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ON PSEUDOLINEAR FRACTIONAL FUNCTIONS – THE IMPLICIT FUNCTION APPROACH

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Dedicated to the memory of Tamás Rapcsák

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Abstract: The aim of the paper is to present a local method called "the implicit function approach" in characterizing pseudolinearity of functions with several variables. The local analysis is applied for special classes of quadratic fractional functions. The classes, investigating in the paper, has already been intensively studied by several authors, including Tamás Rapcsák.

1. Introduction

Fractional programming plays important role in several applied fields, such as economics, engineering, decision sciences, mathematical programming, etc. [5, 12, 24, 28, 29]. A celebrated class of fractional programming problems is the one, where the objective function enjoys certain kind of generalized convexity properties.

It was Martos [20, 21], who observed first that linear fractional functions show similar properties with linear ones, namely, linear fractional functions are pseudolinear over a specific part of their domain of defini-

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tion. This finding resulted in a development of Pseudolinear Programming, as a natural extension of Linear Programming. This development gave a firm impetus for elaborating a general theory of pseudolinearity [1–3, 9, 11, 14, 15, 17–19, 23–25, 31].

In fractional programming the pseudolinearity of the fractional objective function is one of the central questions. There are several nice results and growing interest on the mentioned field [4, 5, 7, 8, 22, 24, 26, 27, 30].

My aim with this paper is to present an alternative approach in investigating pseudolinearity of twice continuously differentiable functions and make use of it in case of fractional functions.

The concept of pseudolinearity comes from that of pseudoconvexity/pseudoconcavity. $f(x)$ is called pseudolinear by definition if $f(x)$ is both pseudoconvex and pseudoconcave.

Definition 1. The differentiable function $f(x)$ is called pseudolinear on the convex set $X \subseteq R^n$ if for all $x, y \in X$

$$\text{(PLIN)} \qquad \begin{aligned} &f(x) < f(y) \text{ implies } \nabla f(y)^T(x-y) < 0 \text{ and} \\ &f(x) > f(y) \text{ implies } \nabla f(y)^T(x-y) > 0. \end{aligned}$$

Condition (PLIN) has an obvious consequence: if $f(x)$ is pseudolinear on X, then either $f(x)$ is constant on X (consequently $\nabla f(x) = 0$ for all $x \in X$) or for all $x \in X$ we have $\nabla f(x) \neq 0$. It follows that if $f(x)$ is pseudolinear on X and there exists $x \in X$ such that $\nabla f(x) = 0$, then $f(x)$ is constant on X. Therefore the nontrivial case for pseudolinearity is the one, where $\nabla f(x) \neq 0$ for all $x \in X$.

2. Local analysis of pseudolinearity – the implicit-function approach

The approach carried out in the present paper for investigating pseudolinearity can be labeled as "the implicit-function approach" elaborated by the author in [15, 16]. This approach carries out a local analysis on the given function. To this end we have to "localize" the global concept of pseudolinearity. The following concept proved to be suitable for this approach.

Definition 2 [15]**.** $f(x)$ is *locally pseudolinear* at x_0 if there exists a neighborhood G of x_0 such that

$$x \in G,\ f(x) < f(x_0) \ \text{ implies } \ \nabla f(x_0)(x - x_0) < 0,$$
$$x \in G,\ f(x) = f(x_0) \ \text{ implies } \ \nabla f(x_0)(x - x_0) = 0,$$
$$x \in G,\ f(x) > f(x_0) \ \text{ implies } \ \nabla f(x_0)(x - x_0) > 0.$$

The usefulness of this concept lies in the fact that global analysis can be carried out via the local one on the basis of the following theorem.

Proposition 1 [15, 16]**.** *Let the differentiable function $f(x)$ be defined on the open convex set $X \subseteq R^n$. Then $f(x)$ is pseudolinear on X if and only if it is locally pseudolinear at each point of X.*

As the next theorem asserts, local pseudolinearity can be analyzed via a more simple condition.

Proposition 2 [15]**.** *Let $\nabla f(x_0) \neq 0$. Then $f(x)$ is locally pseudolinear at x_0 if and only if there exists a neighborhood G of x_0 such that the following condition holds:*

$$\text{(LPLIN)} \qquad x \in G,\ f(x) = f(x_0) \ \textit{ implies } \ \nabla f(x_0)(x - x_0) = 0.$$

Condition (LPLIN) opens the way for applying the well-known implicit-function theorem in our local analysis. Let $f(x)$ be defined and continuously differentiable on a neighborhood of $x_0 \in R^n$ and let $\nabla f(x_0) \neq 0$. Assume that $f'_{x_n}(x_0) \neq 0$. Introduce the following notations:

$$(x_1, x_2, \dots, x_{n-1}) = u, \quad x_n = v, \quad x = (u, v) \text{ and } x_0 = (u_0, v_0).$$

The implicit-function theorem. *The level curve $\{x \in X : f(x) = f(x_0)\}$ can be represented locally (on a certain neighborhood G of x_0) by the help of a uniquely determined implicit function $p_{x_0}(u)$ defined on a suitable neighborhood N of u_0, as follows:*

$$\textit{for all } \ x = (u, v) \in G, \quad f(x) = f(x_0) \ \textit{holds iff } v = p_{x_0}(u),\ u \in N.$$

According to the following theorem local pseudolinearity can be investigated by the help of the implicit function $p_{x_0}(u)$.

Proposition 3 [15]**.** *Let the continuously differentiable function $f(x)$ satisfy condition $\nabla f(x_0) \neq 0$. Then $f(x)$ is locally pseudolinear at x_0 if and only if the implicit function $p_{x_0}(u)$ is linear on N.*

From Prop. 2 and Prop. 3 one can infer the following Rapcsák type characterization of local pseudolinearity, the global version of which was proved in [25].

Proposition 4 [15]**.** *Let the continuously differentiable function $f(x)$ satisfy condition $\nabla f(x_0) \neq 0$. Then $f(x)$ is locally pseudolinear at x_0 if and only if there exists a neighborhood G of x_0, a function $c(x)$ defined* *on G and a nonzero vector $g \in R^n$ such that $c(x) \neq 0$ for each $x \in G$ and*

$$\text{(R)} \qquad x \in G, \quad f(x) = f(x_0) \ \ \textit{implies} \ \ \nabla f(x) = c(x)g.$$

The key tool in our further investigation is a simple consequence of Prop. 3.

Lemma 1. *Let the twice continuously differentiable function $f(x)$ satisfy condition $\nabla f(x_0) \neq 0$. Then $f(x)$ is locally pseudolinear at x_0 if and only if $p_{x_0}(u)$ (in short: $p(u)$) fulfils the following condition: for all $u \in N$*

(1)
$$\nabla^2_{uu} f(u, p(u)) + 2\nabla_u f'_v(u, p(u))\nabla p(u)^T + f''_{vv}(u, p(u))\nabla p(u)\nabla p(u)^T \equiv 0.$$

Proof. If we derivate twice the implicit equation $f(u, p(u)) \equiv f(x_0)$, we get $\nabla^2 p(u)$ the Hessian of the implicit function $p(u)$,

$$f'_v(u, p(u))\nabla^2 p(u) =$$
$$= -\nabla^2_{uu} f(u, p(u)) - 2\nabla_u f'_v(u, p(u))\nabla p(u)^T - f''_{vv}(u, p(u))\nabla p(u)\nabla p(u)^T.$$

It follows that (1) holds if and only if $\nabla^2 p(u) \equiv 0$ for all $u \in N$, which holds true if and only if $p(u)$ is linear on N. $\Diamond$

Theorem 1. *Let us suppose that the twice continuously differentiable function $f(x)$ is locally pseudolinear at $x_0 \in X$, where $\nabla f(x_0) \neq 0$. Assume for the sake of simplicity that $f'_{x_n}(x_0) \neq 0$. Then there exist $r \in R^{n-1}$ and $\lambda, \mu \in R$ such that*

$$\text{(2)} \qquad \nabla^2 f(x_0) = \begin{bmatrix} \lambda r r^T & \mu r \\ \mu r^T & -(\lambda + 2\mu) \end{bmatrix}.$$

Proof. Consider the implicit function $p(u)$ which describes the level curve $f(x) = f(x_0)$ around x_0. By Prop. 3 $p(u)$ is linear and by Lemma 1, the following conditions hold: for all $u \in N$ one has

$$\text{(3)} \qquad \nabla p(u) = -\frac{\nabla_u f(u, p(u))}{f'_v(u, p(u))} \equiv r = \text{const},$$

and

(4)
$$\nabla^2_{uu} f(u, p(u)) \equiv -2\nabla_u f'_v(u, p(u))^T \nabla p(u) - f''_{vv}(u, p(u))\nabla p(u)^T \nabla p(u).$$

Let us consider matrix $A = \nabla^2 f(x_0)$ in its (u, v) decomposition:

$$\nabla^2 f(x_0) = A = \begin{bmatrix} A_{uu} & a_u \\ a_u^T & a_{vv} \end{bmatrix},$$

which is in harmony with the (u, v) decomposition of x. Based on this decomposition, a simple computation yields the following formulas:

$$\nabla^2_{uu} f(u, p(u)) = A_{uu}, \quad \nabla_u f'_v(u, p(u)) \nabla p(u)^T = a_u \nabla p(u)^T = a_u r^T,$$

$$f''_{vv}(u, p(u)) \nabla p(u) \nabla p(u)^T = a_{vv} \nabla p(u) \nabla p(u)^T = a_{vv} r r^T.$$

Taking (4) into account, it follows that $A_{uu} = -(2a_u + a_{vv} r) r^T$. Since A_{uu} is a symmetric submatrix in A therefore $-2a_u - a_{vv} r = \lambda r$ should hold for some $\lambda \in R$ and thus $A_{uu} = \lambda r r^T$, and $a_u = \mu r$ with $\mu = -\frac{\lambda + a_{vv}}{2}$. From the last equation it follows that $a_{vv} = -(\lambda + 2\mu)$. $\diamondsuit$

Investigating further the special form of $A = \nabla^2 f(x_0)$, one can obtain more useful information on it. The analysis is based on the Haynsworth's inertia theorem [13]. The inertia of a symmetric matrix A is defined to be the triple $(\nu_-(A), \nu_0(A), \nu_+(A))$, where $\nu_-(A)$, $\nu_0(A)$, $\nu_+(A)$ are respectively the numbers of negative, zero and positive eigenvalues of A counted with multiplicities. $\mathrm{Iner}(A) = (\nu_-(A), \nu_0(A), \nu_+(A))$.

The Haynsworth's inertia theorem uses the concept of Schur complement, which is linked to the following partitioning of the symmetric matrix A:

$$A = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix},$$

where P is a nonsingular submatrix. Matrix S,

$$S = R - Q^T P^{-1} Q$$

is called the Schur complement of P in A. The Haynsworth's inertia theorem says that if the nonsingular P submatrix is principal in A, then

$$\mathrm{Iner}(A) = \mathrm{Iner}(P) + \mathrm{Iner}(S), \tag{5}$$

where addition means componentwise addition. The principality of P ensures that P and S are symmetric matrices. Since the Schur complement plays a central role in any pivot-algorithms, therefore it is possible, by means of equation (5), to evaluate the inertia of any symmetric matrix by a sequence of pivot transformations, following a special pivoting rule. The details have been elaborated by R. W. Cottle in [10].

Theorem 2. *Let us suppose that the twice continuously differentiable function $f(x)$ is locally pseudolinear at $x_0 \in X$, where $\nabla f(x_0) \neq 0$. Assume that $A = \nabla^2 f(x_0) \neq 0$ and A admits partitioning* (2)*.*

If $r \neq 0$, then the following statements hold true:

(i) *In case of $\lambda + \mu \neq 0$* $\mathrm{Iner}(A) = (1, n-2, 1)$.

(iia) *In case of $\lambda + \mu = 0$ and $\lambda + 2\mu > 0$* $\mathrm{Iner}(A) = (1, n-1, 0)$.

(iib) *In case of* $\lambda + \mu = 0$ *and* $\lambda + 2\mu < 0$ $\mathrm{Iner}(A) = (0, n-1, 1)$.

If $r = 0$, *then* $\mathrm{Iner}(A) = (1, n-1, 0)$ *if* $\lambda + 2\mu > 0$ *and* $\mathrm{Iner}(A) = (0, n-1, 1)$ *if* $\lambda + 2\mu < 0$.

Proof. (ia) Consider first the subcase when $\lambda + 2\mu \neq 0$ and choose $P = [-(\lambda + 2\mu)]$ in the Haynsworth inertia formula. The Schur complement formula is resulted in

$$S = \left[\frac{(\lambda+\mu)^2}{\lambda + 2\mu} rr^T\right].$$

Since $r \neq 0$, therefore $\mathrm{Iner}(rr^T) = (0, n-2, 1)$ and thus

$$\mathrm{Iner}(S) = \mathrm{Iner}\left[\frac{(\lambda+\mu)^2}{\lambda + 2\mu}\right] + (0, n-2, 0).$$

Since

$$\mathrm{Iner}(A) = \mathrm{Iner}(P) + \mathrm{Iner}(S) =$$
$$= \mathrm{Iner}[-(\lambda + 2\mu)] + \mathrm{Iner}\left[\frac{(\lambda+\mu)^2}{\lambda + 2\mu}\right] + (0, n-2, 0),$$

the thesis follows.

(ib) Assume now that $\lambda + 2\mu = 0$. In this case $\lambda \neq 0$ must hold. Choose $P_1 = [\lambda r_i^2]$, with $r_i \neq 0$. For the sake of simplicity we may assume that $i = 1$. The Schur complement formula gives us in this case

$$S_1 = \begin{bmatrix} 0 & 0 \\ 0^T & -\lambda/4 \end{bmatrix}.$$

Choose $P_2 = [-\lambda/4]$. The Schur complement of P_2 in S_1 is the 0 matrix of order $(n-2)$ and thus

$$\mathrm{Iner}(A) = \mathrm{Iner}\lfloor \lambda r_i^2 \rfloor + \mathrm{Iner}[-\lambda/4] + (0, n-2, 0) = (1, n-2, 1).$$

(ii) Choose $P = [-(\lambda + 2\mu)]$ in the Haynsworth inertia formula. The Schur complement formula provides us with

$$S = \left[\frac{(\lambda+\mu)^2}{\lambda + 2\mu} rr^T\right] = 0.$$

Since

$$\mathrm{Iner}(A) = \mathrm{Iner}(P) + \mathrm{Iner}(S) = \mathrm{Iner}[-(\lambda + 2\mu)] + (0, n-1, 0),$$

the thesis follows.

The case with $r = 0$ is similar to that of (ii). $\Diamond$

Theorem 3. *Let us suppose that the twice continuously differentiable function* $f(x)$ *is locally pseudolinear at* $x_0 \in X$, *where* $\nabla f(x_0) \neq 0$. *Assume that* $A = \nabla^2 f(x_0) \neq 0$ *and admits partitioning* (2). *Let* $g^T = [r^T - 1]$. *Then the following statements hold true:*

(i) $\operatorname{rank}(A)$ *equals to either* 1 *or* 2*;*

(ii) *if* $\operatorname{rank}(A) = 1$, *then* $\operatorname{range}(A) = \operatorname{Lin}\{g\}$, *if* $\operatorname{rank}(A) = 2$, *then* $\operatorname{range}(A) = \operatorname{Lin}\{g, Ag\}$ *(here* $\operatorname{range}(A) = A(R^n)$ *and* $\operatorname{Lin}\{\,.\,\}$ *denotes the linear hull of the given vectors);*

(iii) $\nabla f(x_0) \in \operatorname{Lin}\{g\}$*;*

(iv) *if* $\operatorname{rank}(A) = 2$, *then* $Ax = \nabla f(x_0)$ *implies* $x^T \nabla f(x_0) = = x^T Ax = 0$, *and* $x^T \nabla f(x_0) = 0$ *implies* $x^T Ax = 0$.

Proof. (i) Since $\operatorname{rank}(A) = \nu_-(A) + \nu_+(A)$, it is clear from Th. 2 that $\operatorname{rank}(A) = 1$ or 2.

(ii) By Th. 2, $\operatorname{rank}(A) = 1$ is equivalent to any of the following two conditions: either one has $r \neq 0$ and $\lambda + \mu = 0$ or $r = 0$. Simple calculation shows that in both of the two cases it is

$$A = \kappa \begin{bmatrix} rr^T & -r \\ -r^T & 1 \end{bmatrix} = \kappa \begin{bmatrix} r \\ -1 \end{bmatrix} [r^T - 1],$$

with $\kappa = \lambda$ in the first case and with $\kappa = -(\lambda + 2\mu)$ in the second case. It follows that $A = \kappa g g^T$, and $\operatorname{range}(A) = \operatorname{Lin}\{g\}$.

By Th. 2 $\operatorname{rank}(A) = 2$ is equivalent to $r \neq 0$ and $\lambda + \mu \neq 0$. First we show that $g \in \operatorname{range}(A)$, that is the problem of $Ax = g$ is solvable in $x \in R^n$. Consider x, g and A in their (u, v) decomposition, where $u \in R^{n-1}$ and $v \in R$.

$$Ax = \begin{bmatrix} \lambda rr^T & \mu r \\ \mu r^T & -(\lambda + 2\mu) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \big(\lambda(r^T u) + \mu v\big) r \\ \mu(r^T u) - (\lambda + 2\mu) v \end{bmatrix}. \tag{6}$$

From this decomposition it follows that $Ax = g$ holds if and only if $x = (u, v)$ satisfies the following linear system:

$$\begin{aligned} \lambda(r^T u) + \mu v &= 1, \\ \mu(r^T u) - (\lambda + 2\mu) v &= -1. \end{aligned} \tag{7}$$

Since the determinant of this system equals to $\lambda + \mu$, simple computation shows that there is a unique solution in $r^T u$ and v:

$$r^T u = v = \frac{1}{\lambda + \mu}. \tag{8}$$

Condition (8) provides us with one more important fact, namely

$$\text{if} \quad Ax = g, \quad \text{then} \quad g^T x = r^T u - v = 0. \tag{9}$$

Introduce now

$$\hat{g} = \frac{1}{(\lambda+\mu)r^T r}\begin{bmatrix} r \\ r^T r \end{bmatrix}.$$

A simple computation shows that $A\hat{g} = g$ and thus $g \in \mathrm{range}(A)$.

Now we show that g and Ag are linearly independent. As a first step we show that $Ag \neq 0$. Assume for contradiction that $Ag = 0$. Consider this equation in its (u, v) decomposition.

$$Ag = \begin{bmatrix} \lambda r r^T & \mu r \\ \mu r^T & -(\lambda + 2\mu) \end{bmatrix} \begin{bmatrix} r \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda(r^T r)r - \mu r \\ \mu(r^T r) + (\lambda + 2\mu) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This condition holds iff

$$\lambda r^T r - \mu = 0,$$
$$\mu r^T r + \lambda + 2\mu = 0,$$

and

$$\lambda\mu r^T r = \mu^2 = -\lambda^2 - 2\lambda\mu,$$

from which $(\lambda + \mu)^2 = 0$. Since $\lambda + \mu \neq 0$, it is impossible. This contradiction proves the thesis.

As a second step we prove that g and Ag are linearly independent. Assume for the contrary that $Ag = \alpha g$ with some $\alpha \neq 0$. Let $\delta = 1/\alpha \neq 0$. According to our hypothesis $A(\delta g) = g$ should hold. But as we have already proved in (9) it follows that $g^T(\delta g) = \delta g^T g = 0$. Since $g \neq 0$ and $\delta \neq 0$ it is impossible. It proves that g and Ag are linearly independent. Since $\mathrm{rank}(A) = 2$, it follows that $\mathrm{range}(A) = \mathrm{Lin}\{g, Ag\}$.

(iii) From (3) one has that

$$\nabla f(x_0)^T = -f'_v(x_0)[r^T - 1] = -f'_v(x_0)g^T$$

where $f'_v(x_0) \neq 0$ and thus $\nabla f(x_0) \in \mathrm{Lin}\{g\}$.

(iv) Consider now the case when $\mathrm{rank}(A) = 2$. Now we prove that $Ax = \nabla f(x_0)$ implies $\nabla f(x_0)^T x = 0$ and $\nabla f(x_0)^T x = 0$ implies $x^T A x = 0$. Let $x = (u, v)$. If x is a solution of $Ax = g$, then by (8) we have that $g^T x = r^T u - v = 0$. Since $\nabla f(x_0) = \eta g$, $Ax = \nabla f(x_0)$ implies $\nabla f(x_0)^T x = 0$. Assume now that $\nabla f(x_0)^T x = 0$, which is equivalent to $g^T x = r^T u - v = 0$. From (8) it follows that $Ax = (\lambda + \mu)g$ and thus $x^T A x = (\lambda + \mu)g^T x = 0$. $\Diamond$

The following theorem presents a pure matrix algebraic result and prepares the way for further investigations.

Proposition 5. *Let A be a symmetric matrix of order n, with* $\mathrm{Iner}(A) =$ $= (1, n-2, 1)$. *Then for any vector $d \in R^n$, $d \neq 0$, such that $d = A\hat{d}$ and $d^T\hat{d} = 0$ for some $\hat{d} \in R^n$, there exists a unique $p \in R^n$, satisfying*

$$A = pd^T + dp^T. \tag{10}$$

Vector p admits the following properties: $p = A\hat{p}$ and $p^T\hat{p} = 0$, for some $\hat{p} \in R^n$ and $p^T\hat{d} = d^T\hat{p} = 1$.

Proof. Let $s_1, s_2, s_3, \ldots, s_n$ be a basis in R^n consisting of orthonormal eigenvectors of A. Assume that $As_1 = \sigma_1 s_1$, with $\sigma_1 < 0$, $As_2 = \sigma_2 s_2$, with $\sigma_2 > 0$. It follows that range$(A) = \text{Lin}\{s_1, s_2\}$.

Let $d \in R^n$ be given with the requested property. Then $d \in \text{range}(A) = \text{Lin}\{s_1, s_2\}$, and we have a unique decompositions for d as follows:

$$d = \delta_1 s_1 + \delta_2 s_2.$$

Since $d = A\hat{d}$ and $d^T\hat{d} = 0$, we may assume without loss of the generality that $\hat{d} \in \text{Lin}\{s_1, s_2\}$. It follows that

$$\hat{d} = \frac{\delta_1}{\sigma_1}s_1 + \frac{\delta_2}{\sigma_2}s_2 \quad \text{and} \quad d^T\hat{d} = \frac{\delta_1^2}{\sigma_1} + \frac{\delta_2^2}{\sigma_2} = 0.$$

Since $d \neq 0$, it follows that $\delta_1 \neq 0$ and $\delta_2 \neq 0$. Let

$$p = \frac{\sigma_1}{2\delta_1}s_1 + \frac{\sigma_2}{2\delta_2}s_2 \quad \text{and} \quad \hat{p} = \frac{1}{2\delta_1}s_1 + \frac{1}{2\delta_2}s_2.$$

A simple calculation shows that $p = A\hat{p}$,

$$p^T\hat{p} = \frac{\sigma_1}{4\delta_1^2} + \frac{\sigma_2}{4\delta_2^2} = \frac{4\sigma_1\sigma_2}{\delta_1^2\delta_2^2}\left(\frac{\delta_1^2}{\sigma_1} + \frac{\delta_2^2}{\sigma_2}\right) = \frac{4\sigma_1\sigma_2}{\delta_1^2\delta_2^2}d^T\hat{d} = 0,$$

and

$$d^T\hat{p} = \delta_1\frac{1}{2\delta_1} + \delta_2\frac{1}{2\delta_2} = 1 \quad \text{and} \quad p^T\hat{d} = \frac{\sigma_1}{2\delta_1}\frac{\delta_1}{\sigma_1} + \frac{\sigma_2}{2\delta_2}\frac{\delta_2}{\sigma_2} = 1.$$

Now we show that $A = pd^T + dp^T$. Consider the nonsingular matrix $S = \begin{bmatrix} s_1 & s_2 & \ldots & s_n \end{bmatrix}$. By construction one has $S^{-1} = S^T$ and thus $A = pd^T + dp^T$ holds if and only if

$$S^TAS = (S^Tp)(d^TS) + (S^Td)(p^TS).$$

Both matrices can be written in a partitioned form, as follows:

$$S^TAS = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad P = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix},$$

and

$$(S^Tp)(d^TS) + (S^Td)(p^TS) = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix},$$

$$\text{where} \quad Q = \begin{bmatrix} 2\frac{\sigma_1}{2\delta_1}\delta_1 & \frac{\sigma_1}{2\delta_1}\delta_2 + \frac{\sigma_2}{2\delta_2}\delta_1 \\ \frac{\sigma_1}{2\delta_1}\delta_2 + \frac{\sigma_2}{2\delta_2}\delta_1 & 2\frac{\sigma_2}{2\delta_2}\delta_2 \end{bmatrix}.$$

It is quite obvious that $P = Q$ holds if and only if

$$\frac{\sigma_1}{2\delta_1}\delta_2 + \frac{\sigma_2}{2\delta_2}\delta_1 = 0.$$

Let us notice that

$$\frac{\sigma_1}{2\delta_1}\delta_2 + \frac{\sigma_2}{2\delta_2}\delta_1 = \frac{\sigma_1\sigma_2}{2\delta_1\delta_2}\left(\frac{\delta_1^2}{\sigma_1} + \frac{\delta_2^2}{\sigma_2}\right) = \frac{\sigma_1\sigma_2}{2\delta_1\delta_2} d^T\hat{d} = 0,$$

which proves that $P = Q$ and $A = pd^T + dp^T$. $\Diamond$

3. Pseudolinearity of quadratic fractional functions

Consider now the case when

$$f(x) = \frac{\frac{1}{2}x^T Bx + b^T x + \beta}{\frac{1}{2}x^T Cx + c^T x + \gamma}, \quad x \in X, \tag{11}$$

where $X \subseteq R^n$ is an open convex set, B, C are nonzero symmetric matrices of order n; $b, c \in R^n$; $\beta, \gamma \in R$, and $\frac{1}{2}x^T Cx + c^T x + \gamma > 0$ on X. We are going to seek conditions on the input parameters B, C, b, c and β, γ ensuring pseudolinearity of $f(x)$.

3.1. The Cambini–Carosi Theorem

In the literature you can find important results concerning special classes of quadratic fractional functions. The one, where the nominator is linear attracted more attention. [7, 26, 27, 30]. Consider now the case when

$$f(x) = \frac{\frac{1}{2}x^T Ax + a^T x + \alpha}{b^T x + \beta}, \quad x \in X \tag{12}$$

where $X \subseteq R^n$ is an open convex set, $A \neq 0$ is a quadratic and symmetric matrix of order n; $a, b \in R^n$, $b \neq 0$; $\alpha, \beta \in R$ and $b^T x + \beta > 0$ on X.

Riccardo Cambini and Laura Carosi found in [7] the following characterization of pseudolinearity for functions of form (12).

Proposition 6 [7]**.** *Function f defined in* (12) *is pseudolinear on $X = \{x \in R^n : b^T x + \beta > 0\}$ if and only if f is linear or there exist constants $\hat{\alpha} \neq 0$, $\hat{\beta}$ and $\hat{\gamma}$ such that $\hat{\alpha}\hat{\gamma} < 0$ and*

$$f(x) = \hat{\alpha} b^T x + \hat{\beta} + \frac{\hat{\gamma}}{b^T x + \beta}.$$

Later on, using different approach, Tamás Rapcsák presented a different characterization [26].

Proposition 7 [26]**.** *Function f defined in* (12) *is pseudolinear on* $X = \{x \in R^n : b^T x + \beta > 0\}$ *if and only if f is linear or there exist constants* $\widetilde{\alpha} \neq 0$, $\widetilde{\beta}$ *such that*

$$A = \widetilde{\alpha} b b^T, \quad \textit{and} \quad a = \widetilde{\beta} b.$$

Now we show that both characterizations can be obtained by the help of our local analysis, moreover we can prove both statements under milder conditions. Since our analysis is based on Lemma 1, and for this we have to compute and work with the first and second order partial derivatives of function $f(x)$ defined in (12), therefore the following reduction scheme may avoid us from unnecessary technical difficulties.

A reduction scheme for fractional functions. Let us consider the fractional function

$$f(x) = \frac{g(x)}{h(x)}, \quad x \in X,$$

where X is an open convex set in R^n, functions $g(x)$ and $h(x)$ are continuously differentiable over X, and $h(x) > 0$ on X. In case of fractional functions the direct application of condition (1) is rather difficult, but you may overcome the technical difficulties in the following way.

Let us notice that the solution set of the level curve equation $f(x) = f(x_0)$ is the same as the solution set to the level curve equation $\varphi(x) = \varphi(x_0)$, where

$$\varphi(x) = g(x) - \frac{g(x_0)}{h(x_0)} h(x),$$

and thus one can replace function $f(x)$ with $\varphi(x)$ in condition (1). The precise statement based on the previous reasoning reads as follows.

Lemma 2. *The fractional function $f(x)$ is locally pseudolinear at x_0, where $\nabla f(x_0) \neq 0$, if and only if the auxiliary function $\varphi(x)$ is locally pseudolinear at x_0.*

From technical point of view function $\varphi(x)$ admits more simple first and second order partial derivatives than $f(x)$ does. This reduction gives us almost trivially the well-known result on linear fractional functions. The case with quadratic fractional function is not so simple. We shall demonstrate the efficiency and easy applicability of our method in the sequel for quadratic fractional function of form of (12). The only thing we have to take into consideration is that the auxiliary function $\varphi(x)$ is defined as follows:

$$\varphi_{x_0}(x) = \frac{1}{2}x^T Ax + a^T x + \alpha - f(x_0)(b^T x + \beta). \tag{13}$$

The most favorable property of this quadratic function is that its Hessian is independent from the choice of x_0, namely for all x_0, $x \in X$ one has $\nabla^2 \varphi_{x_0}(x) \equiv A$ and thus local analysis can be easily extended to global one.

The next proposition will be of use in the sequel.

Proposition 8. *Let* $f(x)$ *defined in* (12) *be locally pseudolinear at least at two different places with different function values and nonvanishing gradients. Then* $a, b \in \operatorname{range}(A)$.

Proof. Let us suppose that $f(x)$ is locally pseudolinear at x_1, x_2, where the gradients are nonvanishing and the function values are different. By Lemma 2 it follows that $\varphi_{x_i}(x)$ is locally pseudolinear at x_i, $i = 1, 2$. Taking into account (iii) of Th. 3 one has
$\nabla\varphi_{x_i}(x_i) = Ax_i + a - f(x_i)b \in \operatorname{range}(A)$ and thus $a - f(x_i)b \in \operatorname{range}(A)$ for $i = 1, 2$. Since $f(x_1) \neq f(x_2)$ it follows that $a, b \in \operatorname{range}(A)$. $\Diamond$

The next result plays also a crucial role in our further analysis. It has actually been derived by Cambini–Carosi in [7], developing some idea from [4. Prop. 2.3] and assuming (global) pseudolinearity.

Proposition 9. *Let* $f(x)$ *defined in* (12) *be locally pseudolinear at least at three different places with different function values and nonvanishing gradients. Assume that* $\operatorname{rank}(A) = 2$. *Then there exist* $\hat{a}, \hat{b} \in R^n$ *such that* $a = A\hat{a}$, $b = A\hat{b}$ *and*

$$b^T \hat{b} = 0, \quad \beta = a^T \hat{b} \ \text{ and } \ 2\alpha = a^T \hat{a}. \tag{14}$$

Proof. Since by Prop. 8 $a, b \in \operatorname{range}(A)$, therefore there exist $\hat{a}, \hat{b} \in R^n$ with $a = A\hat{a}$, $b = A\hat{b}$. It follows that for all $i = 1, 2, 3$ one has
$\nabla\varphi_{x_i}(x_i) = Ax_i + a - f(x_i)b = Ax_i + A\hat{a} - f(x_i)A\hat{b} = A\big(x_i + \hat{a} - f(x_i)\hat{b}\big)$.
According to (iv) of Th. 3 $Ax = \nabla\varphi_{x_i}(x_i)$ implies $x^T \nabla\varphi_{x_i}(x_i) = 0$, so we have arrived at the following condition:

$$\begin{aligned}&\big(x_i + \hat{a} - f(x_i)\hat{b}\big)^T \big(Ax_i + a - f(x_i)b\big) = \\ &= b^T \hat{b} f^2(x_i) + 2(\beta - a^T \hat{b}) f(x_i) + a^T \hat{a} - 2\alpha = 0,\end{aligned}$$

which shows that $\lambda_i = f(x_i)$, $i = 1, 2, 3$ are three different solutions of the quadratic equation:

$$b^T\hat{b}\lambda^2 + 2(\beta - a^T\hat{b})\lambda + a^T\hat{a} - 2\alpha = 0,$$

which is possible only if when every coefficient is equal to 0, proving (14). $\Diamond$

Theorem 4. *Let $f(x)$ be defined in* (12) *and suppose that* $\mathrm{rank}(A) = 1$. *Assume that $f(x)$ is locally pseudolinear at least at two different places, with different function values and nonvanishing gradients. Then $f(x)$ is locally pseudolinear at x_0, where $\nabla f(x_0) \neq 0$, if and only if there exist constants $\hat{\alpha} \neq 0$, $\hat{\beta}$ and $\hat{\gamma}$ such that*

$$f(x) = \hat{\alpha}b^T x + \hat{\beta} + \frac{\hat{\gamma}}{b^T x + \beta}, \tag{15}$$

and

$$\hat{\gamma} \neq \hat{\alpha}(b^T x_0 + \beta)^2. \tag{16}$$

Proof. *Necessity.* A simple consequence of Th. 3 and Prop. 8 is the existence of constants $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that

$$A = \widetilde{\alpha}bb^T \quad \text{and} \quad a = \widetilde{\beta}b.$$

By using this information we can get (15) with

$$\hat{\alpha} = \frac{\widetilde{\alpha}}{2}, \quad \hat{\beta} = \widetilde{\beta} - \beta\frac{\widetilde{\alpha}}{2} \quad \text{and} \quad \hat{\gamma} = \alpha - \beta\widetilde{\beta} + \beta^2\frac{\widetilde{\alpha}}{2}.$$

Since

$$\nabla f(x) = \left(\hat{\alpha} - \frac{\hat{\gamma}}{(b^T x + \beta)^2}\right)b, \tag{17}$$

$b \neq 0$ and $\nabla f(x_0) \neq 0$, therefore condition (16) should hold.

Sufficiency. (16) and (17) show that $f(x)$ in (15) satisfies Rapcsák's condition (R) of Prop. 4, which is sufficient for local pseudolinearity at x_0. $\Diamond$

Remark. If $f(x)$ is pseudolinear on $X = \{x \in R^n : b^T x + \beta > 0\}$ then condition (16) holds for all $x \in X$ if and only if $\hat{\alpha}\hat{\gamma} < 0$.

Theorem 5. *Let $f(x)$ be defined in* (12) *and suppose that* $\mathrm{rank}(A) = 2$. *Assume that $f(x)$ is locally pseudolinear at least at three different places with different function values and nonvanishing gradients. Assume that* $\mathrm{rank}(A) = 2$. *Then there exist $p \in R^n$ and $\pi \in R$ such that $f(x)$ can be rewritten as*

$$f(x) = p^T x + \pi.$$

Proof. Since by Th. 2 rank$(A) = 2$ is equivalent to have Iner$(A) = = (1, n-2, 1)$, Prop. 9 ensures the existence of $\hat{a}, \hat{b} \in R^n$ satisfying

$$A\hat{a} = a, \quad a^T\hat{a} = 2\alpha, \quad A\hat{b} = b, \quad b^T\hat{b} = 0 \;\text{ and }\; a^T\hat{b} = \beta. \tag{18}$$

By Prop. 5 there exists $p \in R^n$ such that

$$A = bp^T + pb^T. \tag{19}$$

It follows from decomposition (19) and condition (18) that

$$a = A\hat{a} = (p^T\hat{a})b + (b^T\hat{a})p = \pi b + \beta p,$$

and

$$\alpha = \frac{1}{2}a^T\hat{a} = \frac{1}{2}(\pi b^T\hat{a} + \beta p^T\hat{a}) = \frac{1}{2}(\pi\beta + \beta\pi) = \beta\pi,$$

where $\pi = p^T\hat{a}$. Using these equations one can deduce that

$$f(x) = \frac{\frac{1}{2}x^TAx + a^Tx + \alpha}{b^Tx + \beta} = \frac{(b^Tx)(p^Tx) + \pi b^Tx + \beta p^Tx + \beta\pi}{b^Tx + \beta} = p^Tx + \pi,$$

and that was to be proved. $\Diamond$

3.2. The Carosi–Martein Theorem

L. Carosi and L. Martein [8] investigated the pseudoconvexity and pseudolinearity of the following class of quadratic fractional functions

$$f(x) = \frac{\frac{1}{2}x^TBx + b^Tx + \beta}{(c^Tx + \gamma)^2}, \quad x \in X \tag{20}$$

where $X \subseteq R^n$ is an open convex set, $B \neq 0$ is a quadratic and symmetric matrix of order n; $b, c \in R^n$, $c \neq 0$; $\beta, \gamma \in R$ and $(c^Tx + \gamma)^2 > 0$ on X.

Unfortunately the application of the reduction scheme proved to be efficient for investigating fractional functions of form (12) could only be carried out with serious technical difficulties for functions of form (20). Fortunately there exists another possibility of reducing the analysis of function (20) to the analysis of an auxiliary function with much more simple structure. This reduction scheme applies a special transformation of the independent variable, called *the Charnes–Cooper transformation*, which is defined as

$$y(x) = \frac{x}{c^Tx + \gamma}, \tag{21}$$

whose inverse is

$$x(y) = \frac{\gamma y}{1 - c^T y}. \tag{22}$$

This transformation shares a remarkable property, it preserves pseudoconvexity/concavity and thus pseudolinearity [6, 8]. It follows that this transformation preserves local pseudolinearity, as well.

Apply now the Charnes–Cooper transformation on $f(x)$ defined in (20). The result is a pure quadratic function.

$$\varphi(y) = \frac{1}{2} y^T A y + a^T y + \alpha, \tag{23}$$

where

$$A = B + \frac{2\beta}{\gamma^2} cc^T - \frac{1}{\gamma}(cb^T + bc^T), \quad a = \frac{1}{\gamma} b - \frac{2\beta}{\gamma^2} c \ \text{ and } \ \alpha = \frac{\beta}{\gamma^2}. \tag{24}$$

Since the Charnes–Cooper transformation preserves pseudolinearity therefore local pseudolinearity of $f(x)$ in (20) can be tested by testing local pseudolinearity of $\varphi(y)$ in (23).

Proposition 10. *Let us suppose that* $\varphi(y)$ *defined in* (23) *is locally pseudolinear at* $y_0 \in Y$*, where* $\nabla\varphi(y_0) \neq 0$*. Then the following statements hold true:*

(i) $\operatorname{rank}(A)$ *equals to either* 0*,* 1 *or* 2*.*

(ii) *In case of* $\operatorname{rank}(A) = 0$

$$\varphi(y) = a^T y + \alpha.$$

(iii) *In case of* $\operatorname{rank}(A) = 1$*, there exists* $\kappa \in R$ *such that*

$$\varphi(y) = \kappa(a^T y)^2 + a^T y + \alpha.$$

(iv) *In case of* $\operatorname{rank}(A) = 2$ *function* $\varphi(y)$ *can not be locally pseudolinear at two different points with different function values.*

Proof. (i) follows from Th. 3. (ii) holds iff $A = 0$ and it proves the thesis.

(iii) Consider now the case when $\operatorname{rank}(A) = 1$. Then by Th. 3 $A = \lambda g g^T$, $\nabla\varphi(y_0) = \tau g$ and $a = \psi g$ with some $\lambda, \tau, \psi \in R$. It follows that $A = 2\kappa a a^T$ and

$$\varphi(y) = \kappa(a^T y)^2 + a^T y + \alpha.$$

(iv) Consider the case when $\operatorname{rank}(A) = 2$. By Th. 3 $\nabla\varphi(y_0) \in \operatorname{range}(A)$, and it follows that $a = \nabla\varphi(y_0) - Ay_0 \in \operatorname{range}(A)$ and thus $A\hat{a} = a$ holds with some $\hat{a} \in R^n$. Since

$$\nabla\varphi(y_0) = Ay_0 + a = A(y_0 + \hat{a}),$$

therefore from (iv) of Th. 3 it follows that

$$(Ay_0 + a)^T(y_0 + \hat{a}) = y_0^T Ay_0 + 2a^T y_0 + a^T\hat{a} = 2\varphi(y_0) - 2\alpha + a^T\hat{a} = 0,$$

which is equivalent to

$$2\varphi(y_0) = 2\alpha - a^T\hat{a}. \tag{25}$$

Condition (25) has an important consequence. Function $\varphi(y)$ can not be locally pseudolinear at two different points with different function values. Assume on the contrary that there exist y_0 and y_1, with $\nabla\varphi(y_0) \neq 0$, $\nabla\varphi(y_1) \neq 0$ and $\varphi(y_0) \neq \varphi(y_1)$. Then taking into account necessary condition (25) the following equations should hold:

$$2\varphi(y_0) = 2\alpha - a^T\hat{a} \quad \text{and} \quad 2\varphi(y_1) = 2\alpha - a^T\hat{a}.$$

Since these equations hold true iff $\varphi(y_0) = \varphi(y_1)$, the thesis follows. $\Diamond$

Theorem 6. *Let $f(x)$ defined in* (20) *be locally pseudolinear at two different points with nonvanishing gradients and different function values. Then $f(x)$ can be rewritten in one of the following two forms:*

$$f(x) = \frac{a^T x}{c^T x + \gamma} + \frac{\beta}{\gamma^2},$$

$$f(x) = \kappa\frac{(a^T x)^2}{(c^T x + \gamma)^2} + \frac{a^T x}{c^T x + \gamma} + \frac{\beta}{\gamma^2},$$

where

$$a = \frac{1}{\gamma}b - \frac{2\beta}{\gamma^2}c.$$

Proof. Based on Prop. 10 we have to consider only the following two cases: $\operatorname{rank}(A) = 0$ or $\operatorname{rank}(A) = 1$. From Prop. 10 it follows that either

$$\varphi(y) = a^T y + \alpha, \quad \text{or} \quad \varphi(y) = \kappa(a^T y)^2 + a^T y + \alpha.$$

Using (22) and taking into account (24) we obtain that either

$$f(x) = \varphi(y(x)) = \frac{a^T x}{c^T x + \gamma} + \alpha$$

or

$$f(x) = \varphi(y(x)) = \kappa\frac{(a^T x)^2}{(c^T x + \gamma)^2} + \frac{a^T x}{c^T x + \gamma} + \alpha,$$

where $a = \frac{1}{\gamma}b - \frac{2\beta}{\gamma^2}c$ and $\alpha = \frac{\beta}{\gamma^2}$. $\Diamond$

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