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RELATIONSHIPS BETWEEN THE INTERSECTION CONVOLUTION AND OTHER IMPORTANT OPERATIONS ON RELATIONS

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Abstract: We establish some intimate connections between the intersection convolution and the inversion, composition and box product of relations on one groupoid to another.

The intersection convolution $F * G$ of two relations F and G on one groupoid X to another Y is a relation X to Y such that

$$\big(F * G\big)(x) = \bigcap \big\{ F(u) + G(v) : \ x = u + v, \ F(u) \neq \emptyset, \ G(v) \neq \emptyset \big\}$$

for all $x \in X$. The intersection convolution allows of a natural generalization of the Hahn–Banach type extension theorems.

1. A few basic facts on relations and groupoids

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. If in particular, $F \subset X^2$, then we may simply say that F is a relation on X. Thus, a relation F on X to Y is also a relation on $X \cup Y$.

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If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in X : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F, respectively.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the *domain* and *range* of F, respectively. If in particular $D_F = X$ $(R_F = Y)$, then we say that F is a relation of X to Y (on X onto Y).

As usual, a relation F on X is called (1) *reflexive* if $x \in F(x)$ for all $x \in D_F$; (2) *symmetric* if $y \in F(x)$ implies $x \in F(y)$; and (3) *transitive* if $y \in F(x)$ and $z \in F(y)$ imply $z \in F(x)$.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$.

If X is a set and $+$ is a function of X^2 to X, then the function $+$ is called an *operation* in X and the ordered pair $X(+) = (X, +)$ is called a *groupoid* even if X is void.

In this case, we may simply write $x + y$ in place of $+(x, y)$ for any $x, y \in X$. Moreover, we may also simply write X in place of $X(+)$ whenever the operation $+$ is clearly understood.

In the practical applications, instead of groupoids, it is usually sufficient to consider only semigroups. However, several definitions and theorems on semigroups can be naturally extended to groupoids.

For instance, if X is a groupoid, then for any $A, B \subset X$, we may naturally write $A + B = \{a + b : \ a \in A, \ b \in B\}$. Moreover, we may also write $x + A = \{x\} + A$ and $A + x = A + \{x\}$ for any $x \in X$.

Note that if in particular X is a group, then we may also naturally write $-A = \{-a : \ a \in A\}$ and $A - B = A + (-B)$ for any $A, B \subset X$. Though, the family $\mathcal{P}(X)$ of all subsets of X is only a semigroup with zero.

2. Some important operations on relations

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F. Therefore, the *inverse* relation F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in Y : y \in F(x)\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the *composition* relation $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$.

On the other hand, if F is a relation on X to Y and G is a relation on Z to W, then we may also naturally define the *box product* relation $F \boxtimes G$ such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$.

Concerning inversion and composition, we only quote here the following two theorems.

Theorem 2.1. *If F is a relation on X, then*

(1) *F is symmetric if and only if $F^{-1} \subset F$;*

(2) *F is transitive if and only if $F \circ F \subset F$.*

Remark 2.2. Note that if F is symmetric, then we actually have $F^{-1} = F$. Moreover, if F is reflexive and transitive, then under the notation $F^2 = F \circ F$ we also have $F^2 = F$.

Theorem 2.3. *If F is a relation on X to Y and G is a relation on Y to Z, then*

(1) $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$*;*

(2) $(G \circ F)[A] = G\big[F[A]\big]$ *for all $A \subset X$.*

Now, as a counterpart of the latter theorem, we can also easily establish the following

Theorem 2.4. *If F is a relation on X to Y and G is a relation on Z to W, then*

(1) $(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}$*;*

(2) $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ *for all $A \subset X \times Z$.*

Hint. To prove the inclusion $(F \boxtimes G)[A] \subset G \circ A \circ F^{-1}$, we can note that if $(y, w) \in (F \boxtimes G)[A]$, then there exists $(x, z) \in A$ such that

$$(y, w) \in (F \boxtimes G)(x, z) = F(x) \times G(z),$$

and thus $y \in F(x)$ and $w \in G(z)$. Hence, by noticing that $x \in F^{-1}(y)$, we can already see that

$$z \in A(x) \subset A\big[F^{-1}(y)\big] = \big(A \circ F^{-1}\big)(y),$$

and thus $w \in G(z) \subset G\big[\big(A \circ F^{-1}\big)(y)\big] = \big(G \circ \big(A \circ F^{-1}\big)\big)(y)$. Therefore, $(y, w) \in \big(G \circ \big(A \circ F^{-1}\big)\big) = \big(G \circ A \circ F^{-1}\big)$ also holds.

Remark 2.5. Note that the operation $\boxtimes$ and the above assertion (1) can be naturally extended to arbitrary families of relations.

3. The intersection convolution of relations

Definition 3.1. If X is a groupoid, then for any $x \in X$ and $A, B \subset X$, we define

$$\Gamma(x, A, B) = \big\{(u, v) \in A \times B : \; x = u + v\big\}.$$

Remark 3.2. Now, in particular, we may simply write $\Gamma(x)=\Gamma(x,X,X)$. This Γ is just the inverse relation of the operation $+$ in X. Moreover, we have $\Gamma\big(x,A,B\big)=\Gamma(x)\cap(A\times B)$.

Definition 3.3. If F and G are relations on one groupoid X to another Y, then we define a relation $F*G$ on X to Y such that

$$\big(F*G\big)(x)=\bigcap\big\{F(u)+G(v):\ (u,v)\in\Gamma(x,D_F,D_G)\big\}$$

for all $x\in X$. The relation $F*G$ is called the *intersection convolution* of the relations F and G.

Remark 3.4. If in particular F and G are relations of X to Y, then we may simply write

$$\big(F*G\big)(x)=\bigcap_{x=u+v}\big(F(u)+G(v)\big)=\bigcap\big\{F(u)+G(v):\ (u,v)\in\Gamma(x)\big\}.$$

A particular case of Def. 3.3 was already considered in [3]. But, the following theorem has only been proved in [4].

Theorem 3.5. *If F and G are relations on a group X to a groupoid Y, then for any $x\in X$ we have*

$$\begin{aligned}(F*G)(x)&=\bigcap\big\{F(x-v)+G(v):\ v\in(-D_F+x)\cap D_G\big\}=\\&=\bigcap\big\{F(u)+G(-u+x):\ u\in D_F\cap(x-D_G)\big\}.\end{aligned}$$

Hence, by using that $-X+x=X$ and $x-X=X$ for all $x\in X$, we can immediately get

Corollary 3.6. *If F and G are relations on a group X to a groupoid Y, then for any $x\in X$ we have*

(1) $\big(F*G\big)(x)=\bigcap\limits_{v\in D_G}\big(F(x-v)+G(v)\big)$ *whenever F is total;*

(2) $\big(F*G\big)(x)=\bigcap\limits_{u\in D_F}\big(F(u)+G(-u+x)\big)$ *whenever G is total.*

Thus, in particular, we can also state the following

Corollary 3.7. *If F and G are relations of a group X to a groupoid Y, then for any $x\in X$ we have*

$$(F*G)(x)=\bigcap_{v\in X}\big(F(x-v)+G(v)\big)=\bigcap_{u\in X}\big(F(u)+G(-u+x)\big).$$

Remark 3.8. The multiplicative form of the first statement of this corollary closely resembles to the definition of the ordinary convolution of integrable functions.

4. Relationships between the inversion and the intersection convolution

An example given in [1] shows that in general $(F*G)^{-1} \neq F^{-1}*G^{-1}$. However, by using the first part of Cor. 3.6, we can easily prove the following

Theorem 4.1. *If F is a relation of one group X onto another Y and g is a function on X to Y, then*

$$(F*g)^{-1} \subset F^{-1}*g^{-1}.$$

Proof. If $y \in Y$ and $x \in (F*g)^{-1}(y)$, then

$$y \in (F*g)(x) = \bigcap_{v \in D_g} \big(F(x-v)+g(v)\big).$$

Therefore, for any $v \in D_g$, we have $y \in F(x-v)+g(v)$, and thus

$$y-g(v) \in F(x-v)+g(v)-g(v) = F(x-v).$$

Hence, it follows that $x-v \in F^{-1}\big(y-g(v)\big)$, and thus

$$x = x-v+v \in F^{-1}\big(y-g(v)\big)+v \subset F^{-1}\big(y-g(v)\big)+g^{-1}\big(g(v)\big).$$

Now, we can see that

$$x \in \bigcap_{v \in D_g} \Big(F^{-1}\big(y-g(v)\big)+g^{-1}\big(g(v)\big)\Big) =$$
$$= \bigcap_{w \in R_g} \big(F^{-1}(y-w)+g^{-1}(w)\big) =$$
$$= \bigcap_{w \in D_{g^{-1}}} \big(F^{-1}(y-w)+g^{-1}(w)\big) = \big(F^{-1}*g^{-1}\big)(y).$$

Therefore,

$$(F*g)^{-1}(y) \subset \big(F^{-1}*g^{-1}\big)(y)$$

for all $y \in Y$, and thus the required inclusion is also true. $\Diamond$

From the above theorem, we can immediately derive the following

Corollary 4.2. *If F is a relation of one group X onto another Y and g is an injective function on X to Y, then*

$$(F*g)^{-1} = F^{-1}*g^{-1}.$$

Proof. By applying Th. 4.1 to F^{-1} and g^{-1} instead of F and g, we can note that

$$\big(F^{-1}*g^{-1}\big)^{-1} \subset \big(F^{-1}\big)^{-1} * \big(g^{-1}\big)^{-1} = F*g,$$

and thus $F^{-1}*g^{-1} \subset (F*g)^{-1}$ also holds. $\Diamond$

Now, as an immediate consequence of this corollary, we can also state

Corollary 4.3. *If F is a symmetric relation of a group X and g is a symmetric function on X, then $F * g$ is a symmetric relation on X.*

Proof. Now, we have $F = F^{-1}$ and $g = g^{-1}$. Thus,

$$R_F = D_{F^{-1}} = D_F = X$$

and g is injective. Hence, by Cor. 4.2, it is clear that

$$(F * g)^{-1} = F^{-1} * g^{-1} = F * g.$$

Therefore, the required assertion is also true. $\Diamond$

By using the second part of Cor. 3.6, we can quite similarly prove the following

Theorem 4.4. *If f is a function on one group X to another Y and G is a relation of X onto Y, then*

$$(f * G)^{-1} \subset f^{-1} * G^{-1}.$$

Hence, it is clear that in particular we also have the following

Corollary 4.5. *If f is an injective function on one group X to another Y and G is a relation of X onto Y, then*

$$(f * G)^{-1} = f^{-1} * G^{-1}.$$

Thus, in particular, we can also state the following

Corollary 4.6. *If f is a symmetric function on a group X and G is a symmetric relation of X, then $f * G$ is a symmetric relation on X.*

5. Relationships between the composition and the intersection convolution

In contrast to the above results, the following theorem will, in particular, show that the intersection convolution of transitive relations is usually a transitive relation.

Theorem 5.1. *If F and G are relations on one groupoid X to another Y and H and K are relations on Y to a groupoid Z such that $R_F \subset D_H$ and $R_G \subset D_K$, then*

$$\big(H * K\big) \circ \big(F * G\big) \subset \big(H \circ F\big) * \big(K \circ G\big).$$

Proof. If $(x, z) \in \big(H * K\big) \circ \big(F * G\big)$, then

$$z \in \big((H * K) \circ (F * G)\big)(x) = (H * K)[(F * G)(x)].$$

Therefore, there exists $y \in Y$ such that

$$y \in \big(F * G\big)(x) = \bigcap \big\{ F(u) + G(v) : \ (u,v) \in \Gamma(x, D_F, D_G) \big\}$$

and

$$z \in \big(H * K\big)(y) = \bigcap \big\{ H(s) + K(t) : \ (s,t) \in \Gamma(y, D_H, D_K) \big\}.$$

Thus, for any $(u,v) \in \Gamma(x, D_F, D_G)$, we have $y \in F(u)+G(v)$. Therefore, there exist $s \in F(u)$ and $t \in G(v)$ such that $y = s + t$. Hence, we can infer that

$$H(s) \subset H\big[F(u)\big] = \big(H \circ F\big)(u) \quad \text{and} \quad K(t) \subset K\big[G(v)\big] = \big(K \circ G\big)(v).$$

Moreover, by using that

$$s \in F(u) \subset R_F \subset D_H \quad \text{and} \quad t \in G(v) \subset R_G \subset D_K,$$

we can also see that $(s,t) \in \Gamma(y, D_H, D_K)$. Hence, since $z \in \big(H * K\big)(y)$, it is clear that

$$z \in H(s) + K(t) \subset \big(H \circ F\big)(u) + \big(K \circ G\big)(v).$$

Therefore,

$$z \in \bigcap \big\{ (H \circ F)(u) + (K \circ G)(v) : \ (u,v) \in \Gamma(x,\, D_F,\, D_G) \big\} = \\ = \big((H \circ F) * (K \circ G)\big)(x).$$

Thus, $(x,z) \in \big(H \circ F\big) * \big(K \circ G\big)$ also holds. This proves the required inclusion. $\Diamond$

From the above theorem, we can immediately derive the following

Corollary 5.2. *If F and G are relations on a groupoid X such that $R_F \subset D_F$ and $R_G \subset D_G$, then*

$$\big(F * G\big)^2 \subset F^2 * G^2.$$

Proof. By Th. 5.1, we have

$$\big(F * G\big)^2 = \big(F * G\big) \circ \big(F * G\big) \subset \big(F \circ F\big) * \big(G \circ G\big) = F^2 * G^2. \quad \Diamond$$

Now, as an immediate consequence of this corollary, we can also state

Corollary 5.3. *If F and G are transitive relations on a groupoid X such that $R_F \subset D_F$ and $R_G \subset D_G$, then $F*G$ is also a transitive relation on X.*

Proof. Because of the above assumptions, we have

$$D_{F^2} = D_F, \qquad D_{G^2} = D_G \qquad \text{and} \qquad F^2 \subset F, \qquad G^2 \subset G.$$

Hence, by Cor. 5.2 and Def. 3.3, it is clear that

$$\big(F * G\big)^2 \subset F^2 * G^2 \subset F * G.$$

Therefore, the required assertion is also true. $\Diamond$

6. Relationships between the box product and the intersection convolution

Theorem 6.1. *If F and G are relations on one groupoid X to another Y, then for any $x \in X$ and $y \in Y$ the following assertions are equivalent:*

(1) $y \in (F * G)(x)$;

(2) $\Gamma(x, D_F, D_G) \subset G^{-1} \circ \Gamma(y, R_F, R_G) \circ F$;

(3) $\Gamma(x, D_F, D_G) \subset \big(F \boxtimes G\big)^{-1}\big[\Gamma(y, R_F, R_G)\big]$;

(4) $\Gamma(x, D_F, D_G) \subset \big(F^{-1} \boxtimes G^{-1}\big)\big[\Gamma(y, R_F, R_G)\big]$.

Proof. By Th. 2.4, it is clear that (2), (4) and (3) are equivalent. Therefore, it is enough to show only that (1) and (3) are also equivalent.

For this, note that if (1) holds, then

$$y \in \bigcap \big\{ F(u) + G(v) : \ (u, v) \in \Gamma(x, D_F, D_G) \big\}.$$

Thus, for any $(u, v) \in \Gamma(x, D_F, D_G)$, we have $y \in F(u)+G(v)$. Therefore, there exist $s \in F(u)$ and $t \in G(v)$ such that $y = s + t$. Hence, it follows that

$$(s, t) \in \Gamma\big(y, R_F, R_G\big) \quad \text{and} \quad (s, t) \in F(u) \times G(v) = \big(F \boxtimes G\big)(u, v).$$

Therefore,

$$(u, v) \in \big(F \boxtimes G\big)^{-1}(s, t) \subset \big(F \boxtimes G\big)^{-1}\big[\Gamma(y, R_F, R_G)\big],$$

and thus (3) also holds.

Conversely, note that if (3) holds, then for any $(u, v) \in \Gamma(x, D_F, D_G)$ we have $(u, v) \in \big(F \boxtimes G\big)^{-1}\big[\Gamma(y, R_F, R_G)\big]$. Therefore, there exists $(s, t) \in \in \Gamma(y, R_F, R_G)$ such that $(u, v) \in \big(F \boxtimes G\big)^{-1}(s, t)$. Hence, it follows that

$$y = s + t \quad \text{and} \quad (s, t) \in \big(F \boxtimes G\big)(u, v) = F(u) \times G(v).$$

This implies that $s \in F(u)$ and $t \in G(v)$, and thus $y = s + t \in F(u)+ +G(v)$. Therefore,

$$y \in \bigcap \big\{ F(u) + G(v) : \ (u, v) \in \Gamma(x, D_F, D_G) \big\},$$

and thus (1) also holds. $\Diamond$

Remark 6.2. Note that, for any $y \in Y$, we have

$$\begin{aligned} \big(F \boxtimes G\big)^{-1}\big[\Gamma(y)\big] &= \big(F \boxtimes G\big)^{-1}\big[\Gamma(y) \cap R_{F \boxtimes G}\big] = \\ &= \big(F \boxtimes G\big)^{-1}\big[\Gamma(y) \cap \big(R_F \times R_G\big)\big] = \\ &= \big(F \boxtimes G\big)^{-1}\big[\Gamma(y, R_F, R_G)\big]. \end{aligned}$$

Therefore, in the above theorem we may write $\Gamma(y)$ in place of $\Gamma(y, R_F, R_G)$.

Now, as an immediate consequence of Th. 6.1 and Rem. 6.2, we can also state

Corollary 6.3. *If F and G are relations of one groupoid X to another Y, then for any $x \in X$ and $y \in Y$ the following assertions are equivalent:*

(1) $y \in (F * G)(x)$;

(2) $\Gamma(x) \subset G^{-1} \circ \Gamma(y) \circ F$;

(3) $\Gamma(x) \subset \big(F \boxtimes G\big)^{-1}\big[\Gamma(y)\big]$;

(4) $\Gamma(x) \subset \big(F^{-1} \boxtimes G^{-1}\big)\big[\Gamma(y)\big]$.

Hence, it is clear that in particular we also have

Corollary 6.4. *If F and G are symmetric relations of a groupoid X to itself, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $y \in (F * G)(x)$;

(2) $\Gamma(x) \subset G \circ \Gamma(y) \circ F$;

(3) $\Gamma(x) \subset \big(F \boxtimes G\big)\big[\Gamma(y)\big]$.

Remark 6.5. This corollary also strongly suggests that the intersection convolution of symmetric relations is not, in general, a symmetric relation.

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