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MONOMIAL DIFFERENCES MAJORIZED BY GIVEN FUNCTIONS

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Abstract: A function M defined on a semigroup (group, Banach space etc.) and taking values in an Abelian group is called *monomial of order* (*at most*) n whenever

$$\triangle_y^n M(x) = n!M(y).$$

We consider the functional inequality

$$\|n!F(y) - \triangle_y^n F(x)\| \le \Phi(x, y),$$

and we look for conditions ensuring the existence of a nonnegative constant c such that

$$\|F(x)\| \le \frac{1}{n!}\Phi(x, x) + c\|x\|^n.$$

1. Introduction

Given functions F and f satisfying the inequality

$$\|F(x+y) - F(x) - F(y)\| \le f(x) + f(y) - f(x+y)$$

$\big($resp.

$$\|F(x+y)+F(x-y)-2F(x)-2F(y)\| \le 2f(x)+2f(y)-f(x+y)-f(x-y)\big),$$

R. Ger was looking in [6] for conditions implying the existence of a constant c such that

$$\|F(x)\| \le f(x) + c\|x\| \quad \big(\text{resp.} \quad \|F(x)\| \le f(x) + c\|x\|^2\big).$$

Under the assumption that the functions F and f fulfill the inequality

$$\|n!F(y) - \triangle_y^n F(x)\| \le n!f(y) - \triangle_y^n f(x),$$

we were looking in [3] and [4] for conditions ensuring the existence of a nonnegative constant c such that

$$\|F(x)\| \le f(x) + c\|x\|^n.$$

Now we deal with the following functional inequality

$$\|n!F(y) - \triangle_y^n F(x)\| \le \Phi(x,y).$$

We will look for conditions implying the existence of a constant c such that

$$\|F(x)\| \le \frac{1}{n!}\Phi(x,x) + c\|x\|^n.$$

2. Difference operator and monomial functions

Definition 1. Let $(S,+)$ be a semigroup, and let $(G,+)$ stand for an Abelian group. Let $f: S \longrightarrow G$ and $y \in S$ be fixed. Then *a difference operator* $\triangle_y$ is defined by the formula

$$\triangle_y f(x) = f(x+y) - f(x) \text{ for all } x \in S.$$

Let further $y_1, \ldots, y_n \in S$ be given. Then $\triangle_{y_1,\ldots,y_n}$ is defined by

$$\triangle_{y_1,\ldots,y_n} f(x) = \triangle_{y_1} \circ \ldots \circ \triangle_{y_n} f(x)$$

for all $x \in S$.

In the case when $y_1 = \ldots = y_n = y$, we will use the symbol $\triangle_y^n f(x)$ instead of $\triangle_{y,\ldots,y} f(x)$.

We will apply the following, well-known lemmas (see e.g. M. Kuczma [7] or L. Székelyhidi [9].

Lemma 1. *Let $(S,+)$ and $(G,+)$ be Abelian groups, and let $f: S \longrightarrow G$ be a function. For every $n \in \mathbb{N}$ and for every $x, y_1, \ldots, y_n \in S$ we have*

$$\triangle_{y_1,\ldots,y_n} f(x) = \sum_{\varepsilon_1,\ldots,\varepsilon_n=0}^{1} (-1)^{n-(\varepsilon_1+\ldots+\varepsilon_n)} f(x + \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n).$$

In particular,

$$\triangle_y^n f(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+jy) \quad \text{for all} \quad x, y \in S.$$

Lemma 2. *Let $(S,+)$ and $(G,+)$ be Abelian groups. Let $F : S^k \longrightarrow G$ be a symmetric k-additive function, and let $f : S \longrightarrow G$ be the diagonalization of F, i.e. $f(x) = F(x,\dots,x)$ for all $x \in S$. For every $n \in \mathbb{N},\ n \geq k$, and for every $x, y_1, \dots, y_n \in S$ we have*

$$\triangle_{y_1,\dots,y_n} f(x) = \begin{cases} k!F(y_1,\dots,y_k), & \text{if } n = k, \\ 0, & \text{if } n > k. \end{cases}$$

Lemma 3. *Let $(S,+)$ be an Abelian semigroup, and let $(G,+)$ be an Abelian group uniquely divisible by $n!$. Then, for any monomial function $f : S \longrightarrow G$ of order n, there exists exactly one n-additive and symmetric function $F : S^n \longrightarrow G$ such that f coincides with the diagonalization of F.*

Lemma 4. *Let $(X, \|\cdot\|)$ be a real normed linear space. Let $F : X^n \longrightarrow \mathbb{R}$ be a symmetric n-additive function, and let $f : X \longrightarrow \mathbb{R}$ be the diagonalization of F. If the function f is continuous on X, then so is the function F on X^n.*

We will also need the following lemma (see e.g. I. W. Sandberg [8], R. Ger [6] or W. W. Breckner, T. Trif [2]):

Lemma 5. *Let $(X, \|\cdot\|)$ be a Banach space, and let $(Y, \|\cdot\|)$ be a normed linear space. Let further $\{\Phi_\alpha : \alpha \in T\}$ be a nonempty family of n-linear symmetric and continuous operators from X^n into Y. If, for every $(x_1,\dots,x_n) \in X^n$, the set $\{\Phi_\alpha(x_1,\dots,x_n) : \alpha \in T\}$ is bounded in Y, then*

$$\sup_{\alpha \in T} \|\Phi_\alpha\| < \infty.$$

3. Monomial selections of set-valued maps

If S is a nonempty set, then by $\mathcal{B}(S,\mathbb{R})$ we denote the real linear space of all bounded real-valued functions defined on S, equipped with the uniform norm.

Definition 2. A mapping $\mathcal{M} : \mathcal{B}(S,\mathbb{R}) \longrightarrow \mathbb{R}$ is called *a mean* provided that it has the following properties:

(i) $\mathcal{M}$ is linear ;

(ii) $\inf f(S) \leq \mathcal{M}(f) \leq \sup f(S)$ for all $f \in \mathcal{B}(S,\mathbb{R})$.

Definition 3. Let $(S,+)$ be a semigroup. Consider a map $f : S \longrightarrow \mathbb{R}$ and fix arbitrarily a $t \in S$. The function $f_t : S \longrightarrow \mathbb{R}$, given by the formula

$$f_t(x) := f(x+t) \quad \text{for all} \quad x \in S,$$

is called *the right translate* of f.

Definition 4. The semigroup $(S,+)$ is called *right amenable* if there exists a mean $\mathcal{M}$ on $\mathcal{B}(S,\mathbb{R})$ which is invariant with respect to the right translations, i.e., if

$$\mathcal{M}(f_t) = \mathcal{M}(f) \quad \text{for all} \quad f \in \mathcal{B}(S,\mathbb{R}) \quad \text{and all} \quad t \in S.$$

The notions of left invariant mean and left amenability can be defined analogously. If both left and right invariant mean exist, then S is called amenable.

Remark 1. Any Abelian group is amenable.

Remark 2. Let $\mathcal{M} : \mathcal{B}(S,\mathbb{R}) \longrightarrow \mathbb{R}$ be a mean. Then

$$|\mathcal{M}(f)| \le \|\mathcal{M}\| \cdot \|f\| = \|f\| \quad \text{for all} \quad f \in \mathcal{B}(\mathcal{S},\mathbb{R}).$$

R. Badora, Z. Páles and L. Székelyhidi have proved the theorem about monomial selections of multifunctions (see Th. 3 in [1]). In the case when S is an Abelian group, $X = \mathbb{R}$ and $n = 1$, this theorem may be stated as follows

Theorem I. *Let $(S,+)$ be an Abelian group. Let $\Psi : S \longrightarrow 2^{\mathbb{R}}$ be a map with values being compact intervals. Assume that there exists a function $f : S \longrightarrow \mathbb{R}$ such that*

$$\frac{1}{n!} \triangle_t^n f(x) \in \Psi(t) \quad \textit{for all} \quad x,t \in S.$$

Then there exists a monomial function $F : S \longrightarrow \mathbb{R}$ of order n such that $F(x) \in \Psi(x)$ for all $x \in S$.

Remark 3. The function F in Th. I is given by the formula

$$F(t) = \mathcal{M}(\psi_t) \quad \text{for all} \quad t \in S,$$

where $\psi_t : S \longrightarrow \mathbb{R}$ is defined by

$$\psi_t(x) := \frac{1}{n!} \triangle_t^n f(x) \quad \text{for all} \quad x \in S,$$

and $\mathcal{M} : \mathcal{B}(S,\mathbb{R}) \longrightarrow \mathbb{R}$ is an invariant mean.

4. Results

In the proof of our first theorem we shall be using the following version of Taylor's formula (see e.g. J. Dieudonné [5]).

Theorem II. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be real Banach spaces. Further let $F : X \longrightarrow Y$ be an n-times continuously differentiable function, and let $x_0 \in X$. Then, for every $x \in X$, we have*

$$F(x) = \sum_{k=0}^{n-1} \frac{1}{k!} d^k F(x_0)(x - x_0) + R(x),$$

where

$$R(x) = \int_0^1 \frac{(1-\xi)^{n-1}}{(n-1)!} d^n F(x_0 + \xi(x - x_0))(x - x_0) d\xi.$$

Moreover, if there exists a constant α *such that*

$$\|d^n F(x)\| \le \alpha \quad \textit{for all} \quad x \in X,$$

then

$$\|R(x)\| \le \frac{\alpha}{n!} \|x - x_0\|^n \quad \textit{for all} \quad x \in X.$$

In the above-mentioned theorem $D^k F(x)$ denotes the k-th Fréchet differential of the function F at a point x. Clearly, $D^k F(x)$ is a k-additive and symmetric mapping. The monomial generated by $D^k F(x)$ is denoted by $d^k F(x)$. The integral occuring here is understood in the sense of Bochner.

Theorem 1. *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be real Banach spaces. Further, let* $F : X \to Y$ *be an* n*-times continuously differentiable function, and let* $\Phi : X^2 \longrightarrow \mathbb{R}$ *be a function such that the inequality*

$$\|n! F(y) - \triangle_y^n F(x)\| \le \Phi(x, y) \tag{1}$$

holds for all $x, y \in X$*. If the function* $X \ni x \longrightarrow \|d^n F(x)\|$ *is bounded, then there exists a nonnegative constant* c *such that*

$$\|F(x)\| \le c\|x\|^n + \frac{1}{n!} \Phi(x, x) \quad \textit{for all} \quad x \in X.$$

Proof. By virtue of Th. II applied for $x_0 = 0$ we obtain

$$F(x) = \sum_{k=0}^{n-1} \frac{1}{k!} d^k F(0)(x) + R(x) \quad \text{for all} \quad x \in X,$$

with

$$R(x) = \int_0^1 \frac{(1-\xi)^{n-1}}{(n-1)!} d^n F(\xi x)(x) d\xi.$$

Fix arbitrarily $x, y \in X$. By Lemma 2, we infer that

$$\triangle_y^n F(x) = \triangle_y^n R(x). \tag{2}$$

Now Lemma 1 and the subadditivity of the norm imply the inequality

$$\| \triangle_y^n R(x)\| \leq \sum_{k=0}^{n} \binom{n}{k} \|R(x+ky)\|. \tag{3}$$

Then, by (1), (2) and (3), we deduce that

$$\|n!F(y)\| \leq \Phi(x,y) + \| \triangle_y^n F(x)\| = \Phi(x,y) + \| \triangle_y^n R(x)\| \leq$$
$$\leq \Phi(x,y) + \sum_{k=0}^{n} \binom{n}{k} \|R(x+ky)\|.$$

In particular, for $x = y$, one obtains

$$\|n!F(x)\| \leq \Phi(x,x) + \sum_{k=0}^{n} \binom{n}{k} \|R((k+1)x)\|. \tag{4}$$

Since the function $X \ni x \longrightarrow \|d^n F(x)\|$ is bounded, there exists a constant α such that

$$\|d^n F(x)\| \leq \alpha \quad \text{for all} \quad x \in X. \tag{5}$$

Hence, by (4), (5) and Th. II we obtain

$$\|n!F(x)\| \leq \Phi(x,x) + \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n!} \alpha \|(k+1)x\|^n \quad \text{for all} \quad x \in X.$$

Put

$$c := \frac{\alpha}{n!} \sum_{k=0}^{n} \frac{(k+1)^n}{k!(n-k)!}.$$

Then we have

$$\|F(x)\| \leq \frac{1}{n!} \Phi(x,x) + c\|x\|^n \quad \text{for all} \quad x \in X,$$

which completes the proof. ◊

Remark 4. Under the assumptions of Th. 1, we may show that also the following inequality is true:

$$\|F(x)\| \leq \frac{1}{n!} \Phi(0,x) + C\|x\|^n \quad \text{for all} \quad x \in X,$$

where

$$C = \frac{\alpha}{n!} \sum_{k=0}^{n} \frac{(k)^n}{k!(n-k)!}.$$

In fact, it suffices to take $x = 0$ and $y = x$ in the inequality

$$\|n!F(y)\| \leq \Phi(x,y) + \sum_{k=0}^{n} \binom{n}{k} \|R(x+ky)\|.$$

In the case when function Φ depends only upon the second variable, our assumption about the space Y as well as the assumption upon the function F may considerably be weakened. Namely, the following theorem holds true.

Theorem 2. *Let $(X, \|\cdot\|)$ be a real Banach space, and let $(Y, \|\cdot\|)$ be a real normed linear space. Let $F : X \longrightarrow Y$ be a continuous function, and let $\varphi : X \longrightarrow \mathbb{R}$ be a function such that inequality*

$$\|n!F(y) - \triangle_y^n F(x)\| \leq \varphi(y) \tag{6}$$

holds for all $x, y \in X$. Then there exists a nonnegative constant c such that

$$\|F(x)\| \leq \frac{1}{n!}\varphi(x) + c\|x\|^n \quad \textit{for all} \quad x \in X.$$

Proof. For each $y^* \in Y^*$ with $\|y^*\| = 1$ and for all $x, y \in X$ we have

$$-\varphi(y) \leq n!y^* \circ F(y) - \triangle_y^n y^* \circ F(x) \leq \varphi(y). \tag{7}$$

Fix arbitrarily a $y^* \in Y^*$ with $\|y^*\| = 1$ and define the functions $H_{y^*} : X \longrightarrow \mathbb{R}$ and $\Psi_{y^*} : X \longrightarrow 2^{\mathbb{R}}$ by the formulas

$$H_{y^*}(x) := -y^* \circ F(x) \quad \text{for all} \quad x \in X$$

and

$$\Psi_{y^*}(x) := \left[-\frac{1}{n!}\varphi(x) - y^* \circ F(x), \frac{1}{n!}\varphi(x) - y^* \circ F(x)\right] \quad \text{for all} \quad x \in X,$$

respectively. Clearly, the values of the function Ψ_{y^*} are compact intervals.

By (7) and by the definition of the function Ψ_{y^*} we obtain

$$\frac{1}{n!} \triangle_y^n H_{y^*}(x) \in \Psi_{y^*}(y)$$

for all $y^* \in Y^*$ with $\|y^*\| = 1$ and for all $x, y \in X$.

By virtue of Th. I, for every $y^* \in Y^*$ with $\|y^*\| = 1$ there exists a monomial function $M_{y^*} : X \longrightarrow \mathbb{R}$ of order n such that

$$M_{y^*}(x) \in \Psi_{y^*}(x) \quad \text{for all} \quad x \in X.$$

Hence we obtain

$$|y^*(F(x)) + M_{y^*}(x)| \leq \frac{1}{n!}\varphi(x) \quad \text{for all} \quad x \in X, \tag{8}$$

for all $y^* \in Y^*$ with $\|y^*\| = 1$, and, consequently,

$$|M_{y^*}(x)| \leq |y^*(F(x))| + \frac{1}{n!}\varphi(x) \leq \|F(x)\| + \frac{1}{n!}\varphi(x) \text{ for all } x \in X. \tag{9}$$

Moreover, in view of Rem. 3, $M_{y^*}(t) = \mathcal{M}(\psi_{t,y^*})$ for all $t \in X$, where $\mathcal{M} : \mathcal{B}(X,\mathbb{R}) \longrightarrow \mathbb{R}$ is an invariant mean, and for every $t \in X$ we have

$$\psi_{t,y^*}(x) = \frac{1}{n!} \triangle_t^n H_{y^*}(x), \quad x \in X,$$

and

$$H_{y^*}(x) = -y^* \circ F(x) \ \text{ whenever } \ x \in X.$$

Let $y^* \in Y^*$ with $\|y^*\| = 1$ be fixed. We will show that the function M_{y^*} is continuous. Fix arbitrarily a $t_0 \in X$. Then, for any $t \in X$, one obtains

$$\begin{aligned}\big|M_{y^*}(t) - M_{y^*}(t_0)\big| &= \big|\mathcal{M}(\psi_{t,y^*}) - \mathcal{M}(\psi_{t_0,y^*})\big| = \big|\mathcal{M}(\psi_{t,y^*} - \psi_{t_0,y^*})\big| \leq \\ &\leq \|\mathcal{M}\| \cdot \big\|\psi_{t,y^*} - \psi_{t_0,y^*}\big\| \leq \big\|\psi_{t,y^*} - \psi_{t_0,y^*}\big\| = \\ &= \sup_{x\in X}\big|\psi_{t,y^*}(x) - \psi_{t_0,y^*}(x)\big|.\end{aligned}$$

For all $t \in X$ and all $x \in X$ we have

$$\begin{aligned}&\big|\psi_{t,y^*}(x) - \psi_{t_0,y^*}(x)\big| = \\ &= \left|\frac{1}{n!} \triangle_t^n H_{y^*}(x) - \frac{1}{n!} \triangle_{t_0}^n H_{y^*}(x)\right| = \\ &= \frac{1}{n!}\left|\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}H_{y^*}(x+it) - \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}H_{y^*}(x+it_0)\right| \leq \\ &\leq \frac{1}{n!}\sum_{i=1}^{n}\binom{n}{i}\big|H_{y^*}(x+it) - H_{y^*}(x+it_0)\big|.\end{aligned}$$

Fix arbitrarily an $\varepsilon > 0$. By the continuity of function H_{y^*}, there exists for each $i \in \{1,\dots,n\}$ a $\delta_i > 0$ such that for all $x, t \in X$ one has

$$\big\|(x+it) - (x+it_0)\big\| < \delta_i \Longrightarrow \big|H_{y^*}(x+it) - H_{y^*}(x+it_0)\big| < \frac{\varepsilon}{2^{n+1}}.$$

Let $\delta := \min\big\{\frac{\delta_i}{i} : i \in \{1,\dots,n\}\big\}$. Then

$$\begin{aligned}\big|\psi_{t,y^*}(x) - \psi_{t_0,y^*}(x)\big| &\leq \frac{1}{n!}\sum_{i=1}^{n}\binom{n}{i}\big|H_{y^*}(x+it) - H_{y^*}(x+it_0)\big| \leq \\ &\leq \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2n!}\end{aligned}$$

for all $x \in X$ and all $t \in X$ such that $\|t - t_0\| < \delta$.

Hence we deduce that

$$\big|M_{y^*}(t) - M_{y^*}(t_0)\big| \leq \sup_{x\in X}\big|\psi_{t,y^*}(x) - \psi_{t_0,y^*}(x)\big| \leq \frac{\varepsilon}{2n!} < \varepsilon$$

for all $t \in X$ such that $\|t - t_0\| < \delta$. Therefore, the function M_{y^*} is continuous, as claimed.

Since M_{y^*} is a monomial function of order n, by Lemma 3, there exists an n-additive symmetric function $\overline{M}_{y^*} : X^n \longrightarrow \mathbb{R}$ such that

$$M_{y^*}(x) = \overline{M}_{y^*}(x, \ldots, x) \quad \text{for all} \quad x \in X.$$

In view of the continuity of M_{y^*}, it follows by virtue of Lemma 4 that $\overline{M}_{y^*}$ is continuous. Therefore, $\overline{M}_{y^*}$ is n-linear.

Now we will show that the set

$$\left\{\overline{M}_{y^*}(x_1, \ldots, x_n) : y^* \in Y^*, \|y^*\| = 1\right\}$$

is bounded for all $(x_1, \ldots, x_n) \in X^n$. Let $(x_1, \ldots, x_n) \in X^n$ and $y^* \in Y^*$ with $\|y^*\| = 1$ be fixed. Then, by Lemma 2 and (9), we have

$$\begin{aligned}
&\left|\overline{M}_{y^*}(x_1, \ldots, x_n)\right| = \left|\frac{1}{n!} \triangle_{x_1, \ldots, x_n} M_{y^*}(0)\right| = \\
&= \left|\frac{1}{n!} \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0}^{1} (-1)^{n-(\varepsilon_1 + \ldots + \varepsilon_n)} M_{y^*}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)\right| \le \\
&\le \frac{1}{n!} \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0}^{1} \left|M_{y^*}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)\right| \le \\
&\le \frac{1}{n!} \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0}^{1} \left(\|F(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)\| + \frac{1}{n!}\varphi(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)\right).
\end{aligned}$$

Since the family $\left\{\overline{M}_{y^*} : y^* \in Y^*, \|y^*\| = 1\right\}$ satisfies the assumptions of Lemma 5, there exists a nonnegative constant c such that

$$\sup_{\|y^*\|=1} \|\overline{M}_{y^*}\| \le c.$$

Hence, by (8), one obtains

$$\begin{aligned}
|y^* \circ F(x)| &\le \left|M_{y^*}(x)\right| + \frac{1}{n!}\varphi(x) = \left|\overline{M}_{y^*}(x, \ldots, x)\right| + \frac{1}{n!}\varphi(x) \le \\
&\le \left\|\overline{M}_{y^*}\right\| \cdot \|x\|^n + \frac{1}{n!}\varphi(x) \le c\|x\|^n + \frac{1}{n!}\varphi(x)
\end{aligned}$$

for all $x \in X$ and all $y^* \in Y^*$ with $\|y^*\| = 1$. Consequently, for all $x \in X$, the inequality

$$\|F(x)\| \le c\|x\|^n + \frac{1}{n!}\varphi(x)$$

is satisfied, which completes the proof. ♢

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