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A SPECIAL LABOR-MANAGED OLIGOPOLY

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Abstract: A special labor-managed oligopoly without product differentiation is considered. The existence of the equilibrium is first proved and a simple example is presented to show the possibility of multiple equilibrium. The local asymptotic stability of the equilibria is next examined, the stability conditions are derived in both discrete and continuous time scales.

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1. Introduction

The theory of oligopoly is one of the most frequently discussed subjects of mathematical economics. Since the pioneering work of Cournot (1838), many researchers examined the classical Cournot model and its extensions including models without and with product differentiation, multi-product oligopolies, rent-seeking games, labor-managed models, oligopolies with intertemporal demand interaction and models with product adjustment costs. A comprehensive summary of the most significant works up to the mid 70s is given in Okuguchi (1976). Multiproduct models with further extensions and applications are discussed in Okuguchi and Szidarovszky (1999), and nonlinear dynamic oligopolies are examined in detail in Bischi et al. (2009).

In this paper a special labor-managed model is examined. The seminal work of Ward (1958) is considered to be the first to introduce labor-managed oligopolies. Hill and Waterson (1983) have formulated profit-maximizing and labor-managed models without product differentiation and with symmetric firms and compared the long-term behavior of these models. Neary (1984) generalized this work to the nonsymmetric case. The existence and uniqueness of the equilibrium was proved by Okuguchi (1991) under rather restrictive conditions, furthermore in Okuguchi (1993) comparative statics are presented for profit-maximizing and labor-managed oligopolies. The existence of equilibrium was also shown in Okuguchi and Szidarovszky (1999) under slightly more general conditions, and the asymptotical properties of the equilibrium were also investigated with discrete and continuous time scales. These results were further generalized in Li and Szidarovszky (1999). Okuguchi and Szidarovszky (2008) have presented a general existence and uniqueness theorem for the equilibrium of labor-managed oligopolies.

In contrary to earlier works, the payoff functions are not concave and the strategy sets are not compact in the model to be presented in this paper, so general existence theorems cannot be used. This paper is organized as follows. In Sec. 2 the mathematical model will be presented and the existence of the equilibrium will be proved. The uniqueness of the equilibrium cannot be guaranteed in general as it will be demonstrated by a simple example. The local asymptotical stability of the equilibrium will be examined with discrete and continuous time scales in Sec. 3. Sec. 4 concludes the paper.

2. Mathematical model and existence theorem

Consider a single product oligopoly without product differentiation. Let N denote the number of firms, and let L_k be the capacity limit of firm k. If x_k denotes the output of firm k and $Q = \sum_{k=1}^{N} x_k$ is the output of the industry, then the price function is assumed to be $f(Q) = A - BQ$ with $A > 0$ and $B > 0$. In order to guarantee nonnegative price, we assume that $A/B > \sum_{k=1}^{N} L_k$. The amount of labor necessary for producing output x_k by firm k is denoted by $h_k(x_k) = p_k x_k^{\alpha}$, where $\alpha > 0$ determining the production elasticity of labor. Let $d_k > 0$ be the fixed cost of firm k and w the competitive wage rate. The surplus of firm k per unit labor is given as

$$\varphi_k(x_1, \dots, x_N) = \frac{x_k(A - Bx_k - BQ_k) - wp_k x_k^{\alpha} - d_k}{p_k x_k^{\alpha}}, \tag{1}$$

where $Q_k = \sum_{\ell \neq k} x_\ell$ is the output of the rest of the industry. With these notations, an N-person game can be defined in which the set of strategies of player k is the closed interval $[0, L_k]$ and its payoff function is φ_k. Notice that $\varphi_k \to -\infty$ as $x_k \to 0$, so φ_k is undefined for $x_k = 0$.

The best response of firm k can be obtained by maximizing φ_k with fixed value of Q_k. Since

$$\varphi_k(x_1, \dots, x_N) = \frac{A - BQ_k}{p_k} x_k^{1-\alpha} - \frac{B}{p_k} x_k^{2-\alpha} - w - \frac{d_k}{p_k} x_k^{-\alpha},$$

we have

$$\begin{aligned} \frac{\partial \varphi_k}{\partial x_k} &= \frac{A - BQ_k}{p_k}(1-\alpha) x_k^{-\alpha} - \frac{B}{p_k}(2-\alpha) x_k^{1-\alpha} + \frac{d_k}{p_k} \alpha x_k^{-\alpha-1} \\ &= \frac{1}{p_k x_k^{\alpha+1}} \left(-B(2-\alpha) x_k^2 + (A - BQ_k)(1-\alpha) x_k + d_k \alpha \right). \end{aligned} \tag{2}$$

Assume first that $\alpha < 1$, then the bracketed concave quadratic polynomial has a unique positive root:

$$x_k^* = \frac{(A - BQ_k)(1-\alpha) + \sqrt{D}}{2B(2-\alpha)} \tag{3}$$

with

$$D = (A - BQ_k)^2 (1-\alpha)^2 + 4B(2-\alpha) d_k \alpha.$$

Clearly $\partial\varphi_k/\partial x_k$ is positive as $x_k < x_k^*$ and is negative for $x_k > x_k^*$, therefore the best response of firm k is

$$R_k(Q_k) = \begin{cases} x_k^* \text{ if } x_k^* \leq L_k \\ L_k \text{ otherwise.} \end{cases} \tag{4}$$

From eq. (3) it is also clear that x_k^* strictly decreases in Q_k, furthermore

$$0 > \frac{\partial x_k^*}{\partial Q_k} = \frac{1}{2B(2-\alpha)}\left[-B(1-\alpha) + \frac{(A-BQ_k)(-B)(1-\alpha)^2}{\sqrt{D}}\right]$$
$$> \frac{1}{2B(2-\alpha)}\left[-B(1-\alpha) - B(1-\alpha)\right] > -1,$$

so

$$-1 < R_k'(Q_k) \leq 0 \tag{5}$$

except on the border line between the two cases of (4).

In the case of $\alpha = 1$ the bracketed quadratic polynomial has again a unique positive root:

$$x_k^* = \sqrt{\frac{\alpha d_k}{B}},$$

and the best response of firm k is given by (4). Notice that x_k^* does not depend on Q_k, so $R_k'(Q_k) = 0$ for all Q_k.

Assume next that $1 < \alpha < 2$, then the bracketed parabola of (2) has again a unique positive root:

$$x_k^* = \frac{\sqrt{D} - (A-BQ_k)(\alpha-1)}{2B(2-\alpha)} = \frac{4B(2-\alpha)d_k\alpha}{\sqrt{D} + (A-BQ_k)(\alpha-1)}.$$

This relation shows that x_k^* increases in Q_k and since

$$\frac{\partial x_k^*}{\partial Q_k} = \frac{1}{2(2-\alpha)}\left[-\frac{(A-BQ_k)(\alpha-1)^2}{\sqrt{D}} + (\alpha-1)\right],$$

we have

$$0 \leq R_k'(Q_k) < \frac{\alpha-1}{2(2-\alpha)} \tag{6}$$

where $R_k'(Q_k)$ can approach the upper bound arbitrary closely if d_k is sufficiently large. Notice that this upper bound can become arbitrarily large if α tends to 2.

Assume now that $\alpha = 2$. The bracketed term of (2) becomes linear, and the best response of firm can be obtained as

$$R_k(Q_k) = \begin{cases} x_k^* \text{ if } x_k^* \le L_k \\ L_k \text{ otherwise,} \end{cases} \tag{7}$$

where $x_k^* = \dfrac{2d_k}{A - BQ_k}$. The first case occurs when

$$Q_k \le \frac{AL_k - 2d_k}{BL_k}.$$

The graph of R_k is shown in Fig. 1. From eq. (7) we see that $R_k'(Q_k) = 0$ or

$$R_k'(Q_k) = \frac{2d_kB}{(A - BQ_k)^2}. \tag{8}$$

In the second case,

$$R_k'(Q_k) = \frac{x_k^{*2}B}{2d_k} = \frac{2d_k - (A - BQ)x_k^*}{2d_k},$$

where we used eq. (2) with $\alpha = 2$. This expression implies that

$$0 \le R_k'(Q_k) < 1.$$

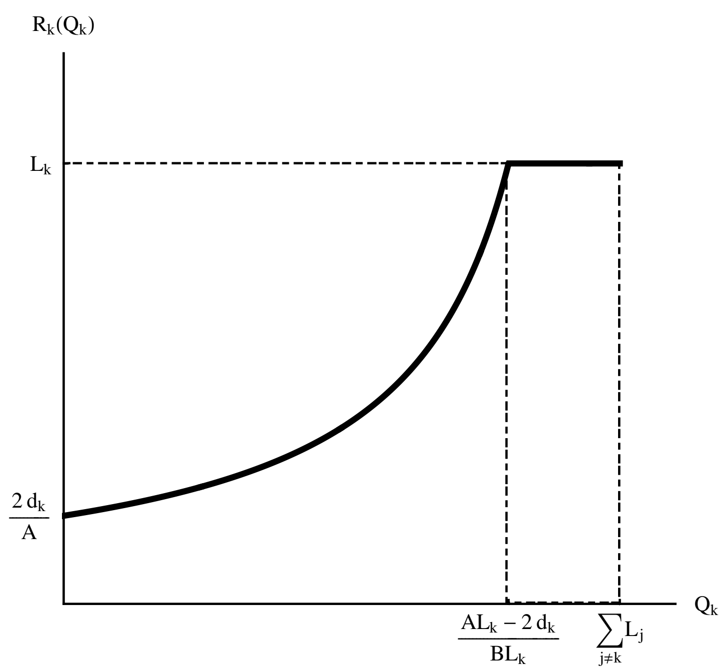


Figure 1. Piecewise continuous best response curve

The case of $\alpha > 2$ is unrealistic from the economic point of view, so we can ignore this case.

Notice that for all values of $\alpha \in (0, 2]$ the best response mapping $\left(R_1\left(\sum_{\ell\neq 1} x_\ell\right), \ldots, R_N\left(\sum_{\ell\neq N} x_\ell\right)\right)$ maps the compact, convex set $\prod_{k=1}^{N}[0, L_k]$ into itself, and since it is continuous, the Brouwer fixed point theorem guarantees the existence of at least one fixed point.

Similarly to classical Cournot models, we can present a constructive existence proof which can be also used to compute the equilibria.

If Q denotes the total output of the industry, then from (2), we have

$$\begin{aligned} 0 &= -B(2-\alpha)x_k^2 + (A - BQ + Bx_k)(1-\alpha)x_k + d_k\alpha \\ &= -Bx_k^2 + (A - BQ)(1-\alpha)x_k + d_k\alpha \end{aligned}$$

with the unique positive root

$$\bar{x}_k^* = \frac{(A - BQ)(1-\alpha) + \sqrt{\bar{D}}}{2B}$$

where

$$\bar{D} = (A - BQ)^2(1-\alpha)^2 + 4Bd_k\alpha.$$

So the best response of firm k can be also obtained as

$$\bar{R}_k(Q) = \begin{cases} \bar{x}_k^* \text{ if } \bar{x}_k^* \leq L_k, \\ L_k \text{ otherwise.} \end{cases}$$

Notice that $\bar{R}_k(Q)$ decreases in Q for $\alpha < 1$, it does not depend on Q if $\alpha = 1$, so if $\alpha \leq 1$, then

$$\bar{R}_k'(Q) \leq 0. \tag{9}$$

If $\alpha > 1$, then

$$\bar{x}_k^* = \frac{2d_k\alpha}{(A - BQ)(\alpha - 1) + \sqrt{\bar{D}}}$$

showing that $\bar{R}_k(Q)$ increases for $\alpha > 1$. Notice that in this case

$$\frac{\partial \bar{x}_k^*}{\partial Q} = \frac{1}{2B}\left(B(\alpha - 1) + \frac{2(A - BQ)(-B)(1-\alpha)^2}{2\sqrt{\bar{D}}}\right) < \frac{\alpha - 1}{2},$$

so

$$0 \leq R_k'(Q) < \frac{\alpha - 1}{2}. \tag{10}$$

In all cases $\bar{R}_k(Q)$ is a continuous function of Q.

The industry output at any equilibrium is the solution of equation

$$\sum_{k=1}^{N} \bar{R}_k(Q) - Q = 0. \tag{11}$$

At $Q = 0$ the left-hand side is positive, at $Q = \sum_{k=1}^{N} L_k$ it is nonpositive and continuous implying the existence of at least on equilibrium. If $\alpha \leq 1$, then the left-hand side is strictly decreasing, so the equilibrium is unique. In the case of $\alpha > 1$, it is also unique if

$$\frac{N(\alpha - 1)}{2} \leq 1,$$

which occurs if

$$\alpha \leq 1 + \frac{2}{N}.$$

Notice that if $\alpha \leq 2$, then this relation always holds for duopolies ($N = = 2$).

The uniqueness of the equilibrium cannot be guaranteed in general as it is shown in the following example.

Example 1. Let $N \geq 3$ be arbitrary, $\alpha = 2,\ L_k = 4,\ d_k = 8.5$ for all k, $A = 12$ and $B = \frac{2}{N-1}$. Clearly all conditions are satisfied: $A > 0$, $B > 0$ and

$$\sum_{k=1}^{N} L_k = 4N \leq \frac{A}{B} = 6(N-1).$$

We can show that both

$$\bar{x}_k^S = 3 - \sqrt{0.5} \quad \text{and} \quad \bar{x}_k^M = 3 + \sqrt{0.5}$$

are symmetric interior equilibria. Notice first that for all k,

$$0 < \bar{x}_k^S < L_k \quad \text{and} \quad 0 < \bar{x}_k^M < L_k,$$

and they satisfy the best response equations $x_k = R_k(Q_k)$, since

$$\bar{Q}_k^S = (N-1)(3 - \sqrt{0.5}) \quad \text{and} \quad \bar{Q}_k^M = (N-1)(3 + \sqrt{0.5}),$$

furthermore in these cases

$$R_k(\bar{Q}_k) = \frac{2d_k}{A - B\bar{Q}_k} = \frac{17}{12 - 2(3 \mp \sqrt{0.5})} = 3 \mp \sqrt{0.5} = \bar{x}_k.$$

It can be also shown that $\bar{x}_k^L = L_k$ is a symmetric boundary equilibrium, since in this case the second case of (3) occurs:

$$x_k^* = \frac{17}{12 - \dfrac{2}{N-1}4(N-1)} = \frac{17}{4} > 4 = L_k.$$

3. Local stability analysis

In the literature of dynamic economics there are two major dynamic processes: partial adjustment toward best responses and adaptive expectations. In an N-firm oligopoly the first process is N-dimensional, while the other can be described by a $2N$-dimensional system. In Bischi et al. (2009) the equivalence of the local asymptotical properties of the two systems is verified, so we will consider the first adjustment process.

Consider first continuous time scales and assume that the firms use partial adjustment toward best responses. Then their outputs satisfy the differential equation system

$$\dot{x}_k = K_k \left(R_k \left(\sum_{\ell \neq k}^{N} x_\ell \right) - x_k \right) \quad \text{for} \quad k = 1, 2, \ldots, N, \tag{12}$$

where $K_k > 0$ is the speed of adjustment of firm k. The equilibria of the labor-managed oligopoly game coincide with the steady states of this system. The local asymptotic stability of this system can be examined by linearization. The Jacobian has the special form

$$\boldsymbol{J} = \begin{pmatrix} -K_1 & K_1\bar{r}_1 & \cdot & K_1\bar{r}_1 \\ K_2\bar{r}_2 & -K_2 & \cdot & K_2\bar{r}_2 \\ \cdot & \cdot & \cdot & \cdot \\ K_N\bar{r}_N & K_N\bar{r}_N & \cdot & -K_N \end{pmatrix} = \boldsymbol{D} + \boldsymbol{a}\boldsymbol{1}^T,$$

where $\bar{r}_k$ is the value of R'_k at the equilibrium,

$$\boldsymbol{D} = \operatorname{diag}\left(-K_1(1+\bar{r}_1), \ldots, -K_N(1+\bar{r}_N)\right),$$

$$\boldsymbol{a} = (K_1\bar{r}_1, \ldots, K_N\bar{r}_N)^T$$

and

$$\boldsymbol{1}^T = (1, \ldots, 1).$$

The characteristic polynomial of this matrix can be written in the following form:

$$\begin{aligned} \varphi(\lambda) &= \det(\boldsymbol{D} + \boldsymbol{a}\boldsymbol{1}^T - \lambda \boldsymbol{I}) \\ &= \det(\boldsymbol{D} - \lambda\boldsymbol{I}) \det(\boldsymbol{I} + (\boldsymbol{D} - \lambda\boldsymbol{I})^{-1}\boldsymbol{a}\boldsymbol{1}^T) \\ &= \prod_{k=1}^{N} (-K_k(1+\bar{r}_k) - \lambda) \left(1 - \sum_{k=1}^{N} \frac{K_k\bar{r}_k}{K_k(1+\bar{r}_k) + \lambda} \right). \end{aligned} \tag{13}$$

Assume first that $\alpha \le 1$, then $-1 < \bar{r}_k \le 0$ for all k. For the sake of simplicity, we assume that the $K_k(1+\bar{r}_k)$ values are different and $\bar{r}_k \neq 0$ for all k. The other case can be discussed in the same way. The eigenvalues are $\lambda = -K_k(1+\bar{r}_k) < 0$ and the roots of equation

$$g(\lambda) = \sum_{k=1}^{N} \frac{K_k \bar{r}_k}{K_k(1+\bar{r}_k)+\lambda} - 1 = 0. \tag{14}$$

It is easy to see that

$$\lim_{\lambda\to\pm\infty} g(\lambda) = -1, \quad \lim_{\lambda\to -K_k(1+\bar{r}_k)\pm 0} g(\lambda) = \mp\infty$$

and

$$g'(\lambda) = \sum_{k=1}^{N} \frac{-K_k \bar{r}_k}{\left(K_k(1+\bar{r}_k)+\lambda\right)^2} > 0,$$

so g strictly increases locally. All poles are negative, and since eq. (14) is equivalent to a polynomial equation of degree N, there are N real or complex roots. There is one root between each adjacent pair of poles and one additional root before the smallest pole. Therefore all roots are real and negative implying the local asymptotical stability of the equilibrium.

Assume next that $\alpha > 1$. Then $\bar{r}_k > 0$, and the eigenvalues of the Jacobian are $\lambda = -K_k(1+\bar{r}_k)$ and the roots of eq. (14). In this case $g'(\lambda) < 0$ so g strictly decreases locally. There is one root between each adjacent pair of poles and one additional root after the largest pole. All roots are negative if $g(0) < 0$ or

$$\sum_{k=1}^{N} \frac{\bar{r}_k}{1+\bar{r}_k} < 1. \tag{15}$$

Consider next discrete time scales. Then system (13) is replaced by the following system of difference equations:

$$x_k(t+1) = x_k(t) + K_k\left(R_k\left(\sum_{\ell\neq k}^{N} x_\ell(t)\right) - x_k(t)\right) \quad \text{for} \quad k = 1, 2, \ldots, N, \tag{16}$$

where $0 < K_k \le 1$. The Jacobian of this system is $\boldsymbol{I}+\boldsymbol{J}$, the eigenvalues of which are inside the unit circle if the eigenvalues of $\boldsymbol{J}$ are between -2 and 0. Assume first that $\alpha \le 1$. All eigenvalues of $\boldsymbol{J}$ are between -2 and 0 if for all k,

$$-K_k(1+\bar{r}_k) > -2$$

and $g(-2) < 0$. Hence the equilibrium is locally asymptotically stable if for all k,

$$K_k < \frac{2}{1+\bar{r}_k} \tag{17}$$

and

$$\sum_{k=1}^{N} \frac{K_k \bar{r}_k}{K_k(1+\bar{r}_k)-2} < 1. \tag{18}$$

In the case of $\alpha > 1$, all eigenvalues of $\boldsymbol{J}$ are between -2 and 0 if for all k,

$$-K_k(1+\bar{r}_k) > -2$$

and $g(0) < 0$. So the equilibrium is locally asymptotically stable if for all k,

$$K_k < \frac{2}{1+\bar{r}_k} \tag{19}$$

and

$$\sum_{k=1}^{N} \frac{\bar{r}_k}{1+\bar{r}_k} < 1. \tag{20}$$

We can summarize the above two cases by requiring that the speeds K_k of adjustments are sufficiently small.

Assume now that $N = 2$ and $\alpha = 2$. Since $0 \le \bar{r}_k < 1$,

$$\frac{\bar{r}_k}{1+\bar{r}_k} < \frac{\bar{r}_k}{2\bar{r}_k} = \frac{1}{2},$$

so relations (15) and (20) are satisfied. The additional condition for the discrete case, $K_k < 2/(1+\bar{r}_k)$ is also satisfied, since usually it is assumed that $K_k \le 1$. We have already seen that the equilibrium is unique, so this unique equilibrium is locally asymptotically stable.

If $N \ge 3$, the model may have multiple equilibria and the local stability of these equilibria may not be guaranteed as shown in the following example.

Example 2. Take $N = 3$. The model specified in Ex. 1 has three equilibria:

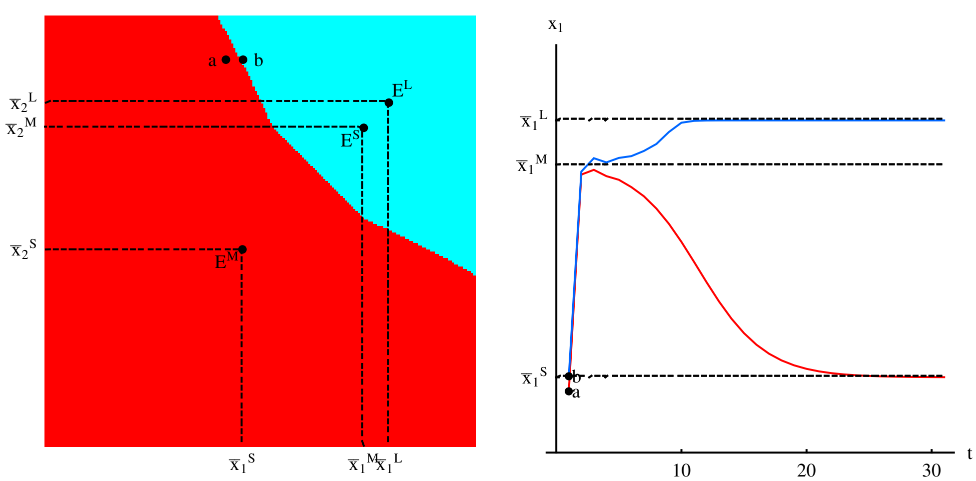
$$\bar{x}_k^S = 3 - \sqrt{0.5},\ \bar{x}_k^M = 3 + \sqrt{0.5} \ \text{ and } \ \bar{x}_k^L = L_k$$

which we call the smallest, middle and largest equilibria. It is apparent that the largest equilibrium is locally stable as $R_k'(Q_k) = 0$. To examine the stability of the other two equilibria, we calculate the derivative values evaluated at the equilibria,

$$\bar{r}_k^S = \frac{17}{(6+\sqrt{2})^2} \simeq 0.31 \ \text{ and } \ \bar{r}_k^M = \frac{17}{(-6+\sqrt{2})^2} \simeq 0.81.$$

According to (15) (or (20)), the smallest equilibrium is locally stable whereas the middle equilibrium is locally unstable.

Ex. 2 confirms the local stability of the smallest and the largest equilibrium but does not say anything about the global behavior of the trajectories. The basin of attraction for the discrete dynamic system (16) with $N = 3$, $K_k = 0.8$ for all k and $x_3(0) = 5.1$ is illustrated in Fig. 2. The shape of the basin depends on the value of $x_3(0)$. It is the set of points in the output space (x_1, x_2) such that initial points chosen in the red (darker) region converges to the smallest equilibrium and those in the blue (lighter) region evolve to the largest equilibrium. We choose two points, denoted as a in the red region and b in the blue region in Fig. 2A, and perform numerical simulations to confirm the convergence, which are depicted in Fig. 2B. In particular, although the initial point a is close to b, the trajectory starting at point a eventually converges to the smallest equilibrium and so does the trajectory starting at point b to the largest equilibrium.



(A) Basin of attraction (B) Globally stable time trajectories

Figure 2. Global stability

4. Conclusions

The existence of a special labor-managed oligopoly is proved by using fixed point theorems and also by introducing a solution algorithm. The uniqueness of the equilibrium and the local asymptotical stability of the equilibrium are not guaranteed in general. Conditions are derived for the local asymptotical stability of the equilibrium, and a simple numerical example illustrates the theoretical results.

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