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CERTAIN RELATIONS CONCERNING BICENTRIC POLYGONS AND 2-PARAMETRIC PRESENTATION OF FUSS' RELATIONS

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Abstract: Using one favorable algorithm in research of bicentric n-gons we have established certain important properties and relations concerning bicentric n-gons. First part of the article deals with certain relations concerning bicentric n-gons where n is odd. The second part deals with 2-parametric presentation of Fuss' relations.

1. Introduction

First about some terms which will be used.

A polygon $A_1 \dots A_n$ is called chordal polygon if there is a circle which contains each of the points (vertices) $A_1, \dots, A_n$. A polygon $A_1 \dots A_n$ is called tangential polygon if there is a circle such that segments $A_1A_2, \dots, A_nA_1$ are tangential segments of the circle.

A polygon which is both chordal and tangential is shortly called bicentric polygon. If $A_1 \dots A_n$ is a bicentric polygon then it is usually that radius of its circumcirlce is denoted by R, radius od incircle by r

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and distance between centers of circumcircle and incircle by d.

Let $A_1 \dots A_n$ be a bicentric polygon and let $A_1T_1, T_1A_2, \dots, A_nT_n, T_nA_1$ be segments such that $T_1, \dots, T_n$ are tangential points of the segments $A_1A_2, \dots, A_nA_1$. Then the lengths $|A_1T_1|, \dots, |T_nA_1|$ are called tangent lengths of the polygon $A_1 \dots A_n$.

The first one that was concerned with bicentric polygons is German mathematician Nicolaus Fuss (1755–1826). He found relations (conditions) for bicentric quadrilaterals, pentagons, hexagons, heptagons and octagons, These relations can be written as

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2) \tag{1.1a}$$

$$p^3q^3 + p^2q^2r(p+q) - pqr^2(p+q)^2 - r^3(p+q)(p-q)^2 = 0, \tag{1.1b}$$

$$3p^4q^4 - 2p^2q^2r^2(p^2+q^2) - r^4(p^2-q^2)^2 = 0, \tag{1.1c}$$

$$\begin{aligned}(pq-r(p-q)-2r^2)2pqr\sqrt{(p-r)(p+q)}+(p^2q^2-r^2(p^2+q^2))\times \\ \times 2r\sqrt{(q-r)(p+q)} = \pm(pq-r(p-q))(p^2q^2+r^2(p^2-q^2)),\end{aligned} \tag{1.1d}$$

$$(r^2(p^2+q^2) - p^2q^2)^4 - 16p^4q^4r^4(p^2-r^2)(q^2-r^2) = 0, \tag{1.1e}$$

where $p = R + d$, $q = R - d$.

The corresponding relation for triangle is

$$R^2 - d^2 - 2Rr = 0, \tag{1.2}$$

and had already been given by Euler.

Although Fuss found relations for R, d, r only for bicentric n-gons, $4 \le n \le 8$, it is in his honor to call such relations Fuss' relations also in the case $n > 8$.

The very remarkable theorem concerning bicentric polygons is given by French mathematician Poncelet (1788–1867). This theorem, so called Poncelet's closure theorem for circles, can be stated as follows.

Let C_1 and C_2 be two circles, where C_2 is inside of C_1. If there is a bicentric n-gon $A_1 \dots A_n$ such that C_1 is its circumcircle and C_2 its incircle then for every point P_1 on C_1 there are points $P_1, \dots, P_n$ on C_1 such that $P_1 \dots P_n$ is a bicentric n-gon whose circumcircle is C_1 and incircle C_2.

Although this famous Poncelet's closure theorem dates from nineteenth century, many mathematicians have been working on number of problems in connection with this theorem. In this article we deal with certain important properties and relations in this connection. The following notation and known facts will be used.

A bicentric n-gon $A_1 \dots A_n$ is called k-outscribed if

$$2\sum_{i=1}^{n} \arctan \frac{t_i}{r} = k \cdot 360^\circ, \tag{1.3}$$

where k is a positive integer and $t_1, \dots, t_n$ are tangent lengths of the n-gon $A_1 \dots A_n$.

As it is known, the following holds. If $n \geq 3$ is an odd integer, then for each positive integer $k \leq \frac{n-1}{2}$ which is relatively prime to n, there is a bicentric n-gon which is k-outscribed. In the case when n is even, then for each positive integer $k \leq \frac{n-2}{2}$ which is relatively prime to n, there is a bicentric n-gon which is k-outscribed.

Let, for brevity, Fuss' relation for k-outscribed bicentric n-gons be denoted by

$$F_n^{(k)}(R, d, r) = 0. \tag{1.4}$$

As it is known, for each positive solution (R, d, r) of Fuss' relation $F_n^{(k)}(R, d, r) = 0$ there is a class $C_n^{(k)}(R, d, r)$ of bicentric polygons such that all polygons from this class have the same circumcircle and the same incircle and that the following is valid. If by C_1 is denoted circumcircle and by C_2 is denoted incircle then for every point P_1 on C_1 there are points $P_2, \dots, P_n$ on C_1 such that $P_1 \dots P_n$ is a k-outscribed bicentric n-gon whose circumcircle is C_1 and incircle C_2. Of course, also it is valid

$R =$ radius of C_1, $r =$ radius of C_2,

$d =$ distance between centers of C_1 and C_2.

For example, if $n = 5$ and $k = 2$, then by Poncelet's closure theorem every pentagon from the class $C_5^{(2)}(R, d, r)$ is 2-outscribed (see Fig. 1). Thus, for every point P_1 on C_1 we can construct a 2-outscribed bicentric pentagon. Conversely, if a 2-outscribed bicentric pentagon is given, we can construct circles C_1 and C_2 using lines of symmetry of the segment P_1P_2 and P_2P_3, respectively, of angles $\sphericalangle P_1$ and $\sphericalangle P_2$.

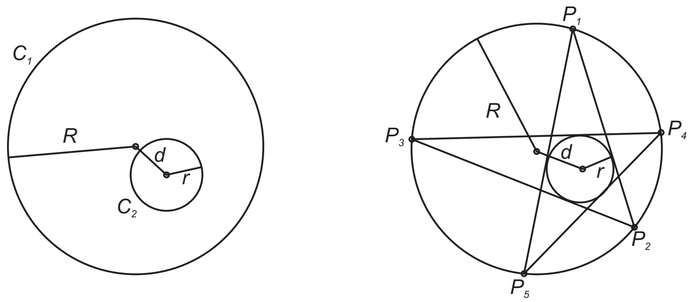

Figure 1

In [4] it is shown that $F_5^{(1)}(R,d,r)=0$ can be written as

$$r = r\sqrt{\frac{R-d-r}{2R}} + (R-d)\sqrt{1-\left(\frac{r}{R+d}\right)^2}\sqrt{\frac{R+d+r}{2R}} \tag{1.5}$$

and that $F_5^{(2)}(R,d,r)=0$ can be written as

$$r = (R-d)\sqrt{1-\left(\frac{r}{R+d}\right)^2}\sqrt{\frac{R+d-r}{2R}} - r\sqrt{\frac{R-d+r}{2R}}. \tag{1.6}$$

Also it is shown that relations $F_7^{(k)}(R,d,r)=0$, $k=1,2,3$, can be written, respectively, as

$$\left(\sqrt{1-\left(\frac{r}{R+d}\right)^2}\sqrt{\frac{R-d-r}{2R}} - \frac{r}{R+d}\sqrt{\frac{R+d+r}{2R}}\right)^2 = \left(\frac{r}{R-d}\right)^2 \cdot \tag{1.7}$$

$$\cdot\left(\sqrt{\frac{R-d-r}{2R}} - \frac{r}{R+d}\right)^2 + \left(\frac{r}{R+d}\right)^2\left(\sqrt{1-\left(\frac{r}{R+d}\right)^2} - \sqrt{\frac{R+d+r}{2R}}\right)^2,$$

$$\left(\sqrt{1-\left(\frac{r}{R+d}\right)^2}\sqrt{\frac{R-d+r}{2R}} + \frac{r}{R+d}\sqrt{\frac{R+d-r}{2R}}\right)^2 = \left(\frac{r}{R-d}\right)^2 \cdot \tag{1.8}$$

$$\cdot\left(\sqrt{\frac{R-d+r}{2R}} + \frac{r}{R+d}\right)^2 + \left(\frac{r}{R+d}\right)^2\left(\sqrt{1-\left(\frac{r}{R+d}\right)^2} - \sqrt{\frac{R+d-r}{2R}}\right)^2,$$

$$(1.9)\quad \left(\sqrt{1-\left(\frac{r}{R+d}\right)^2}\sqrt{\frac{R-d-r}{2R}}-\frac{r}{R+d}\sqrt{\frac{R+d+r}{2R}}\right)^2=\left(\frac{r}{R-d}\right)^2\cdot$$
$$\cdot\left(\sqrt{\frac{R-d-r}{2R}}+\frac{r}{R+d}\right)^2+\left(\frac{r}{R+d}\right)^2\left(\sqrt{1-\left(\frac{r}{R+d}\right)^2}+\sqrt{\frac{R+d+r}{2R}}\right)^2.$$

From (1.5), after rationalization and factorization, we get relation $F_5^{(1)}(R,d,r)=0$ written as

$$(1.10)\qquad \begin{aligned}&3R^4d^2-R^6-3R^2d^4+d^6-2R^5r+4R^3d^2r-\\&-2Rd^4r+4R^4r^2-4R^2d^2r^2+8Rd^2r^3=0,\end{aligned}$$

and from (1.6) we get relation $F_5^{(2)}(R,d,r)=0$ written as

$$(1.11)\qquad \begin{aligned}&R^6-3R^4d^2+3R^2d^4-d^6-2R^5r+4R^3d^2r-\\&-2Rd^4r-4R^4r^2+4R^2d^2r^2+8Rd^2r^3=0.\end{aligned}$$

Also from (1.7), (1.8) and (1.9) can be obtained relations for bicentric heptagons written in this form.

Let, for brevity in the following, by

$$(1.12)\qquad C_n^{(k)}(R,d,r)$$

be denoted a class of k-outscribed bicentric n-gons such that

$R=$ radius of circumcircle of $C_n^{(k)}(R,d,r)$,

$r=$ radius of incircle of $C_n^{(k)}(R,d,r)$,

$d=$ distance between centers of circumcircle and incirle.

Important role in the following will play lengths t_m and t_M given by

$$(1.13)\qquad t_m=\sqrt{(R-d)^2-r^2},\quad t_M=\sqrt{(R+d)^2-r^2}.$$

See Fig. 2, where by C_1 is denoted circumcircle of the polygons from the class $C_n^{(k)}(R,d,r)$ and by C_2 is denoted incircle of the polygons from the class $C_n^{(k)}(R,d,r)$. As can be easily seen, t_m is the length of the least tangent that can be drawn from C_1 to C_2 and t_M is the length of the largest tangent that can be drawn from C_1 to C_2.

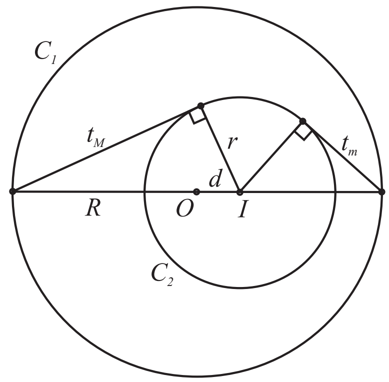

Figure 2: $t_M^2 = (R+d)^2 - r^2,\ t_m^2 = (R-d)^2 - r^2$

The lengths t_m and t_M can be called minimal and maximal tangent lengths of the class $C_n^{(k)}(R, d, r)$.

It is clear from Poncelet's closure theorem that the following holds. If t_1 is any given length such that $t_m \le t_1 \le t_M$, where t_m and t_M are given by (1.13), then there is a bicentric n-gon from the class $C_n^{(k)}(R, d, r)$ such that its first tangent has the length t_1. Such an n-gon will be denoted by

$$P_n^{(k)}(R, d, r; t_1). \tag{1.14}$$

For calculation of tangent lengths of bicentric polygons can be used the following known formula

$$(t_2)_{1,2} = \frac{(R^2 - d^2)t_1 \pm \sqrt{D}}{r^2 + t_1^2}, \tag{1.15}$$

where $D = t_1^2(R^2 - d^2)^2 + (r^2 + t_1^2)[4R^2d^2 - r^2t_1^2 - (R^2 + d^2 - r^2)^2]$. If t_1 is given then its consequent is $(t_2)_1$ or $(t_2)_2$.

Concerning signs $+$ and $-$ in expression $\pm\sqrt{D}$ it does not matter, since for each integer i such that $1 < i < n$, the following is valid. If $t_{i+1} = \frac{(R^2-d^2)t_i+\sqrt{D}}{r^2+t_i^2}$ then $t_{i-1} = \frac{(R^2-d^2)t_i-\sqrt{D}}{r^2+t_i^2}$ and vice versa.

Analogously holds for $i = 1$.

Important role in the following will play Cor. 3 of Th. 1 given in [5] here something modified and written as Th. A.

Theorem A. *Let $A_1 \ldots A_n$ be any given bicentric n-gon and let*

R_0 = radius of the circumcircle of $A_1 \ldots A_n$,

r_0 = radius of the incircle of $A_1 \ldots A_n$,

d_0 = distance between centers of the circumcircle and incircle.

Then there are lengths (in fact positive numbers) R, r, d such that

$$R^2 + d^2 - r^2 = R_0^2 + d_0^2 - r_0^2, \tag{1.16}$$

$$Rd = R_0 d_0, \tag{1.17}$$

$$R^2 - d^2 = 2R_0 r. \tag{1.18}$$

Using computer algebra it can be easily found that(positive) solutions of the above system in R, d, r are given by

$$R_1^2 = R_0(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}), \tag{1.19}$$

$$d_1^2 = R_0(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}), \quad r_1^2 = (R_0 + r_0)^2 - d_0^2 \tag{1.20}$$

and

$$R_2^2 = R_0(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2}), \tag{1.21}$$

$$d_2^2 = R_0(R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - d_0^2}), \quad r_2^2 = (R_0 - r_0)^2 - d_0^2. \tag{1.22}$$

Thus, it is valid

$$R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_0^2 + d_0^2 - r_0^2$$

$$R_1 d_1 = R_2 d_2 = R_0 d_0,$$

$$R_1^2 - d_1^2 - 2R_0 r_1 = R_2^2 - d_2^2 - 2R_0 r_2 = 0.$$

That R_1, d_1, r_1 and R_2, d_2, r_2 given by (1.19)–(1.22) are two solutions of the system in Th. A can be also easily seen by hand (without using computer algebra). So, using relations (1.19) and (1.20) we can write

$$R_1^2 + d_1^2 - r_1^2 = 2R_0(R_0 + r_0) - r_1^2 = R_0^2 - d_0^2 - r_0^2,$$

$$R_1^2 d_1^2 = R_0^2((R_0 + r_0)^2 - (R_0 + r_0)^2 + d_0^2) = R_0^2 d_0^2,$$

$$R_1^2 - d_1^2 = 2R_0\sqrt{(R_0 + r_0)^2 - d_0^2} = 2R_0 r_1.$$

Also can be easily seen that

$$\frac{R_1^2 - d_1^2}{2r_1} = \frac{R_2^2 - d_2^2}{2r_2} = R_0, \quad \frac{2R_1 d_1 r_1}{R_1^2 - d_1^2} = \frac{2R_2 d_2 r_2}{R_2^2 - d_2^2} = d_0, \tag{1.23}$$

$$\begin{aligned} &- (R_1^2 + d_1^2 - r_1^2) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}\right)^2 = \\ &= -(R_2^2 + d_2^2 - r_2^2) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}\right)^2 = r_0^2. \end{aligned} \tag{1.24}$$

So using relations (1.19) and (1.20) we can write

$$\frac{R_1^2-d_1^2}{2r_1}=\frac{2R_0r_1}{2r_1}=R_0,\qquad \frac{2R_1d_1r_1}{R_1^2-d_1^2}=\frac{2R_0d_0r_1}{2R_0r_1}=d_0.$$

In the same way can be seen that (1.24) is valid.

Now let R_0, d_0, r_0 be as in Th. A and let $A_1 \ldots A_n$ be k-outscribed. Then by Poncelet's closure theorem every n-gon from the class $C_n^{(k)}(R_0, d_0, r_0)$ is a k-outscribed bicentric n-gon. We state the following conjecture.

Conjecture 1. *Let R_1, d_1, r_1 and R_2, d_2, r_2 be given by* (1.19)–(1.22) *and let n be even. Then there are classes*

$$C_{2n}^{(k)}(R_1, d_1, r_1),\quad C_{2n}^{(n-k)}(R_2, d_2, r_2) \tag{1.25}$$

where the first is a class of k-outscribed bicentric $2n$-gons and the second is a class of $(n-k)$-outscribed bicentric $2n$-gons.

Now let n be odd. Then only one of the classes given by (1.25) *is a class of bicentric $2n$-gons and the other is a class of double k-outscribed bicentric n-gons.*

It is not difficult to prove that this conjecture is a true one for even $n = 4, 6, 8$ and for odd $n = 3, 5, 7$. Here is an example for which this conjecture is proved. From this example can be seen the way of proving for each other example.

Let $n = 4$. Then $k = 1$ since there are only 1-outscribed bicentric quadrilaterals. Let R_0, d_0, r_0 be any given positive solution of Fuss' relation for bicentric quadrilaterals given by (1.1a) and for brevity of writing here denoted by $F_4^{(1)}(R, d, r) = 0$. Let R_0, d_0, r_0 in relation $F_4^{(1)}(R_0, d_0, r_0) = 0$ be replaced, respectively, by corresponding expressions given by (1.23) and (1.24), that is, by

$$\frac{R_1^2-d_1^2}{2r_1},\quad \frac{2R_1d_1r_1}{R_1^2-d_1^2},\quad -(R_1^2+d_1^2-r_1^2)+\left(\frac{R_1^2-d_1^2}{2r_1}\right)^2+\left(\frac{2R_1d_1r_1}{R_1^2-d_1^2}\right)^2$$

where R_1, d_1, r_1 are given by (1.19) and (1.20). It is not difficult to show that obtained equation can be written as $F_8^{(1)}(R_1, d_1, r_1) = 0$, where $F_8^{(1)}(R, d, r) = 0$ is Fuss' relation for 1-outscribed bicentric octagons. (See later stated relation given by (2.29).) Thus, (R_1, d_1, r_1) is a solution of Fuss' relation $F_8^{(1)}(R, d, r) = 0$.

In the same way can be seen that (R_2, d_2, r_2), where R_2, d_2, r_2 are given by (1.21) and (1.22) is a solution of Fuss' relation $F_8^{(3)}(R, d, r) = 0$ given by (2.30).

From this it is clear that $C_8^{(1)}(R_1, d_1, r_1)$ is a class of 1-outscribed bicentric octagons and $C_8^{(3)}(R_2, d_2, r_2)$ is a class of 3-outscribed bicentric octagons and that for their outscriptions 1 and 3 it is valid $1+3=4=\frac{8}{2}$.

It is easy to show that $F_8^{(1)}(R,d,r)\cdot F_8^{(3)}(R,d,r) = F_8(R,d,r)$, where $F_8(R,d,r)=0$ is Fuss' relation for both 1-outscribed and for 3-outscribed bicentric octagons.

In this connection let us remark that from above said can be concluded that the following holds good.

If R,d,r in relation $F_4^{(1)}(R,d,r)=0$ we replace, respectively, by

$$\frac{R^2-d^2}{2r},\quad \frac{2Rdr}{R^2-d^2},\quad -(R^2+d^2-r^2)+\left(\frac{R^2-d^2}{2r}\right)^2+\left(\frac{2Rdr}{R^2-d^2}\right)^2$$

we get Fuss' relation $F_8(R,d,r)=0$. (It is not difficult to establish even by hand (without using computer algebra.)

Of course, now instead of starting from $F_4^{(1)}(R,d,r)=0$ we can start from $F_8^{(1)}(R,d,r)=0$ or from $F_8^{(3)}(R,d,r)=0$. Analogy is complete.

More about this will be in connection with Conj. 2 which is connected with Conj. 1. For this purpose the following corollaries of Th. A will be used. (These corollaries are not stated in [5].)

The following corollaries of Th. A (which are not given in [5]) will be used.

Corollary A.1. *Let instead of R_0, d_0, r_0 in the system stated in Th. A be put, respectively, R_1, r_1, d_1 given by* (1.19) *and* (1.20)*. Then positive solutions of the so obtained system in R, d, r, that is, of the system*

$$R^2+d^2-r^2=R_1^2+d_1^2-r_1^2,\quad Rd=R_1d_1,\quad R^2-d^2=2R_1r,$$

are $(R,d,r)_1=(R_{11},d_{11},r_{11})$ *and* $(R,d,r)_2=(R_{12},d_{12},r_{12})$*, where*

$$(1.26)\quad R_{11}^2=R_1\left(R_1+r_1+\sqrt{(R_1+r_1)^2-d_1^2}\right),$$

$$(1.27)\quad d_{11}^2=R_1\left(R_1+r_1-\sqrt{(R_1+r_1)^2-d_1^2}\right),\quad r_{11}^2=(R_1+r_1)^2-d_1^2$$

and

$$(1.28)\quad R_{12}^2=R_1\left(R_1-r_1+\sqrt{(R_1-r_1)^2-d_1^2}\right),$$

$$(1.29)\quad d_{12}^2=R_1\left(R_1-r_1-\sqrt{(R_1-r_1)^2-d_1^2}\right),\quad r_{12}^2=(R_1-r_1)^2-d_1^2.$$

Corollary A.2. *Let instead of R_0, d_0, r_0 in the system stated in Th. A be put, respectively, R_2, d_2, r_2 given by* (1.21) *and* (1.22)*. Then positive solutions of the so obtained system in R, r and d are*

$$(1.30)\quad R_{21}^2 = R_2(R_2 + r_2 + \sqrt{(R_2 + r_2)^2 - d_2^2}),$$

$$(1.31)\quad d_{21}^2 = R_2(R_2 + r_2 - \sqrt{(R_2 + r_2)^2 - d_2^2}), \quad r_{21}^2 = (R_2 + r_2)^2 - d_2^2$$

and

$$(1.32)\quad R_{22}^2 = R_2(R_2 - r_2 + \sqrt{(R_2 - r_2)^2 - d_2^2}),$$

$$(1.33)\quad d_{22}^2 = R_2(R_2 - r_2 - \sqrt{(R_2 - r_2)^2 - d_2^2}), \quad r_{22}^2 = (R_2 - r_2)^2 - d_2^2.$$

In the same way can be proceed and seen that the following *algorithm* is valid.

Algorithm determined by Th. A

Let $i_1, \ldots, i_k$ be any given integers such that each of them belongs to the set $\{1, 2\}$. Let, for brevity, $i_1 \ldots i_{k-1}$ be denoted by u and $i_1 \ldots i_k$ be denoted by v. Then, if $i_k = 1$,

$$(1.34)\quad R_v^2 = R_u\big(R_u + r_u + \sqrt{(R_u + r_u)^2 - d_u^2}\big),$$

$$(1.35)\quad d_v^2 = R_u\big(R_u + r_u - \sqrt{(R_u + r_u)^2 - d_u^2}\big), \quad r_v^2 = (R_u + r_u)^2 - d_u^2.$$

But, if $i_k = 2$, then

$$(1.36)\quad R_v^2 = R_u\big(R_u - r_u + \sqrt{(R_u - r_u)^2 - d_u^2}\big),$$

$$(1.37)\quad d_v^2 = R_u\big(R_u - r_u - \sqrt{(R_u - r_u)^2 - d_u^2}\big), \quad r_v^2 = (R_u - r_u)^2 - d_u^2.$$

Concerning indices, let us remark that the situation is in some way connected with the fact that there are 2^k k-digit integers with digits from the set $\{1, 2\}$. For example, if $k = 3$, we have indices

$$(1.38)\qquad 111, 112, 121, 122, 211, 212, 221, 222$$

and we have

$$R_{111}^2 = R_{11}\Big(R_{11} + r_{11} + \sqrt{(R_{11} + r_{11})^2 - d_{11}^2}\Big),$$

$$R_{112}^2 = R_{11}\Big(R_{11} - r_{11} + \sqrt{(R_{11} - r_{11})^2 - d_{11}^2}\Big), \text{ and so on.}$$

Sometimes in using this algorithm can be convenient a sketch as

$$C_n^{(k)}(R_0, d_0, r_0) \quad \begin{matrix} \nearrow & C_{2n}^{(k)}(R_1, d_1, r_1) \\ \searrow & C_{2n}^{(n-k)}(R_2, d_2, r_2). \end{matrix}$$

The following two corollaries will also be useful in the following.

Corollary A.3. *Let h be given by $h = t_m t_M$, where*

$$t_m = \sqrt{(R_0 - d_0)^2 - r_0^2}, \quad t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}.$$

Then

$$r_1 r_2 = r_{11} r_{12} = r_{21} r_{22} = r_{111} r_{112} = r_{121} r_{122} = \cdots = h,$$

that is, generally holds

$$r_{u1} r_{u2} = h \tag{1.39}$$

where

$$r_{u1}^2 = (R_u + r_u)^2 - d_u^2, \quad r_{u2}^2 = (R_u - r_u)^2 - d_u^2$$

and $u = i_1 \dots i_k, \ \ i_1, \dots, i_k \in \{1, 2\}$.

Corollary A.4. *Let R_u, d_u, r_u and R_v, d_v, r_v be given by (1.34) and (1.35) or by (1.36) and (1.37). Then*

$$\frac{R_v^2 - d_v^2}{2r_v} = R_u, \quad \frac{2R_v d_v r_v}{R_v^2 - d_v^2} = d_u, \tag{1.40}$$

$$-(R_v^2 + d_v^2 - r_v^2) + \left(\frac{R_v^2 - d_v^2}{2r_v}\right)^2 + \left(\frac{2R_v d_v r_v}{R_v^2 - d_v^2}\right)^2 = r_u^2. \tag{1.41}$$

Conjecture 2. *Let R_0, d_0, r_0 be any given lengths (in fact positive numbers) such that $F_n^{(k)}(R_0, d_0, r_0) = 0$. Let R_0, d_0, r_0 in $F_n^{(k)}(R_0, d_0, r_0) = 0$ be replaced, respectively, by*

(1.42)

$$\frac{R^2-d^2}{2r}, \quad \frac{2Rdr}{R^2-d^2}, \quad \sqrt{-(R^2 + d^2 - r^2) + \left(\frac{R^2-d^2}{2r}\right)^2 + \left(\frac{2Rdr}{R^2-d^2}\right)^2}.$$

Then, if n is even, the equation

(1.43)

$$F_n^{(k)}\left(\frac{R^2-d^2}{2r}, \frac{2Rdr}{R^2-d^2}, \sqrt{-(R^2+d^2-r^2) + \left(\frac{R^2-d^2}{2r}\right)^2 + \left(\frac{2Rdr}{R^2-d^2}\right)^2}\right) = 0$$

becomes Fuss' relation for both k-outscribed and $(n-k)$-outscribed bicentric $2n$-gons and can be denoted by

$$F_{2n}^{(k,n-k)}(R,d,r)=0. \tag{1.44}$$

But, if n is odd, and $GCD(k,2n)=1$, then equation (1.43) *is only Fuss' relation for k-outscribed bicentric $2n$-gons. If $GCD(k,2n)=2$, then equation* (1.43) *is only Fuss' relation for $(n-k)$-outscribed bicentric $2n$-gons. (Here let us remark that from $GCD(k,n)=1$ and $GCD(k,2n)>1$ it follows $GCD(k,2n)=2$.)*

Of course, if n is odd then corresponding Fuss' relation is also a relation for corresponding double bicentric n-gons.

Remark 1. We have proved this conjecture for many n. Anyone, using computer algebra, can easily check this conjecture for, say, even $n=4,6,8,10$ and odd $n=3,5,7,9$. (In the case when $n=3$ or $n=4$ it can be easily checked even by hand, without using computer algebra.)

Conj. 2 is very connected with Conj. 1 where we have already considered example starting from $n=4$ and using relations (1.19)–(1.22). Here we shall start from $n=8$ and use relations (1.26)–(1.33).

Let R_1,d_1,r_1 in $F_8^{(1)}(R_1,d_1,r_1)=0$ be replaced, respectively, by

$$\frac{R^2-d^2}{2r}\,,\quad \frac{2Rdr}{R^2-d^2}\,,\quad -(R^2+d^2-r^2)+\left(\frac{R^2-d^2}{2r}\right)^2+\left(\frac{2Rdr}{R^2-d^2}\right)^2. \tag{1.45}$$

Let the obtained equation be denoted by $F_{16}^{(1,7)}(R,d,r)=0$. Using computer algebra it is easy to show that (R_{11},d_{11},r_{11}) and (R_{12},d_{12},r_{12}) are solutions of the equation $F_{16}^{(1,7)}(R,d,r)=0$.

Now let R_2,d_2,r_2 in $F_8^{(3)}(R_2,d_2,r_2)=0$ be replaced respectively by expressions given by (1.45) and let obtained equation be denoted by $F_{16}^{(3,5)}(R,d,r)=0$. Using computer algebra it is easy to show that (R_{21},d_{21},r_{21}) and (R_{22},d_{22},r_{22}) are two solutions of the equation $F_{16}^{(3,5)}(R,d,r)=0$.

Using computer algebra also can be shown that the following holds.

If R,d,r in $F_8(R,d,r)=0$ is replaced respectively by expressions given by (1.45) we get Fuss' relation $F_{16}(R,d,r)=0$ for $1,3,5,7$-outscribed bicentric 16-gons.

As can be seen, it is valid $1+7=3+5=8=\dfrac{16}{2}$ (analogously as in the starting from $n=4$).

We have found that analogously holds for many n what strongly suggests that stated conjectures must be true ones.

From this can be concluded that Th. A has important role in investigation of bicentric polygons.

Corollary A.5. Let in Conj. 1 be $d_0 = 0$. Then such conjecture can be relatively easily proved.

Proof. Let in Th. A be $d_0 = 0$ and let R_1, d_1, r_1 and R_2, d_2, r_2 be given by (1.19)–(1.23), where now $d_0 = 0$. Then, using algorithm determined by Th. A, we have

$$R_1^2 = 2R_0(R_0 + r_0), \quad r_1^2 = (R_0 + r_0)^2, \tag{1.46}$$

$$R_2^2 = 2R_0(R_0 - r_0), \quad r_2^2 = (R_0 - r_0)^2. \tag{1.47}$$

Let $P_1 \dots P_{2n}$ and $Q_1 \dots Q_{2n}$ be two bicentric $2n$-gons if n is even or one bicentric and the other be a double bicentric n-gon if n is odd. Let

R_1 = radius of circumcircle of $P_1 \dots P_{2n}$,
r_1 = radius of incircle of $P_1 \dots P_{2n}$,
R_2 = radius of circumcircle of $Q_1 \dots Q_{2n}$,
r_2 = radius of incircle of $Q_1 \dots Q_{2n}$.

Let by t be denoted the length of the tangents of $P_1 \dots P_{2n}$. Then, since (1.46) and (1.47) are valid, we have

$$t = \sqrt{R_1^2 - r_1^2} = \sqrt{R_2^2 - r_2^2} = \sqrt{R_0^2 - r_0^2}$$

and

$$\frac{t}{r_1} = \sqrt{\frac{R_0 - r_0}{R_0 + r_0}} = \tan \frac{k_1 \cdot 2\pi}{4n}, \quad \frac{t}{r_2} = \sqrt{\frac{R_0 + r_0}{R_0 - r_0}} = \tan \frac{k_2 \cdot 2\pi}{4n}, \tag{1.48}$$

where k_1 is outscription of $P_1 \dots P_{2n}$ and k_2 is outscription of $Q_1 \dots Q_{2n}$. Thus

$$\arctan \frac{t}{r_1} = \frac{k_1 \pi}{2n}, \quad \arctan \frac{t}{r_2} = \frac{k_2 \pi}{2n}. \tag{1.49}$$

We have to prove that $k_1 + k_2 = n$. To prove this we shall use the equality

$$\arctan \sqrt{\frac{R_0 - r_0}{R_0 + r_0}} + \arctan \sqrt{\frac{R_0 + r_0}{R_0 - r_0}} = \frac{\pi}{2}, \tag{1.50}$$

since, as it is well known, for every positive real number a it is valid

$$\arctan a + \arctan \frac{1}{a} = \frac{\pi}{2}.$$

So, using relations (1.48) and (1.49), we can write

$$(1.51) \qquad \frac{k_1\pi}{2n} + \frac{k_2\pi}{2n} = \frac{\pi}{2} \qquad \text{or} \qquad k_1 + k_2 = n.$$

Cor. A.5 is proved. $\Diamond$

Now in short about the following fact. If k is outscription of the n-gon $A_1 \ldots A_n$ in Th. A, then k is also outscription of the $2n$-gon $P_1 \ldots P_{2n}$ in Cor. A.5, that is, it is easy to show that

$$2n \arctan \frac{\sqrt{R_0^2 - r_0^2}}{r_0} = 4n \arctan \frac{\sqrt{R_0^2 - r_0^2}}{R_0 + r_0}.$$

Remark 2. Cor. A.5 strongly suggests that the Conj. 1 is a true one. Even from this corollary can be concluded, at least intuitively, that Conj. 1 is valid. See, for example, Fig. 3. If $d \to 0$, then because of continuity with regard to Poncelet's closure theorem, 2-outscribed pentagon becomes 2-outscribed pentagon with $d = 0$ and vice versa.

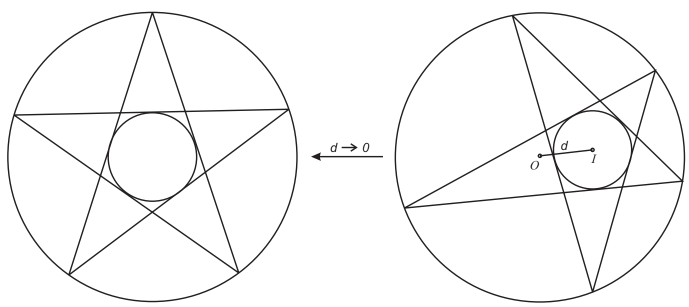

Figure 3

Cor. A.5 can also be very useful in proving Conj. 2 for some given n.

2. One algorithm in research of bicentric polygons

Using algorithm explained in Introduction we shall first deal with certain relations concerning bicentric n-gons where n is odd. In the second part we shall deal with 2-parametric presentation of Fuss'relations.

The case when $n = 5$. First we have the following theorem.

Theorem 1. *Let R_1, d_1, r_1 be any given lengths (in fact positive numbers) such that $F_5^{(1)}(R_1, d_1, r_1) = 0$ and let R_2, d_2, r_2 be given by*

$$\begin{aligned} R_2 &= \frac{R_1^2 - d_1^2}{2r_1}, \quad d_2 = \frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}, \\ r_2^2 &= -(R_1^2 + d_1^2 - r_1^2) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}\right)^2. \end{aligned} \tag{2.1}$$

Then

$$F_5^{(2)}(R_2, d_2, r_2) = 0. \tag{2.2}$$

Reversely, let R_2, d_2, r_2 be given any lengths such that $F_5^{(2)}(R_2, d_2, r_2) = 0$ and let R_1, d_1, r_1 be given by

$$\begin{aligned} R_1 &= \frac{R_2^2 - d_2^2}{2r_2}, \quad d_1 = \frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}, \\ r_1^2 &= -(R_2^2 + d_2^2 - r_2^2) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}\right)^2. \end{aligned} \tag{2.3}$$

Then

$$F_5^{(1)}(R_1, d_1, r_1) = 0. \tag{2.4}$$

In other words, it is valid

$$F_5^{(1)}(R_1, d_1, r_1) = 0 \Leftrightarrow F_5^{(2)}(R_2, d_2, r_2) = 0. \tag{2.5}$$

Proof. Using computer algebra it can be found that

$$F_5^{(2)}(R_2, d_2, r_2) = F_5^{(1)}(R_1, d_1, r_1) F_5^{(2)}(R_1, d_1, r_1),$$

where R_2, d_2, r_2 are given by (2.1). Thus, (2.2) is valid since $F_5^{(1)}(R_1, d_1, r_1) = 0$. Starting from (2.3), it can be found that

$$F_5^{(1)}(R_1, d_1, r_1) = F_5^{(2)}(R_2, d_2, r_2) F_5^{(1)}(R_2, d_2, r_2).$$

(Relations (1.10) and (1.11) can be used.)

This proves Th. 1. $\Diamond$

Corollary 1.1. *From relations*

$$\begin{aligned} R_1^2 &= R_2(R_2 + r_2 + \sqrt{(R_2 + r_2)^2 - d_2^2}), \\ d_1^2 &= R_2(R_2 + r_2 - \sqrt{(R_2 + r_2)^2 - d_2^2}), \\ r_1^2 &= (R_2 + r_2)^2 - d_2^2, \end{aligned}$$

we can get relations given by (2.1)*. So, we have*

$$\frac{R_1^2 - d_1^2}{2r_1} = \frac{2R_2 r_1}{2r_1} = R_2.$$

Corollary 1.2. *From the following relations*

$$R_2^2 = R_1(R_1 - r_1 + \sqrt{(R_1 - r_1)^2 - d_1^2}),$$
$$d_2^2 = R_1(R_1 - r_1 - \sqrt{(R_1 - r_1)^2 - d_1^2}),$$
$$r_2^2 = (R_1 - r_1)^2 - d_1^2,$$

we can get relations given by (2.3)*. So, we have*

$$\frac{R_2^2 - d_2^2}{2r_2} = \frac{2R_1 r_2}{2r_2} = R_1.$$

Remark 3. Relations

$$R_{22}^2 = R_2(R_2 - r_2 + \sqrt{(R_2 - r_2)^2 - d_2^2}),$$
$$d_{22}^2 = R_2(R_2 - r_2 - \sqrt{(R_2 - r_2)^2 - d_2^2}),$$
$$r_{22}^2 = (R_2 - r_2)^2 - d_2^2,$$

refer to 3-outscribed bicentric 10-gons, and the relations

$$R_{11}^2 = R_1(R_1 + r_1 + \sqrt{(R_1 + r_1)^2 - d_1^2}),$$
$$d_{11}^2 = R_1(R_1 + r_1 - \sqrt{(R_1 + r_1)^2 - d_1^2}),$$
$$r_{11}^2 = (R_1 + r_1)^2 - d_1^2,$$

refer to 1-outscribed bicentric 10-gons.

It can be established using computer algebra that

$$F_{10}^{(1)}(R_{11}, d_{11}, r_{11}) = 0, \quad F_{10}^{(3)}(R_{22}, d_{22}, r_{22}) = 0.$$

Of course, for relations in Cor. 1.1 and Cor. 1.2 it is valid

$$F_5^{(1)}(R_1, d_1, r_1) = 0, \quad F_5^{(2)}(R_2, d_2, r_2) = 0.$$

As can be seen, the important role in this has Th. A and its corollaries listed in Introduction.

Corollary 1.3. *Let* R_1, d_1, r_1 *and* R_2, d_2, r_2 *be as in Th. 1. Then, by algorithm determined by Th. A, we have*

$$C_5^{(1)}(R_1, d_1, r_1) \quad \begin{array}{l} \nearrow \quad C_{10}^{(1)}(R_{11}, d_{11}, r_{11}) \\ \searrow \quad C_{10}^{(4)}(R_{12}, d_{12}, r_{12}),\ R_{12} = R_2,\ d_{12} = d_2,\ r_{12} = r_2 \end{array}$$

$$C_5^{(2)}(R_2,d_2,r_2) \quad \begin{matrix} \nearrow & C_{10}^{(2)}(R_{21},d_{21},r_{21}),\ R_{21}=R_1,\ d_{21}=d_1,\ r_{21}=r_1 \\ \searrow & C_{10}^{(3)}(R_{22},d_{22},r_{22}) \end{matrix}$$

Thus $C_{10}^{(4)}(R_{12},d_{12},r_{12})$ is a class of double 2-outscribed bicentric pentagons and $C_{10}^{(2)}(R_{21},d_{21},r_{21})$ is class of double 1-outscribed pentagons. (Here may be interesting that R_1,d_1,r_1 and R_2,d_2,r_2 are as in Th. 1.)

Corollary 1.4. *It is valid*

$$R_1d_1 = R_2d_2, \quad R_1^2+d_1^2-r_1^2 = R_2^2+d_2^2-r_2^2. \tag{2.6}$$

Proof. It follows from (2.1) and (2.3). ♢

Corollary 1.5. *There are lengths t_m and t_M such that*

$$(R_1-d_1)^2-r_1^2 = (R_2-d_2)^2-r_2^2 = t_m^2, \tag{2.7}$$

$$(R_1+d_1)^2-r_1^2 = (R_2+d_2)^2-r_2^2 = t_M^2. \tag{2.8}$$

Definition 1. Let R_1,d_1,r_1 and R_2,d_2,r_2 be as in Th. 1. Then classes such as $C_5^{(1)}(R_1,d_1,r_1)$ and $C_5^{(2)}(R_2,d_2,r_2)$ will be called conjugate classes of bicentric pentagons.

Since (2.7) and (2.8) hold we can also state the following definition.

Definition 2. Let t_1 be any given length such that $t_m \le t_1 \le t_M$, where t_m and t_M are given by (2.7) and (2.8). Then pentagons

$$P_5^{(1)}(R_1,d_1,r_1;t_1), \quad P_5^{(2)}(R_2,d_2,r_2;t_1) \tag{2.9}$$

will be called conjugate bicentric pentagons.

From Th. 1 it is clear that so defined binary relation *be conjugate* in the set of bicentric pentagons is a symmetric relation but neither reflexive nor transitive.

Theorem 2. *Conjugate bicentric pentagons have the same tangent lengths only with different ordering, that is, if $t_1,\dots,t_5$ are tangent lengths of the pentagon $P_5^{(1)}(R_1,d_1,r_1;t_1)$ and $u_1,\dots,u_5$ are tangent lengths of the pentagon $P_5^{(2)}(R_2,d_2,r_2;t_1)$ then*

$$u_i = t_{1+(i-1)2},\ i=1,2,\dots,5.$$

Proof. Using relation given by (1.15) and computer algebra, it is not difficult to establish that above theorem holds good. ♢

Example 1.

$R_1 = 7,\qquad d_1 = 2,\qquad r_1 = 4.789111662\ldots$

$R_2 = 4.698157318\ldots,\quad d_2 = 2.979891701\ldots,\quad r_2 = 0.942351978\ldots$

Since in this case $t_m = 1.436805307\ldots$, $t_M = 7.620000623\ldots$, we can take, say, $t_1 = 2$. Using formula given by (1.15) we get

$t_2 = 5.160129225\ldots,\quad t_3 = 7.370217485\ldots,\quad t_4 = 3.425898801\ldots$

$t_5 = 1.522479047\ldots,\quad$ where $t_1 = 2$,

$$u_i = t_{1+(i-1)2},\ i = 1, 2, \ldots, 5$$

The case when $n = 7$. First we have the following theorem.

Theorem 3. *Let R_1, d_1, r_1 be any given lengths (in fact positive numbers) such that $F_7^{(1)}(R_1, d_1, r_1) = 0$ and let*

$$\begin{aligned} R_2 &= \frac{R_1^2 - d_1^2}{2r_1}, \quad d_2 = \frac{2R_1d_1r_1}{R_1^2 - d_1^2}, \\ r_2^2 &= -(R_1^2 + d_1^2 - r_1^2) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1d_1r_1}{R_1^2 - d_1^2}\right)^2, \end{aligned} \tag{2.10}$$

$$\begin{aligned} R_3 &= \frac{R_2^2 - d_2^2}{2r_2}, \quad d_3 = \frac{2R_2d_2r_2}{R_2^2 - d_2^2}, \\ r_3^2 &= -(R_2^2 + d_2^2 - r_2^2) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2d_2r_2}{R_2^2 - d_2^2}\right)^2. \end{aligned} \tag{2.11}$$

Then

$$F_7^{(2)}(R_2, d_2, r_2) = 0, \quad F_7^{(3)}(R_3, d_3, r_3) = 0.$$

Also it is valid

$$\begin{aligned} &\frac{R_3^2 - d_3^2}{2r_3} = R_1, \quad \frac{2R_3d_3r_3}{R_3^2 - d_3^2} = d_1, \\ &-(R_3^2 + d_3^2 - r_3^2) + \left(\frac{R_3^2 - d_3^2}{2r_3}\right)^2 + \left(\frac{2R_3d_3r_3}{R_3^2 - d_3^2}\right)^2 = r_1^2. \end{aligned} \tag{2.12}$$

Proof. Analogous to the proof of Th. 1, but here needs something more calculation to establish that

$$\begin{aligned} F_7^{(2)}(R_2, d_2, r_2) &= F_7^{(1)}(R_1, d_1, r_1)F_7^{(2)}(R_1, d_1, r_1), \\ F_7^{(3)}(R_3, d_3, r_3) &= F_7^{(2)}(R_2, d_2, r_2)F_7^{(3)}(R_2, d_2, r_2), \\ F_7^{(1)}(R_1, d_1, r_1) &= F_7^{(3)}(R_3, d_3, r_3)F_7^{(1)}(R_3, d_3, r_3), \end{aligned}$$

where R_2, d_2, r_2 are given by (2.10), R_3, d_3, r_3 by (2.11) and R_1, d_1, r_1 by (2.12). $\Diamond$

Corollary 3.1. *Let R_2, d_2, r_2, R_3, d_3, r_3 and R_1, d_1, r_1 be as in Th. 3. Then the following relations are valid:*

$$R_1^2 = R_2\Big(R_2 + r_2 + \sqrt{(R_2+r_2)^2 - d_2^2}\Big),$$
$$d_1^2 = R_2\Big(R_2 + r_2 - \sqrt{(R_2+r_2)^2 - d_2^2}\Big),$$
$$r_1^2 = (R_2+r_2)^2 - d_2^2,$$
$$R_2^2 = R_3\Big(R_3 - r_3 + \sqrt{(R_3-r_3)^2 - d_3^2}\Big),$$
$$d_2^2 = R_3\Big(R_3 - r_3 - \sqrt{(R_3-r_3)^2 - d_3^2}\Big),$$
$$r_2^2 = (R_3-r_3)^2 - d_3^2,$$
$$R_3^2 = R_1\Big(R_1 - r_1 + \sqrt{(R_1-r_1)^2 - d_1^2}\Big),$$
$$d_3^2 = R_1\Big(R_1 - r_1 - \sqrt{(R_1-r_1)^2 - d_1^2}\Big),$$
$$r_3^2 = (R_1-r_1)^2 - d_1^2.$$

Remark 4. The following three relations refer to 5-outscribed bicentric 14-gons

$$R_{22}^2 = R_2\Big(R_2 - r_2 + \sqrt{(R_2-r_2)^2 - d_2^2}\Big),$$
$$d_{22}^2 = R_2\Big(R_2 - r_2 - \sqrt{(R_2-r_2)^2 - d_2^2}\Big),$$
$$r_{22}^2 = (R_2-r_2)^2 - d_2^2.$$

The following three relations refer to 3-outscribed bicentric 14-gons

$$R_{31}^2 = R_3\Big(R_3 + r_3 + \sqrt{(R_3+r_3)^2 - d_3^2}\Big),$$
$$d_{31}^2 = R_3\Big(R_3 + r_3 - \sqrt{(R_3+r_3)^2 - d_3^2}\Big),$$
$$r_{31}^2 = (R_3+r_3)^2 - d_3^2.$$

The following three relations refer to 1-outscribed bicentric 14-gons

$$R_{11}^2 = R_1\Big(R_1 + r_1 + \sqrt{(R_1+r_1)^2 - d_1^2}\Big),$$
$$d_{11}^2 = R_1\Big(R_1 + r_1 - \sqrt{(R_1+r_1)^2 - d_1^2}\Big),$$
$$r_{11}^2 = (R_1+r_1)^2 - d_1^2.$$

Important role in this has Th. A and its corollaries.

Corollary 3.2. *Let $R_1, d_1, r_1,\ R_2, d_2, r_2$ and R_3, d_3, r_3 be as in Th. 3. Then, by algorithm determined by Th. A, we have*

$$C_7^{(1)}(R_1,d_1,r_1)\ \begin{array}{l}\nearrow\ C_{14}^{(1)}(R_{11},d_{11},r_{11})\\ \searrow\ C_{14}^{(6)}(R_{12},d_{12},r_{12}),\ R_{12}=R_3,\ d_{12}=d_3,\ r_{12}=r_3\end{array}$$

$$C_7^{(2)}(R_2,d_2,r_2)\ \begin{array}{l}\nearrow\ C_{14}^{(2)}(R_{21},d_{21},r_{21}),\ R_{21}=R_1,\ d_{21}=d_1,\ r_{21}=r_1\\ \searrow\ C_{14}^{(5)}(R_{22},d_{22},r_{22})\end{array}$$

$$C_7^{(3)}(R_3,d_3,r_3)\ \begin{array}{l}\nearrow\ C_{14}^{(3)}(R_{31},d_{31},r_{31})\\ \searrow\ C_{14}^{(4)}(R_{32},d_{32},r_{32}),\ R_{32}=R_2,\ d_{32}=d_2,\ r_{32}=r_2\end{array}$$

There is a complete analogy with the case when $n = 5$. So $C_{14}^{(6)}(R_{12}, d_{12}, r_{12})$ is a class of double 3-outscribed bicentric heptagons, $C_{14}^{(2)}(R_{21}, d_{21}, r_{21})$ is a class of double 1-outscribed bicentric heptagons, and $C_{14}^{(4)}(R_{32}, d_{32}, r_{32})$ is a class of double 2-outscribed bicentric heptagons.

Corollary 3.3. *It is valid*

$$R_1d_1 = R_2d_2 = R_3d_3, \tag{2.13}$$
$$R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_3^2 + d_3^2 - r_3^2. \tag{2.14}$$

Proof. As in Cor. 1.1. ◊

Corollary 3.4. *There are lengths t_m and t_M such that*

$$(R_1 - d_1)^2 - r_1^2 = (R_2 - d_2)^2 - r_2^2 = (R_3 - d_3)^2 - r_3^2 = t_m^2, \tag{2.15}$$
$$(R_1 + d_1)^2 - r_1^2 = (R_2 + d_2)^2 - r_2^2 = (R_3 + d_3)^2 - r_3^2 = t_M^2. \tag{2.16}$$

Definition 3. Let $R_1, d_1, r_1,\ R_2, d_2, r_2$ and R_3, d_3, r_3 be as in Th. 3. Then classes such as $C_7^{(1)}(R_1, d_1, r_1)$, $C_7^{(2)}(R_2, d_2, r_2)$, $C_7^{(3)}(R_3, d_3, r_3)$ will be called conjugate classes of bicentric heptagons.

Since (2.15) and (2.16) hold we can also state the following defnition

Definition 4. Let t_1 be any given length such that $t_m \le t_1 \le t_M$, where t_m and t_M are given by (2.15) and (2.16). Then heptagons

$$P_7^{(1)}(R_1, d_1, r_1; t_1), \quad P_7^{(2)}(R_2, d_2, r_2; t_1), \quad P_7^{(3)}(R_3, d_3, r_3; t_1) \tag{2.17}$$

will be called conjugate bicentric heptagons.

Theorem 4. *Conjugate bicentric heptagons have the same tangent lengths only with different ordering, that is, if* $t_1, \dots, t_7$ *are tangent lengths of the heptagon* $P_7^{(1)}(R_1, d_1, r_1; t_1)$, $u_1, \dots, u_7$ *are tangent lengths of the heptagon* $P_7^{(2)}(R_2, d_2, r_2; t_1)$ *and* $v_1, \dots, v_7$ *are tangent lengths of the heptagon* $P_7^{(3)}(R_3, d_3, r_3; t_1)$*, then*

$$u_i = t_{1+(i-1)2},\ v_i = t_{i+(i-1)3},\ i = 1, 2, \dots, 7.$$

Proof. Using formula given by (1.15) and computer algebra it can be established that above theorem holds good. ◊

Example 2.

$$\begin{aligned} &R_1 = 7, && d_1 = 2, && r_1 = 4.979113505\dots, \\ &R_2 = 4.518876699\dots, && d_2 = 3.098115069\dots, && r_2 = 1.345412540\dots, \\ &R_3 = 4.021788600\dots, && d_3 = 3.481038261\dots, && r_3 = 0.289796869\dots \end{aligned}$$

Since $t_m = 0.456539926\dots$, $t_M = 7.497228068\dots$, we can take, say, $t_1 = 4$. Using formula (1.15) it can be found that

$$\begin{aligned} &t_2 = 7.488438928\dots, && t_3 = 4.334039372\dots, && t_4 = 1.463009127\dots, \\ &t_5 = 0.554976771\dots, && t_6 = 0.526981475\dots, && t_7 = 1.336913675\dots \end{aligned}$$

$$u_i = t_{1+(i-1)2},\ v_i = t_{1+(i-1)3},\ i = 1, 2, \dots, 7.$$

Also can be found that

$$2\sum_{i=1}^{7} \arctan \frac{t_i}{r_1} = 360^\circ, \quad 2\sum_{i=1}^{7} \arctan \frac{u_i}{r_2} = 2 \cdot 360^\circ, \quad 2\sum_{i=1}^{7} \arctan \frac{v_i}{r_3} = 3 \cdot 360^\circ.$$

Theorem 5. *Let* $n = 9$ *and let* $F_9^{(1)}(R_1, d_1, r_1) = 0$*. Let*

$$\begin{aligned} &R_2 = \frac{R_1^2 - d_1^2}{2r_1}, \quad d_2 = \frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}, \\ &r_2^2 = -(R_1^2 + d_1^2 - r_1^2) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}\right)^2, \\ &R_4 = \frac{R_2^2 - d_2^2}{2r_2}, \quad d_4 = \frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}, \\ &r_4^2 = -(R_2^2 + d_2^2 - r_2^2) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 d_2 r_2}{R_2^2 - d_2^2}\right)^2, \\ &R_1 = \frac{R_4^2 - d_4^2}{2r_4}, \quad d_1 = \frac{2R_4 d_2 r_4}{R_4^2 - d_4^2}, \\ &r_1^2 = -(R_4^2 + d_4^2 - r_4^2) + \left(\frac{R_4^2 - d_4^2}{2r_4}\right)^2 + \left(\frac{2R_4 d_4 r_4}{R_4^2 - d_4^2}\right)^2. \end{aligned}$$

Then $F_9^{(2)}(R_2, d_2, r_2) = 0,\ F_9^{(4)} = (R_4, d_4, r_4) = 0.$

The proof is complete analogous with the proof of Th. 3. Here let us remark that there are no R_3, d_3, r_3 since there is no 3-outscribed bicentric 9-gon.

Corollary 5.1. *Let* $R_1, d_1, r_1,\ R_2, d_2, r_2$ *and* R_4, d_4, r_4 *be as in Th. 5. Then*

$$C_9^{(1)}(R_1, d_1, r_1) \quad \begin{matrix} \nearrow & C_{18}^{(1)}(R_{11}, d_{11}, r_{11}) \\ \searrow & C_{18}^{(8)}(R_{12}, d_{12}, r_{12}),\ R_{12} = R_4,\ d_{12} = d_4,\ r_{12} = r_4 \end{matrix}$$

$$C_9^{(2)}(R_2, d_2, r_2) \quad \begin{matrix} \nearrow & C_{18}^{(2)}(R_{21}, d_{21}, r_{21}),\ R_{21} = R_1,\ d_{21} = d_1,\ r_{21} = r_1 \\ \searrow & C_{18}^{(7)}(R_{22}, d_{22}, r_{22}) \end{matrix}$$

$$C_9^{(3)}(R_4, d_4, r_4) \quad \begin{matrix} \nearrow & C_{18}^{(4)}(R_{41}, d_{41}, r_{41}),\ R_{41} = R_2,\ d_{41} = d_2,\ r_{41} = r_2 \\ \searrow & C_{18}^{(5)}(R_{42}, d_{42}, r_{42}). \end{matrix}$$

The following conjecture is strongly suggested.

Conjecture 3. *Let* $n > 9$ *be an odd integer. Then similarly holds as in the cases when* $n = 5, 7, 9$. *Only ordering of outscription may be varied and may be that one solution is not enough for obtaining all other solutions as in the case when* $n = 5, 7, 9$.

We have found that this conjecture is connected with one partition of the set $\left\{1, 2, \dots, \frac{n-1}{2}\right\}$, where n is an odd integer. About this and connection with Fuss' relations we have recently submitted for *Comptes Rendus Mathématique Acad. Sci. Paris*, the following manuscript:

One way of establishing Fuss' relations.

Conj. 1 in this article is a complement to the above stated Conj. 3.

The following part of the article deals with 2-parametric presentation of Fuss' relations. First we prove the following theorem.

Theorem 6. *Let* R_v, d_v, r_v *be given by* (1.34) *and* (1.35) *or by* (1.36) *and* (1.37). *Let* t_M *and* t_m *be given by*

$$t_M^2 = (R_0 + d_0)^2 - r_0^2, \quad t_m^2 = (R_0 - d_0)^2 - r_0^2 \tag{2.18}$$

where R_0, d_0, r_0 are as in Th. A. Then

(2.19)
$$R_v = \sqrt{R_u(R_u + r_u + \sqrt{(R_u + r_u)^2 - d_u^2})} = \frac{1}{2}\Big(\sqrt{r_v^2 + t_M^2} + \sqrt{r_v^2 + t_m^2}\Big),$$
(2.20)
$$d_v = \sqrt{R_u(R_u + r_u - \sqrt{(R_u + r_u)^2 - d_u^2})} = \frac{1}{2}\Big(\sqrt{r_v^2 + t_M^2} - \sqrt{r_v^2 + t_m^2}\Big),$$

where in the case when $v = u1$ we have

$$(2.21)\quad r_v^2 = (R_u + r_u)^2 - d_u^2 = = r_u^2 + \sqrt{(r_u^2 + t_M^2)(r_u^2 + t_m^2)} + r_u\Big(\sqrt{r_u^2 + t_M^2} + \sqrt{r_u^2 + t_m^2}\Big),$$

but in the case when $v = u2$ we have

$$(2.22)\quad r_v^2 = (R_u - r_u)^2 - d_u^2 = = r_u^2 + \sqrt{(r_u^2 + t_M^2)(r_u^2 + t_m^2)} - r_u\Big(\sqrt{r_u^2 + t_M^2} + \sqrt{r_u^2 + t_m^2}\Big).$$

Proof. That (2.19) and (2.20) are valid is clear from
$$\sqrt{r_v^2 + t_M^2} = \sqrt{r_v^2 + (R_v + d_v)^2 - r_v^2} = \sqrt{(R_v + d_v)^2} = R_v + d_v,$$
$$\sqrt{r_v^2 + t_m^2} = \sqrt{r_v^2 + (R_v - d_v)^2 - r_v^2} = \sqrt{(R_v - d_v)^2} = R_v - d_v,$$
$$\Big(\sqrt{r_v^2 + t_M^2} + \sqrt{r_v^2 + t_m^2}\Big) = (R_v + d_v) + (R_v - d_v) = 2R_v,$$
$$\Big(\sqrt{r_v^2 + t_M^2} - \sqrt{r_v^2 + t_m^2}\Big) = (R_v + d_v) - (R_v - d_v) = 2d_v.$$

To prove that (2.21) is valid we can write
$$r_v^2 = (R_u + r_u)^2 - d_u^2 = r_u^2 + R_u^2 - d_u^2 + 2R_u r_u,$$
from which, using relations
$$R_u = \frac{1}{2}\Big(\sqrt{r_u^2 + t_M^2} + \sqrt{r_u^2 + t_m^2}\Big), \quad d_u = \frac{1}{2}\Big(\sqrt{r_u^2 + t_M^2} - \sqrt{r_u^2 + t_m^2}\Big),$$
we get (2.21).

In the same way can be proved that (2.22) is valid. ◊

Theorem 7. *Let $(R,\ d,\ r)$ be any given triple from $\boldsymbol{R}_+^3$ such that $F_4(R, d, r) = 0$ and let t_M and t_m be given by*

$$(2.23)\qquad t_M^2 = (R + d)^2 - r^2, \quad t_m^2 = (R - d)^2 - r^2.$$

Then

(2.24)
$$R=\frac{1}{2}\Big(\sqrt{r^2+t_M^2}+\sqrt{r^2+t_m^2}\Big),\quad d=\frac{1}{2}\Big(\sqrt{r^2+t_M^2}-\sqrt{r^2+t_m^2}\Big),\quad r^2=t_mt_M.$$

Proof. From (2.23) it follows

$$\begin{aligned} &R+d=\sqrt{r^2+t_M^2},\quad R-d=\sqrt{r^2+t_m^2}\\ &R=\frac{1}{2}\Big(\sqrt{r^2+t_M^2}+\sqrt{r^2+t_m^2}\Big),\quad d=\frac{1}{2}\Big(\sqrt{r^2+t_M^2}-\sqrt{r^2+t_m^2}\Big).\end{aligned} \tag{2.25}$$

Replacing R and d in Fuss' relation for bicentric quadrilaterals

$$(R^2-d^2)^2-2r^2(R^2+d^2)=0 \tag{2.26}$$

by expressions given by (2.25) we get the following equation in r^2

$$(r^2+t_m^2)(r^2+t_M^2)-r^2(2r^2+t_m^2+t_M^2)=0 \tag{2.27}$$

whose positive root is only $r^2=t_Mt_m$. $\Diamond$

Corollary 7.1. *Let a and b be any given positive numbers such that $a\geq b$ and let R and d be given by*

$$R=\frac{1}{2}\Big(\sqrt{r^2+a^2}+\sqrt{r^2+b^2}\Big),\quad d=\frac{1}{2}\Big(\sqrt{r^2+a^2}-\sqrt{r^2+b^2}\Big). \tag{2.28}$$

Replacing R and d in Fuss' relation (2.26) *by expressions given by* (2.28) *we get the following equation in r^2*

$$(r^2+a^2)(r^2+b^2)-r^2(2r^2+a^2+b^2)=0$$

whose positive root is only $r^2=ab$.

From Th. 7 and its Cor. 7.1 can be concluded that the following theorem holds good.

Theorem 8. *2-parametric presentation of Fuss' relation* (2.26) *can be written as*

$$\begin{aligned} R&=\frac{1}{2}\Big(\sqrt{r^2+t_M^2}+\sqrt{r^2+t_m^2}\Big),\\ d&=\frac{1}{2}\Big(\sqrt{r^2+t_M^2}-\sqrt{r^2+t_m^2}\Big),\quad r=\sqrt{t_mt_M}.\end{aligned}$$

where t_M and t_m are parameters from $\boldsymbol{R}_+$ and $t_M\geq t_m$.

Thus for every two positive numbers t_M and t_m, where $t_M\geq t_m$, we get one solution (R,d,r) of equation (2.26). *This solution is completely* *determined by t_M and t_m. In other words, for every two lengths t_M and t_m, where $t_M \geq t_m$, there exist two circles, circumcircle and incircle, such that t_M is the length of the largest tangent that can be drawn from circumcircle to incircle, and t_m is the length of the least tangent that can be drawn from circumcircle to incircle.*

Before stating 2-parametric presentation for 1-outscribed bicentric octagons and 3-outscribed bicentric octagons let us remark that Fuss' relation for bicentric octagons can be written as $PQ = 0$, where

$$P = r^2(p^2+q^2) - p^2q^2 - 2pqr\sqrt[4]{(p^2-r^2)(q^2-r^2)},$$
$$Q = r^2(p^2+q^2) - p^2q^2 + 2pqr\sqrt[4]{(p^2-r^2)(q^2-r^2)}$$

and $p = R+d$, $q = R-d$. Putting $P = 0$ and $Q = 0$ we get Fuss' relation for 1-outscribed bicentric octagons and Fuss' relation for 3-outscribed bicentric octagons

$$2r^2(R^2+d^2) - (R^2-d^2)^2 - 2r(R^2-d^2)\sqrt[4]{[(R+d)^2-r^2][(R-d)^2-r^2]} = 0, \tag{2.29}$$

$$2r^2(R^2+d^2) - (R^2-d^2)^2 + 2r(R^2-d^2)\sqrt[4]{[(R+d)^2-r^2][(R-d)^2-r^2]} = 0. \tag{2.30}$$

Theorem 9. *2-parametric presentation of Fuss' relation for 1-outscribed bicentric octagons is given by*

$$R = \frac{1}{2}\left(\sqrt{r^2+t_M^2} + \sqrt{r^2+t_m^2}\right), \quad d = \frac{1}{2}\left(\sqrt{r^2+t_M^2} - \sqrt{r^2+t_m^2}\right), \quad r, \tag{2.31}$$

where

$$r^2 = r_0^2 + \sqrt{(r_0^2+t_M^2)(r_0^2+t_m^2)} + r_0\left(\sqrt{r_0^2+t_M^2} + \sqrt{r_0^2+t_m^2}\right), \tag{2.32}$$

$r_0^2 = t_M t_m$ and parameters t_M and t_m are from $\boldsymbol{R}_+$ such that $t_M \geq t_m$. Analogously holds for 3-outscribed bicentric octagons. In this case r^2 is given by

$$r^2 = r_0^2 + \sqrt{(r_0^2+t_M^2)(r_0^2+t_m^2)} - r_0\left(\sqrt{r_0^2+t_M^2} + \sqrt{r_0^2+t_m^2}\right). \tag{2.33}$$

Proof. Replacing R and d in relation (2.29) by their expressions given by (2.31) and using computer algebra we get equation whose only positive root is r^2 given by (2.32).

Here let us remark that r^2 given by $r^2 = (R_0 + r_0)^2 - d_0^2$ can be, by Th. 6, written as r^2 given by (2.32).

In the same way can be proved that analogously holds for 3-outscribed bicentric octagons. In this case r^2 given by (2.33) can be written as $r^2 = (R_0 - r_0)^2 - d_0^2$. ♢

That this theorem holds good can be directly concluded from the following theorem.

Theorem 10. *Let R_0, d_0, r_0 be as in Th. A and let R_0, d_0, r_0 can be expressed as*

$$R_0 = f_0(t_M, t_m), \quad d_0 = g_0(t_M, t_m), \quad r_0 = h_0(t_M, t_m) \tag{2.34}$$

where t_M and t_m are parameters from $\boldsymbol{R}_+$ and $f_0(t_M, t_m)$, $g_0(t_M, t_m)$, $h_0(t_M, t_m)$ are corresponding expressions of t_M and t_m. Then R_u, d_u, r_u for every $u = 1, 2, 11, 12, 21, 22, \ldots,$ can be expressed as

$$R_u = f_u(t_M, t_m), \quad d_u = g_u(t_M, t_m), \quad r_u = h_u(t_M, t_m)$$

where $f_u(t_M, t_m), g_u(t_M, t_m), h_u(t_M, t_m)$ are corresponding expressions of t_M and t_m.

Proof. It follows from Cor. A.3. Of course, Algorithm determined by Th. A has important role. ♢

Thus, Th. 9 is in fact a corollary of Th. 8 and Th. 10.

For convenience in stating the following corollaries of Th. 8 and Th. 10 we shall, in accordance with notation used in Th. A and its corollaries, instead of notation r^2 in (2.32) use notation r_1^2 and instead of notation r^2 in (2.33) use notation r_2^2. Also $r_0^2 = t_M t_m$. Thus

$$r_0^2 = t_M t_m \tag{2.35}$$

$$r_1^2 = r_0^2 + \sqrt{(r_0^2 + t_M^2)(r_0^2 + t_m^2)} + r_0\left(\sqrt{r_0^2 + t_M^2} + \sqrt{r_0^2 + t_m^2}\right), \tag{2.36}$$

$$r_2^2 = r_0^2 + \sqrt{(r_0^2 + t_M^2)(r_0^2 + t_m^2)} - r_0\left(\sqrt{r_0^2 + t_M^2} + \sqrt{r_0^2 + t_m^2}\right). \tag{2.37}$$

The following two theorems are corollaries of Th. 8 and Th. 10.

Theorem 11. *For 2-parametric presentation of Fuss' relations for k-outscribed bicentric 16-gons, $k = 1, 7, 3, 5$, we have the following relations*

$$R_u = \frac{1}{2}\left(\sqrt{r_u^2 + t_M^2} + \sqrt{r_u^2 + t_m^2}\right), \quad d_u = \frac{1}{2}\left(\sqrt{r_u^2 + t_M^2} - \sqrt{r_u^2 + t_m^2}\right), \quad r_u$$

where $u = 11, 12, 21, 22$ *and*

$$r_{11}^2 = r_1^2 + \sqrt{(r_1^2+t_M^2)(r_1^2+t_m^2)} + r_1\Big(\sqrt{r_1^2+t_M^2}+\sqrt{r_1^2+t_m^2}\Big), \tag{2.38}$$

$$r_{12}^2 = r_1^2 + \sqrt{(r_1^2+t_M^2)(r_1^2+t_m^2)} - r_1\Big(\sqrt{r_1^2+t_M^2}+\sqrt{r_1^2+t_m^2}\Big), \tag{2.39}$$

$$r_{21}^2 = r_2^2 + \sqrt{(r_2^2+t_M^2)(r_2^2+t_m^2)} + r_2\Big(\sqrt{r_2^2+t_M^2}+\sqrt{r_2^2+t_m^2}\Big), \tag{2.40}$$

$$r_{22}^2 = r_2^2 + \sqrt{(r_2^2+t_M^2)(r_2^2+t_m^2)} - r_2\Big(\sqrt{r_2^2+t_M^2}+\sqrt{r_2^2+t_m^2}\Big), \tag{2.41}$$

r_1^2 *and* r_2^2 *are given by* (2.36) *and* (2.37)*. Parameters* t_M *and* t_m *are from* $\boldsymbol{R}_+$ *and* $t_M \geq t_m$.

Relations $r_{11}^2, r_{12}^2, r_{21}^2, r_{22}^2$ *can be obtained, by Th. 6, starting from relations*

$$r_{11}^2 = (R_1+r_1)^2 - d_1^2, \quad r_{12}^2 = (R_1-r_1)^2 - d_1^2,$$
$$r_{21}^2 = (R_2+r_2)^2 - d_2^2, \quad r_{22}^2 = (R_2-r_2)^2 - d_2^2.$$

Theorem 12. *For 2-parametric presentation of Fuss' relations for k-outscribed* 32*-gons,* $k = 1, 15, 7, 9, 3, 13, 5, 11$*, we have the following relations*

$$R_u = \frac{1}{2}\Big(\sqrt{r_u^2+t_M^2}+\sqrt{r_u^2+t_m^2}\Big), \quad d_u = \frac{1}{2}\Big(\sqrt{r_u^2+t_M^2}-\sqrt{r_u^2+t_m^2}\Big), \quad r_u$$

where $u = 111, 112, 121, 122, 211, 212, 221, 222$ *and*

$$r_{111}^2 = r_{11}^2 + \sqrt{(r_{11}^2+t_M^2)(r_{11}^2+t_m^2)} + r_{11}\Big(\sqrt{r_{11}^2+t_M^2}+\sqrt{r_{11}^2+t_m^2}\Big),$$

$$r_{112}^2 = r_{11}^2 + \sqrt{(r_{11}^2+t_M^2)(r_{11}^2+t_m^2)} - r_{11}\Big(\sqrt{r_{11}^2+t_M^2}+\sqrt{r_{11}^2+t_m^2}\Big),$$

$$r_{121}^2 = r_{12}^2 + \sqrt{(r_{12}^2+t_M^2)(r_{12}^2+t_m^2)} + r_{12}\Big(\sqrt{r_{12}^2+t_M^2}+\sqrt{r_{12}^2+t_m^2}\Big),$$

$$r_{122}^2 = r_{12}^2 + \sqrt{(r_{12}^2+t_M^2)(r_{12}^2+t_m^2)} - r_{12}\Big(\sqrt{r_{12}^2+t_M^2}+\sqrt{r_{12}^2+t_m^2}\Big),$$

$$r_{211}^2 = r_{21}^2 + \sqrt{(r_{21}^2+t_M^2)(r_{21}^2+t_m^2)} + r_{21}\Big(\sqrt{r_{21}^2+t_M^2}+\sqrt{r_{21}^2+t_m^2}\Big),$$

$$r_{212}^2 = r_{21}^2 + \sqrt{(r_{21}^2+t_M^2)(r_{21}^2+t_m^2)} - r_{21}\Big(\sqrt{r_{21}^2+t_M^2}+\sqrt{r_{21}^2+t_m^2}\Big),$$

$$r_{221}^2 = r_{22}^2 + \sqrt{(r_{22}^2+t_M^2)(r_{22}^2+t_m^2)} + r_{22}\Big(\sqrt{r_{22}^2+t_M^2}+\sqrt{r_{22}^2+t_m^2}\Big),$$

$$r_{222}^2 = r_{22}^2 + \sqrt{(r_{22}^2+t_M^2)(r_{22}^2+t_m^2)} - r_{22}\Big(\sqrt{r_{22}^2+t_M^2}+\sqrt{r_{22}^2+t_m^2}\Big),$$

$r_{11}^2, r_{12}^2, r_{21}^2, r_{22}^2$ *are given by* (2.38)–(2.41) *and parameters* t_M *and* t_m *are from* $\boldsymbol{R}_+$ *and* $t_M \geq t_m$.

In the following theorem we start from $n=3$.

Theorem 13. *Let instead letters R, d, r in Euler's relation* (1.2) *be written R_0, d_0, r_0, that is, let Euler's relation for triangles be written as*

$$R_0^2 - d_0^2 - 2r_0R_0 = 0. \tag{2.42}$$

Then its 2-parametric presentation can be written as

(2.43)

$$R_0 = \frac{1}{2}\left(\sqrt{r_0^2+t_M^2}+\sqrt{r_0^2+t_m^2}\right), \quad d_0 = \frac{1}{2}\left(\sqrt{r_0^2+t_M^2}-\sqrt{r_0^2+t_m^2}\right), \quad r_0$$

where t_M and t_m are parameters from $\boldsymbol{R}_+$ such that $t_M \geq t_m$ and r_0^2 is given by

$$\text{(2.44)}\quad r_0^2 = -\frac{1}{3}(t_m^2+t_M^2)+\frac{1}{2}\sqrt{-\frac{4}{3}t_m^2t_M^2+\frac{4}{9}(t_m^2+t_M^2)^2+\frac{2}{3}\lambda}\,+$$
$$+\frac{1}{2}\sqrt{-\frac{8}{3}t_m^2t_M^2+\frac{8}{9}(t_m^2+t_M^2)^2-\frac{2}{3}\lambda+\frac{\frac{32}{3}t_m^2t_M^2(t_m^2+t_M^2)-\frac{64}{27}(t_m^2+t_M^2)^3}{4\sqrt{-\frac{4}{3}t_m^2t_M^2+\frac{4}{9}(t_m^2+t_M^2)^2+\frac{2}{3}\lambda}}}$$

where $\lambda = \sqrt[3]{-2t_m^4t_M^4(t_m^2-t_M^2)^2}$.

Proof. Replacing R_0 and d_0 in relation (2.42) with their expressions given by (2.43) we get equation in r_0^2 which can be written as

$$3r_0^8 + 4r_0^6(t_M^2+t_m^2) + 6r_0^4t_M^2t_m^2 - t_M^4t_m^4 = 0.$$

It is not difficult to see that this equation has only one positive root and that it is r_0^2 given by (2.44). ♢

The following three theorems are corollaries of Th. 10 and Th. 13.

Theorem 14. *Let, for convenience in the following theorems, instead of R, d, r in Fuss' relation* (1.1c) *for bicentric hexagons be written R_1, d_1, r_1. Then its 2-parametric presentation can be written as*

(2.45)

$$R_1 = \frac{1}{2}\left(\sqrt{r_1^2+t_M^2}+\sqrt{r_1^2+t_m^2}\right), \quad d_1 = \frac{1}{2}\left(\sqrt{r_1^2+t_M^2}-\sqrt{r_1^2+t_m^2}\right), \quad r_1$$

where

$$r_1^2 = r_0^2 + \sqrt{(r_0^2+t_M^2)(r_0^2+t_m^2)} + r_0\left(\sqrt{r_0^2+t_M^2}+\sqrt{r_0^2+t_m^2}\right) \tag{2.46}$$

and r_0^2 is given by (2.44).

Theorem 15. *For 2-parametric presentation of Fuss' relations for k-outscribed bicentric 12-gons, $k = 1, 5$, we have the following relations*

$$R_u = \frac{1}{2}\Big(\sqrt{r_u^2 + t_M^2} + \sqrt{r_u^2 + t_m^2}\Big), \quad d_u = \frac{1}{2}\Big(\sqrt{r_u^2 + t_M^2} - \sqrt{r_u^2 + t_m^2}\Big), \quad r_u$$

where $u = 11, 12$ and

$$r_{11}^2 = r_1^2 + \sqrt{(r_1^2 + t_M^2)(r_1^2 + t_m^2)} + r_1\Big(\sqrt{r_1^2 + t_M^2} + \sqrt{r_1^2 + t_m^2}\Big), \tag{2.47}$$

$$r_{12}^2 = r_1^2 + \sqrt{(r_1^2 + t_M^2)(r_1^2 + t_m^2)} - r_1\Big(\sqrt{r_1^2 + t_M^2} + \sqrt{r_1^2 + t_m^2}\Big), \tag{2.48}$$

r_1^2 is given by (2.46).

Theorem 16. *For 2-parametric presentation of Fuss' relations for k-outscribed bicentric 24-gons, $k = 1, 11, 5, 7$, we have the following relations*

$$R_u = \frac{1}{2}\Big(\sqrt{r_u^2 + t_M^2} + \sqrt{r_u^2 + t_m^2}\Big), \quad d_u = \frac{1}{2}\Big(\sqrt{r_u^2 + t_M^2} - \sqrt{r_u^2 + t_m^2}\Big), \quad r_u$$

where $u = 111, 112, 121, 122$ and

$$r_{111}^2 = r_{11}^2 + \sqrt{(r_{11}^2 + t_M^2)(r_{11}^2 + t_m^2)} + r_{11}\Big(\sqrt{r_{11}^2 + t_M^2} + \sqrt{r_{11}^2 + t_m^2}\Big),$$

$$r_{112}^2 = r_{11}^2 + \sqrt{(r_{11}^2 + t_M^2)(r_{11}^2 + t_m^2)} - r_{11}\Big(\sqrt{r_{11}^2 + t_M^2} + \sqrt{r_{11}^2 + t_m^2}\Big),$$

$$r_{121}^2 = r_{12}^2 + \sqrt{(r_{12}^2 + t_M^2)(r_{12}^2 + t_m^2)} + r_{12}\Big(\sqrt{r_{12}^2 + t_M^2} + \sqrt{r_{12}^2 + t_m^2}\Big),$$

$$r_{122}^2 = r_{12}^2 + \sqrt{(r_{12}^2 + t_M^2)(r_{12}^2 + t_m^2)} - r_{12}\Big(\sqrt{r_{12}^2 + t_M^2} + \sqrt{r_{12}^2 + t_m^2}\Big),$$

r_{11}^2, r_{12}^2 are given by (2.47) and (2.48).

From the above given theorems concerning bicentric polygons it can be concluded that the following very interesting theorem holds good.

Theorem 17. *Let r_u^2, where $u \in \{0, 1, 2, 11, 12, 21, 22, \dots\}$, be written as in Th. 10, that is*

$$r_u^2 = h_u(t_M, t_m).$$

Then this relation becomes corresponding Fuss' relation if t_M and t_m are replaced, respectively, by

$$(R_u + d_u)^2 - r_u^2, \qquad (R_u - d_u)^2 - r_u^2.$$

So, for example, starting from $r_0^2 = t_M t_m$ we can write

$$r_0^4 - t_M^2 t_m^2 = r_0^4 - [(R_0 + d_0)^2 - r_0^2][(R_0 - d_0)^2 - r_0^2] =$$
$$= (R_0^2 - d_0^2) - 2r_0^2(R_0^2 + d_0^2) = 0.$$

Thus, we get Fuss' relation for bicentric quadrilaterals only instead of letters R, d, r we have symbols R_0, d_0, r_0.

Starting from relation (2.36) *we get Fuss' relation for* 1*-outscribed bicentric octagon. And so on.*

Such way of obtaining Fuss' relations may be sometimes more convenient than that when Cor. A.4 is used. (See Conj. 2.)

In the end we state the following conjecture.

Conjecture 4. 2*-parametric presentations of Fuss' relations for $n \geq 5$ can be obtained in analogous way as those obtained starting from $n = 4$ and $n = 3$.*

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