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ON THE DISTRIBUTION OF THE SQUARE-FULL AND *k*-FULL PARTS OF INTEGERS FOR SHORT INTERVALS

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Abstract: Local distribution of values of some multiplicative functions over short intervals is investigated.

§ 1

Let $\mathcal{P}$ be the set of all prime numbers, $\mathcal{B}_k$ be the set of k-full numbers, and $\mathcal{M}_k$ be the set of k-free numbers. As we know, $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ belongs to $\mathcal{M}_k$, if $\max \alpha_j \leq k-1$, and it belongs to $\mathcal{B}_k$ if $\min \alpha_j \geq k$.

Theorem A (Filaseta)**.** *Given an integer $k \geq 2$, let $g(x)$ be a function satisfying $1 \leq g(x) \leq \log x$ for x sufficiently large, and set*
$$h(x) = x^{1/(2k+1)} g(x)^3.$$
Then the number of k-free numbers belonging to the interval $(x, x+h(x)]$ is
$$\frac{h(x)}{\zeta(k)} + \mathcal{O}\left(\frac{h(x)\log x}{g(x)^3}\right) + \mathcal{O}\left(\frac{h(x)}{g(x)}\right),$$
where ζ stands for the Riemann zeta function.

Th. A is a result due to Professor Michael Filaseta, who kindly communicated it to Professor J. M. De Koninck. A complete proof (provided by Filaseta) is given in [2]. See also [4].

Let $\tau^{(e)}(n)$ be the number of exponential divisors of n. $\tau^{(e)}(n)$ is a multiplicative function, for prime powers p^α, $\tau^{(e)}(p^\alpha) = \tau(\alpha)$, $\tau(n) =$ = number of divisors of n.

By using Th. A in [1] we proved:

Let $h(x) = x^{1/5}(\log x)^{3/2}u(x)$, where $u(x) \to \infty$ as $x \to \infty$, and $h(x) \le x$. Then, for each fixed real number $\alpha > 0$, there exists a constant $c = c(\alpha) > 0$ such that

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} \tau^{(e)}(n)^\alpha = c.$$

As we noted, the method we used can be applied to show

Theorem B. *Let f be a non-negative multiplicative function such that $f(p) = 1$ for each prime p, $f(n) = \mathcal{O}(n^\varepsilon)$ for every $\varepsilon > 0$. Let also $h(x) = x^{1/5}(\log x)^{3/2}u(x)$, where $u(x) \to \infty$ as $x \to \infty$, with $h(x) \le x$. Then*

$$\frac{1}{h(x)} \sum_{n\in[x,x+h(x)]} f(n) = c + o(1),$$

where $c = c(f)$ is a positive constant given by

$$c(f) = \sum_{k\in\mathcal{B}_2} \frac{f(k)}{k} \prod_{p|k} \left(1 + \frac{1}{p}\right)^{-1}.$$

Let $M(x) = \sum_{n\le x} |\mu(n)|$, and $M(x|r) = \sum_{\substack{n\le x\\(n,r)=1}} |\mu(n)|$. According to (8) in [1] we have

$$(1)\qquad M\left(\frac{x+h(x)}{k}|k\right) - M\left(\frac{x}{k}\,|k\right) =$$

$$= \frac{6}{\pi^2}\frac{h(x)}{k}\prod_{p|k}\frac{1}{1+1/p} + \mathcal{O}\left(\frac{h(x)}{k\sqrt{\log x}}\exp\left\{\sum_{p|k}\frac{2}{\sqrt{p}}\right\}\right)$$

if $k < \log x$.

For some integer n, let $E(n)$ be the square-full, and $F(n)$ be the square-free part of n. Then $n = E(n)F(n)$, $(E(n), F(n)) = 1$, $E(n)$ is the largest divisor of n which belongs to $\mathcal{B}_2$. Let $\mathcal{R}_b$ be the set of those integers n for which $E(n) = b$.

Let

$$\nu(b) := \lim_{x\to\infty} \frac{1}{x}\#\{n \le x, E(n) = b\}.$$

It is clear that

$$\nu(b) = \lim_{x\to\infty} \frac{1}{x} M(x \mid b) = \frac{1}{\zeta(2)b} \prod_{p\mid b} \frac{1}{1+1/p}. \tag{2}$$

The following theorem is an immediate consequence of (1).

Theorem 1. *Let $h(x) = x^{1/5}(\log x)^{3/2}u(x)$, $u(x) \to \infty$ as $x \to \infty$, $h(x) \le x$. Then*

$$\frac{1}{h(x)}\#\{n\in[x, x+h(x)],\ \ E(n)=b\} = \nu(b) + \mathcal{O}\left(\frac{1}{b\sqrt{\log x}} \exp\left\{\sum_{p\mid b} \frac{2}{\sqrt{p}}\right\}\right)$$

uniformly as $b < \log x$, $b \in \mathcal{B}_2$.

Remark. This is an improvement of our Th. 3 in [4], according to:

$$\frac{1}{H}\#\{n \in [x, x+H] \mid n \in \mathcal{R}_b\} = \nu(b) + \mathcal{O}\left(H^{-1}\cdot x^{\theta+\varepsilon}\cdot 2^{\omega(b)}\right) + \\ + \mathcal{O}\left(H^{-1/2}x^{\varepsilon} \prod_{p\mid b}\left(1+\frac{1}{\sqrt{p}}\right)\right)$$

uniformly as $0 < H < x$. Here $\theta = 0,2204$ and ε is an arbitrary positive constant. The implied constants in the error terms may depend on ε.

We deduced this from a theorem of P. Varbanets [6]: Let $\phi(d)$ be a multiplicative function, such that $\phi(d) = \mathcal{O}(d^{\varepsilon})$ for $\varepsilon > 0$. Let

$$f(n) = \sum_{d^2\mid n} \phi(d).$$

Then

$$\sum_{x\le n\le x+h} f(n) = h\sum_{d=1}^{\infty} \frac{\phi(d)}{d^2} + \mathcal{O}(h^{1/2}x^{\varepsilon}) + \mathcal{O}(x^{\theta+\varepsilon}),$$

uniformly in h, $(0 <)h < x$, $0 < \varepsilon$ is an arbitrary constant, $\theta = 0,2204$.

§ 2

Let $k \ge 2$ be a fixed integer. For some integer $n > 0$ let $E_k(n)$ be the k-full part and $F_k(n)$ be the k-free part of n. Thus $n = E_k(n)F_k(n)$, $(E_k(n), F_k(n)) = 1$. Let S_b be the set of those integers n, for which $E_k(n) = b$.

Let

$$\sigma(b) := \lim_{x\to\infty} \frac{1}{x}\#\{n \le x,\ E_k(n) = b\}. \tag{3}$$

Let

$$Q_k(x) = \#\{n \le x,\ n \in \mathcal{M}_k\}, \tag{4}$$

$$Q_k(x|r) = \#\{n \le x,\ n \in \mathcal{M}_k,\ (n,r)=1\}. \tag{5}$$

A. Walfisz [7] proved that
$R_k(x) := Q_k(x) - \dfrac{x}{\zeta(k)} = \mathcal{O}\big(x^{1/k}\exp\big(-A\cdot k^{-8/5}(\log x)^{3/5}\cdot(\log\log x)^{-1/5}\big)\big)$.
D. Suryanarayana [8] proved that

$$Q_k(x \mid r) = \frac{r^{k-1}\varphi(r)x}{J_k(r)\zeta(k)} + \mathcal{O}\left(\frac{\varphi(r)\cdot 2^{\omega(r)}}{r}x^{1/k}\right). \tag{6}$$

Here $J_k(r)$ is Jordan's totient. Observe that $\frac{r^{k-1}\varphi(r)}{K_k(r)} = \prod\limits_{p|r}\frac{1-\frac{1}{p}}{1-\frac{1}{p^k}}$.

Let η be a strongly multiplicative function, $\eta(p) = \frac{1-\frac{1}{p}}{1-\frac{1}{p^k}}$ for $p \in \mathcal{P}$. Then

$$\sigma(b) = \frac{\eta(b)x}{b\zeta(k)} \quad (b \in \mathcal{M}_k).$$

By using Th. A we can estimate

$$U_b(x) := \#\{n \in [x, x+h(x)], \quad E_k(n) = b\} \tag{7}$$

if $h(x) = x^{\frac{1}{2k+1}}g(x)^3(\log x)^{2-\frac{2}{2k+1}}$, $g(x)$ tends to infinity monotonically, $g(x) \le \log x$, uniformly as $b < \log x,\ b \in \mathcal{B}_k$.

It is clear that

$$U_b(x) = Q_k\left(\frac{x+h(x)}{b} \mid b\right) - Q_k\left(\frac{x}{b} \mid b\right).$$

Since

$$\begin{aligned} F_b(s) = \sum_{\substack{m\in\mathcal{M}_k\\(m,b)=1}}\frac{1}{m^s} &= \prod_{p\nmid b}\left(1+\frac{1}{p^s}+\ldots+\frac{1}{p^{(k-1)s}}\right)\\ &= F_1(s)\prod_{p|b}\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{ks}}}, \qquad F_1(s) = \frac{\zeta(s)}{\zeta(ks)}, \end{aligned}$$

we can obtain that

$$U_b(x) = \sum_{v\in\mathcal{D}_b} f(v)\left\{Q_k\left(\frac{x+h(x)}{bv}\right) - Q_k\left(\frac{x}{bv}\right)\right\}, \tag{8}$$

where $\mathcal{D}_b$ is the multiplicative semigroup generated by the prime divisors of b, i.e.

$\mathcal{D}_b = \{p_1^{\varepsilon_1}\dots p_r^{\varepsilon_r} \mid \varepsilon_j = 0,1,\dots,\quad j=1,\dots,r\}$ if $b = p_1^{a_1}\dots p_r^{a_r}$,

furthermore f is a multiplicative function,

$$f(p^\alpha) = \begin{cases} -1, & \text{if } \alpha \equiv 1 \pmod k,\\ \ \ 1, & \text{if } \alpha\equiv 0 \pmod k,\\ \ \ 0 & \text{otherwise.}\end{cases}$$

From Th. A we have

$$U_b(x) = \sum_{\substack{v\le v(x)\\ v\in\mathcal{D}_b}} f(v)\left\{\frac{h(x)}{bv\zeta(k)} + \mathcal{O}\left(\frac{h(x)\log x}{bv\cdot g(x)^3}\right) + \mathcal{O}\left(\frac{h(x)}{bv\cdot g(x)}\right)\right\} +$$
$$+\mathcal{O}\left(\sum_{\substack{v\in\mathcal{D}_b\\ vb\le h(x)\\ v>v(x)}} |f(v)|\frac{h(x)}{bv}\right) + \mathcal{O}(S_b),$$

where

$$S_b = \sum_{\substack{v\in\mathcal{D}_b\\ x=vbm\le x+h(x)\\ vb>h(x)}} |f(v)|. \tag{9}$$

Thus

$$U_b(x) = h(x)\sigma(b) + \mathcal{O}\left(\frac{h(x)\log x}{g(x)^3 b}\right) + \mathcal{O}\left(\frac{h(x)}{b\cdot g(x)}\right) +$$
$$+\mathcal{O}\left(\frac{h(x)}{b\sqrt{v(x)}}\sum_{v\in\mathcal{D}_b}\frac{1}{\sqrt{v}}\right) + \mathcal{O}(S_b).$$

Let $v(x) = \log x$. If $b = p_1^{a_1}\dots p_r^{a_r}$ $(\le \log x)$ and $v\in\mathcal{D}_b$ is counted in (9), $v = p_1^{\gamma_1}\dots p_r^{\gamma_r}$, then $p_j^{\gamma_j} \le x + h(x) (< 2x)$, and so $\gamma_j \le \frac{\log 2x}{\log p_j}$.

Thus

$$S_b \le \prod_{j=1}^{r}\left(\frac{\log 2x}{\log p_j} + 1\right) \le (2\log 2x)^{\omega(b)}.$$

Since $\omega(b) \le \log b$, therefore $S_b = \mathcal{O}((2\log 2x)^{\log\log x}) = \mathcal{O}(x^{\varepsilon})$, for every fixed $\varepsilon > 0$.

Furthermore,

$$\sum_{v\in\mathcal{D}_b} \frac{1}{\sqrt{v}} \le \exp\left\{\sum_{p|b} \frac{2}{\sqrt{p}}\right\}.$$

We proved

Theorem 2. *Let $k \ge 2$ be fixed. Then for $b \in \mathcal{B}_k$, $b < \log x$*

$$U_b(x) = h(x)\sigma(b) + \mathcal{O}\left(\frac{h(x)\log x}{g(x)^3 b}\right) + \mathcal{O}\left(\frac{h(x)}{b\sqrt{\log x}}\exp\left(\sum_{p|b}\frac{2}{\sqrt{p}}\right)\right),$$

where $h(x) = x^{\frac{1}{2k+1}} \cdot g(x)^3 \cdot (\log x)^{\frac{4k}{2k+1}}$, $\sqrt{\log x} \le g(x) \le \log x$, $g(x) \to \infty$ as $x \to \infty$.

§ 3

Theorem 3. *Let $k \ge 2$, f be multiplicative, $f(p) = \ldots = f(p^{k-1}) = 1$ for every prime p, $f(n) = \mathcal{O}(n^{\varepsilon})$ for every $\varepsilon > 0$.*

Let

$$D(f) := \sum_{b\in\mathcal{B}_k} \sigma(b) f(b).$$

Assume that $h(x)$ is the same as in Th. 2.

Then

$$\frac{1}{h(x)} \sum_{x<n\le x+h(x)} f(n) = D(f) + \mathcal{O}\left(\frac{\log x}{g(x)^3} + \frac{1}{\sqrt{\log x}}\right).$$

This assertion is a simple consequence of Th. 2.

A positive integer n is called unitary k-free, if $p^{\alpha} \parallel n$, $\alpha > 0$, implies that $\alpha \not\equiv 0 \pmod{k}$. Let $U_k(x) = \#\{n \le x \mid n \text{ is unitary } k\text{-free}\}$.

D. Suryanarayana and R. S. R. C. Rao [9] proved that

$$U_k(x) = \alpha_k \cdot x + \mathcal{O}(x^{1/k} \exp(-A(\log x)^{3/5} \cdot (\log\log x)^{-1/5})), \tag{10}$$

$\alpha_k > 0$, constant.

Let f be multiplicative, $f(p^{\alpha}) = 1$, if $\alpha \not\equiv 0 \pmod{k}$, and $f(p^{kl}) = 0$ $(l = 1, 2, \ldots)$. Then

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is unitary } k\text{-free}, \\ 0, & \text{if } n \text{ is not unitary } k\text{-free}. \end{cases}$$

Consequently we have a short interval version of (10). The conditions of Th. 3 are satisfied.

§ 4

Let Δ be the additive function, $\Delta(p^\alpha) = \alpha - 1$. Thus $\Delta(n) = = \Omega(n) - \omega(n)$. An easy consequence of Th. 1 is the following assertion:

Let $k \geq 0$ be an integer,

$$d_k := \sum_{\substack{\Delta(b)=k \\ b\in\mathcal{B}_2}} \nu(b).$$

Then

$$\frac{1}{h(x)} \sum_{\substack{n\in[x,x+h(x)] \\ \Delta(n)=k}} 1 = (1+o_x(1))d_k, \tag{11}$$

$h(x)$ is the same as in Th. 1. This is a refinement of a theorem of J. M. De Koninck and A. Ivič [10].

§ 5

In our paper written with Subbarao [11] we have given the local distribution of $\tau(\tau(n))$ when we have taken the summation only over a short interval of type $[x, x + x^{7/12+\varepsilon}]$.

We can consider the local distribution of $\tau_2^{(e)}(n) = \tau^{(e)}(\tau^{(e)}(n))$ when we sum on intervals of type $[x, x+h(x)]$, $h(x)$ as in Th. 1. Observe that $\tau^{(e)}(n) = 1$ if and only if n is square-free.

We can write every n as $u \cdot v^2 \cdot m$, where u, v, m are mutually coprime integers, u, v are square-free, and m is cubefull.

Let v, m be fixed, $\mathcal{E}_{v,m} = \{n \mid n = u \cdot v^2 \cdot m\}$. Let $\tau^{(e)}(m) = = 2^{\beta_0} \cdot Q_1^{\beta_1} \dots Q_t^{\beta_t}$, $(2 <)Q_1 < \dots < Q_t$ be primes, $\beta_j \geq 1$ $(j = 1, \dots, t)$, $\beta_0 \geq 0$ ($\beta_0 = 0$ can occur. Then, for $n = u \cdot v^2 \cdot m$ we have $\tau^{(e)}(n) = = 2^{\omega(v)}\tau^{(e)}(m) = \tau[\omega(v) + \beta_0]\tau(\beta_1) \dots \tau(\beta_t))$.

From (1) we have

$$\#\{n \in \mathcal{E}_{v,m} \mid n \in [x, x+h(x)]\} = \tag{12}$$

$$= \frac{6}{\pi^2} \frac{h(x)}{v^2 m} \prod_{p|vm} \frac{1}{1+1/p} + \mathcal{O}\left(\frac{h(x)}{v^2 m\sqrt{\log x}} \exp\left(\sum_{p|vm} 2/\sqrt{p}\right)\right)$$

if $v^2m < \log x$.

Let $\mathcal{T}$ be the set of those k for which there exists $n \in \mathbb{N}$ such that $\tau_2^{(e)}(n) = k$. It is clear that $\mathcal{T} = \mathbb{N}$. Indeed, if $p_1, \dots, p_l$ are distinct primes, then $\tau^{(e)}(p_1^2 \dots p_l^2) = 2^l$, and if $\tau(l) = k$, i.e. $l = 2^{k-1}$, then $\tau_2^{(e)}(p_1^2 \dots p_l^2) = k$.

Let

$$\eta_k = \frac{6}{\pi^2} \sum_{\tau_2^{(e)}(v^2m)=k} \frac{1}{v^2m} \prod_{p|vm} \frac{1}{1+1/p}.$$

From Th. 1 we obtain

Theorem 4. *Let $h(x)$ be as in Th. 1. Then for every $k \in \mathbb{N}$:*

$$\lim_{x\to\infty} \frac{1}{h(x)} \#\{n \in [x, x+h(x)] \mid \tau_2^{(e)}(n) = k\} = \eta_k.$$

Let $\Pi(y \mid b)$ be the number of those primes $p \leq y$, which can be written as $p = -1 + mb$, $(m, b) = 1$, m square-free.

In [2] (Lemma 4) we proved the following

Lemma 1. *Let $\varepsilon > 0$ be a small number. Let ν and r be fixed positive integers and let $y^{7/12\varepsilon} \leq H \leq y$. Then, as $y \to \infty$,*

$$\Pi(y+H \mid b) - \Pi(y \mid b) = \frac{li(y+H) - li(y)}{b} \prod_{q \nmid b} \left(1 - \frac{1}{q(q-1)}\right) + \\ + \mathcal{O}\left(\frac{H}{b} \cdot \frac{1}{(\log y)^{r+2}}\right)$$

uniformly for $b \leq (\log y)^\nu$.

Arguing as in the proof of Th. 4, we obtain

Theorem 5. *Let*

$$\xi_k := \sum_{\tau_2^{(e)}(v^2m)=k} \frac{1}{v^2m} \prod_{q \nmid vm} \left(1 - \frac{1}{q(q-1)}\right).$$

Here v, m run over those integers for which v, m are coprimes, m is cubefull, v is square-free.

Let $\varepsilon > 0$ be an arbitrary small number, $H = H(x) \in [y^{7/12+\varepsilon}, y]$. Then

$$\sum_{\substack{\tau_2^{(e)}(p+1)=k \\ p\in[x,x+H]}} 1 = (1 + o_x(1))\xi_k(li(x+H) - lix)$$

for every $k = 0, 1, 2, \dots$.

By using the argument applied in [5] (see the proof of Th. 6,7) we can prove that by $Y = Y(x) = x^{2/3+\varepsilon}$, $\varepsilon > 0$ the limits

$$\lim_{x\to\infty} \frac{1}{li(x+Y) - li(x)} \#\{p \in [x, x+Y] \mid \tau_2^{(e)}(p^2+1) = k\} = t_k$$

exist, $\sum_{k=0}^{\infty} h_k = 1$, $\sum_{k=0}^{\infty} t_k = 1$.

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