \frac{2^{2m_{k-1}d/p}}{m_{k-1}}. \tag{11}$$

For $2^{2m_k} \le j < q_{m_k,s}$ we have the following

$$\widetilde{S}^{(t)}_{j,\dots,j} f\left(x_1,\dots,x_d\right) =$$
$$= \sum_{l=0}^{k-1} r_{2m_l}\left(t\right) \sum_{v_1=2^{2m_l}}^{2^{2m_l+1}-1} \cdots \sum_{v_d=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}\left(v_1,\dots,v_d\right) \prod_{q=1}^{d} w_{v_q}\left(x_q\right) +$$
$$+ r_{2m_k}\left(t\right) \sum_{v_1=2^{2m_k}}^{j-1} \cdots \sum_{v_d=2^{2m_k}}^{j-1} \widehat{f}\left(v_1,\dots,v_d\right) \prod_{q=1}^{d} w_{v_q}\left(x_q\right) =$$
$$= \sum_{l=0}^{k-1} r_{2m_l}\left(t\right) \sum_{v_1=2^{2m_l}}^{2^{2m_l+1}-1} \cdots \sum_{v_d=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2d(1/p-1)m_l}}{m_l} \prod_{q=1}^{d} w_{v_q}\left(x_q\right) +$$
$$+ \frac{r_{2m_k}\left(t\right) 2^{2d(1/p-1)m_k}}{m_k} \sum_{v_1=2^{2m_k}}^{j-1} \cdots \sum_{v_d=2^{2m_k}}^{j-1} \prod_{q=1}^{d} w_{v_q}\left(x_q\right) =$$

$$= \sum_{l=0}^{k-1} \frac{r_{2m_l}(t)\, 2^{2d(1/p-1)m_l}}{m_l} \prod_{q=1}^{d} \left[D_{2^{2m_l+1}}(x_q) - D_{2^{2m_l}}(x_q) \right] +$$

$$+ \frac{r_{2m_k}(t)\, 2^{2d(1/p-1)m_k}}{m_k} \prod_{q=1}^{d} \left[D_j(x_q) - D_{2^{2m_k}}(x_q) \right],$$

This gives that

(12)

$$II = \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} \sum_{l=0}^{k-1} \frac{r_{2m_l}(t)\, 2^{2d(1/p-1)m_l}}{m_l} \times$$

$$\times \prod_{q=1}^{d} \left[D_{2^{2m_l+1}}(x_q) - D_{2^{2m_l}}(x_q) \right] +$$

$$+ \frac{r_{2m_k}(t)\, 2^{2d(1/p-1)m_k}}{m_k} \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} \prod_{q=1}^{d} \left[D_j(x_q) - D_{2^{2m_l}}(x_q) \right]$$

$$= II_1 + II_2.$$

To discuss II_1, we use (6) and $D_{2^n} \leq 2^n$. Thus we can write

$$|II_1| \leq c(\alpha) \sum_{l=0}^{k-1} \frac{2^{2d(1/p-1)m_l}}{m_l} \prod_{q=1}^{d} \left| D_{2^{2m_l+1}}(x_q) - D_{2^{2m_l}}(x_q) \right| \tag{13}$$

$$\leq c(\alpha) \sum_{l=0}^{k-1} \frac{2^{2d(1/p-1)m_l}}{m_l} 2^{2m_l d} \leq c(\alpha) \frac{2^{2dm_{k-1}/p}}{m_{k-1}}.$$

From $\widetilde{\sigma}^{\alpha,(t)}_{q_{m_k,s}} f(x_1, \ldots, x_d) = I + II_1 + II_2$ and (11), (13) we have

$$\left| \widetilde{\sigma}^{\alpha,(t)}_{q_{m_k,s}} f(x_1, \ldots, x_d) \right| \geq |II_2| - |I| - |II_1| \geq |II_2| - c(\alpha) \frac{2^{2dm_{k-1}/p}}{m_{k-1}}. \tag{14}$$

Since $D_{j+2^{2m_k}} = D_{2^{2m_k}} + w_{2^{2m_k}} D_j$ for II_2 we have

(15)

$$II_2 = \frac{r_{2m_k}(t)\, 2^{2d(1/p-1)m_k}}{m_k} \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \sum_{j=0}^{2^{2s}} A^{\alpha-1}_{2^{2s}-j} \prod_{q=1}^{d} D_j(x_q)\, w_{2^{2m_k}}(x_q) =$$

$$= \frac{r_{2m_k}(t)\, 2^{2d(1/p-1)m_k}}{m_k} \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \prod_{q=1}^{d} w_{2^{2m_k}}(x_q) A^{\alpha}_{2^{2s}-1} K^{\alpha}_{2^{2s}}(x_1,...,x_d).$$

Combining (14) and (15) we can write

$$\left|\widetilde{\sigma}^{\alpha,(t)}_{q_{m_k,s}} f(x_1,\ldots,x_d)\right| \geq \tag{16}$$

$$\geq c(\alpha) \frac{2^{2d(1/p-1)m_k - 2m_k\alpha}}{m_k} A^{\alpha}_{2^{2s}-1} \left|K^{\alpha}_{2^{2s}}(x_1,\ldots,x_d)\right| - c(\alpha) \frac{2^{2dm_{k-1}/p}}{m_{k-1}}.$$

Let $(x_1,\ldots,x_d) \in (I_{2s} \backslash I_{2s+1}) \times \cdots \times (I_{2s} \backslash I_{2s+1})$. Then it is evident that

$$A^{\alpha}_{2^{2s}-1} \left|K^{\alpha}_{2^{2s}}(x_1,\ldots,x_d)\right| \geq c(\alpha)\, 2^{2s(d+\alpha)}.$$

Consequently, from (7) and (16) we have

$$\left|\widetilde{\sigma}^{\alpha,(t)}_{q_{m_k,s}} f(x_1,\ldots,x_d)\right| \geq c(\alpha) \frac{2^{2d(1/p-1)m_k - 2m_k\alpha}}{m_k} 2^{2s(d+\alpha)} - c(\alpha) \frac{2^{m_k}}{m_k},$$

$$\int_{G\times\cdots\times G} \left(\widetilde{\sigma}^{\alpha,(t)}_{*} f(x_1,\ldots,x_d)\right)^p d\mu \geq$$

$$\geq \sum_{s=\left[\frac{m_k}{2}\right]}^{m_k-1} \int_{(I_{2s}\backslash I_{2s+1})\times\cdots\times(I_{2s}\backslash I_{2s+1})} \left(\widetilde{\sigma}^{\alpha,(t)}_{*} f(x_1,\ldots,x_d)\right)^p d\mu \geq$$

$$\geq \sum_{s=\left[\frac{m_k}{2}\right]}^{m_k-1} \int_{(I_{2s}\backslash I_{2s+1})\times\cdots\times(I_{2s}\backslash I_{2s+1})} \left(\widetilde{\sigma}^{\alpha,(t)}_{q_{m_k,s}} f(x_1,\ldots,x_d)\right)^p d\mu \geq$$

$$\geq c(\alpha) \sum_{s=\left[\frac{m_k}{2}\right]}^{m_k-1} \frac{1}{2^{2sd}} \left[\frac{2^{2m_k(d/p-(\alpha+d))}}{m_k} 2^{2s(d+\alpha)}\right]^p \geq$$

$$\geq c(\alpha) \sum_{s=\left[\frac{m_k}{2}\right]}^{m_k-1} 2^{2s((d+\alpha)p-d)} \frac{2^{2m_k(d-p(d+\alpha))}}{m_k^p} \geq$$

$$\geq \begin{cases} c(\alpha)\, m_k^{1-p}, & p = \frac{d}{d+\alpha} \\ c(\alpha) \frac{2^{m_k(d-p(d+\alpha))}}{m_k^p}, & 0 < p < \frac{d}{d+\alpha} \end{cases} \to \infty \quad \text{as } k \to \infty.$$

The proof of Th. 1 is complete. $\Diamond$

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