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A FUNCTIONAL EQUATION RELATED TO CHARACTERIZATION PROBLEMS

Fruzsina **Mészáros**

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P. O. Box 12, Hungary

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Abstract: The functional equation

$$f\,(x)\,g\,(y) = h\,(ax+by)\,k\,(cx+dy)$$

is investigated for almost all $(x, y) \in \mathbb{R}^2$ for the measurable functions $f,\ g,\ h,\ k\ :\ \mathbb{R} \to \mathbb{R}_+$, where $a,\ b,\ c,\ d \in \mathbb{R} \setminus \{0\}$ are arbitrary constants. This equation is related to the characterization of the family of normal distributions and it has important role in the characterization of distributions, whose conditionals belong to given location families.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ be the set of positive real numbers.

The functional equation

$$f\,(x)\,g\,(y) = h\,(ax+by)\,k\,(cx+dy)\,, \qquad (x, y) \in \mathbb{R}^2 \tag{1}$$

where $f, g, h, k : \mathbb{R} \to \mathbb{C}$ and a, b, c, d are fixed non-zero real numbers with $ad - bc \neq 0$, was investigated several times, for example by J. A. Baker [2] and K. Lajkó [6].

E-mail address: mefru@math.klte.hu
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Baker determined all measurable and not almost everywhere zero functions $f, g, h, k : \mathbb{R} \to \mathbb{C}$ satisfying the functional eq. (1), Lajkó gave all solutions f, g, h, k of (1), which are not almost everywhere zero.

The purpose of this paper is to determine all measurable solutions $f, g, h, k : \mathbb{R} \to \mathbb{R}_+$ of eq. (1) satisfied almost everywhere. This result can be used in characterization of distributions, hence eq. (1) related to the characterization of the family of normal distributions (see Sec. 3) and it has important role in the characterization of distributions, whose conditionals belong to given location families (see Sec. 4). (It also has applications to the quantum mechanical three-body problem.)

Our proof is based on the following theorem of Járai (see [4], [5]):

Theorem 1 (Járai)**.** *Let Z be a regular topological space, Z_i $(i=1,2,...,n)$ be topological spaces and T be a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$, $(i=1,2,\dots,n)$ and D an open subset of $T \times Y$. Let furthermore $T' \subset T$ be a dense subset, $F\colon T' \to Z$, $g_i\colon D \to X_i$ and $H\colon D \times Z_1 \times \ldots \times Z_n \to Z$. Suppose that the function f_i is almost everywhere defined on X_i (with respect to the r_i-dimensional Lebesgue measure) with values in Z_i $(i = 1,2,\dots n)$ and the following conditions are satisfied:*

1) *for all $t \in T'$ and for almost all $y \in D_t = \{y \in Y : (t,y) \in D\}$*

$$F(t) = H\big(t, y, f_1(g_1(t,y)), \dots, f_n(g_n(t,y))\big); \tag{2}$$

2) *for each fixed y in Y, the function H is continuous in the other variables;*
3) *f_i is Lebesgue measurable on $\mathbb{R}^{r_i}$ $(i = 1,2,\dots,n)$;*
4) *g_i and the partial derivative $\frac{\partial g_i}{\partial y}$ are continuous on D $(i = = 1,2,\dots,n)$;*
5) *for each $t \in T$ there exist a y such that $(t,y) \in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has the rank r_i at $(t,y) \in D$ $(i = 1,2,\dots,n)$.*

Then there exists a unique continuous function $\widetilde{F}$ such that $F = \widetilde{F}$ almost everywhere on T, and if F is replaced by $\widetilde{F}$ then eq. (2) is satisfied almost everywhere on D.

2. The measurable solution of (1) for a.e. $(x,y) \in \mathbb{R}^2$

First we prove the following

Lemma 1. *If the measurable functions $f, g, h, k : \mathbb{R} \to \mathbb{R}_+$ satisfy eq. (1) for almost all $(x,y) \in \mathbb{R}^2$, where $a, b, c, d \in \mathbb{R} \setminus \{0\}$ are arbitrary* *constants with* $\Delta = ad - bc \neq 0$*, then there exist unique continuous functions* $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{k} : \mathbb{R} \to \mathbb{R}_+$ *such that* $\tilde{f} = f$*,* $\tilde{g} = g$*,* $\tilde{h} = h$ *and* $\tilde{k} = k$ *almost everywhere, and if* f, g, h, k *are replaced by* $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{k}$*, respectively, then* (1) *is satisfied everywhere on* $\mathbb{R}^2$*.*

Proof. First, by the help of Járai's Theorem, we prove that there exist unique continuous function $\tilde{h}$ which is almost everywhere equal to h on $\mathbb{R}$ and replacing h by $\tilde{h}$, eq. (1) is satisfied almost everywhere.

With the substitution $t = ax + by$ we get from (1) the equation

$$h(t) = \frac{f\left(\frac{t-by}{a}\right) g(y)}{k\left(\frac{c}{a}(t-by) + dy\right)} \tag{3}$$

which is satisfied for almost all $(t, y) \in D$, where $D = \mathbb{R}^2$. By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}$ of full measure such that for all $t \in T'$ eq. (3) is satisfied for almost every $y \in D_t = \mathbb{R}$.

Let us define the functions g_1, g_2, g_3, H in the following way:

$$g_1(t, y) = \frac{t - by}{a}, \quad g_2(t, y) = y,$$

$$g_3(t, y) = \frac{c}{a}(t - by) + dy, \quad H(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3},$$

and let us now apply Th. 1 of Járai to (3) with a suitable casting.

Hence the first assumption in Th. 1 with respect to (3) holds. In the event of fixed y, the function H is continuous in the other variables, so the second assumption holds too. Because the functions in eq. (3) are measurable, the third assumption is trivial.

The functions g_i are continuous, the partial derivatives

$$D_2 g_1(t, y) = -\frac{b}{a}, \quad D_2 g_2(t, y) = 1, \quad D_2 g_3(t, y) = \frac{\Delta}{a}$$

are also continuous, so the fourth assumption holds too.

For each $t \in \mathbb{R}$ there exist a $y \in \mathbb{R}$ such that $(t, y) \in D = \mathbb{R}^2$ and the partial derivatives don't equal zero in (t, y), so they have the rank 1. Thus the last assumption is satisfied in Th. 1.

So we get from Th. 1 that there exists unique continuous function $\tilde{h}$ which is almost everywhere equal to h on $\mathbb{R}$ and $f, g, \tilde{h}, k$ satisfy eq. (1) almost everywhere, which is equivalent to the equation

$$f(x) g(y) = \tilde{h}(ax + by) k(cx + dy) \tag{4}$$

for almost all $(x, y) \in \mathbb{R}^2$.

By a similar argument we can prove the same for the function k. From eq. (4) with the substitution $t = cx + dy$ we get the equation

$$k(t) = \frac{f\left(\frac{t-dy}{c}\right) g(y)}{\tilde{h}\left(\frac{a}{c}(t-dy) + by\right)},$$

which with a suitable casting

$$g_1(t,y) = \frac{t-dy}{c}, \quad g_2(t,y) = y,$$

$$g_3(t,y) = \frac{a}{c}(t-dy) + by, \quad H(t,y,z_1,z_2,z_3) = \frac{z_1 z_2}{z_3},$$

by Fubini's Theorem, and the fact that the assumptions of Th. 1 are fulfilled again, gives us that there exists unique continuous function $\tilde{k}$ which is almost everywhere equal to k on $\mathbb{R}$ and $f, g, \tilde{h}, \tilde{k}$ satisfy eq. (1) almost everywhere, i.e.

$$f(x) g(y) = \tilde{h}(ax+by) \tilde{k}(cx+dy) \tag{5}$$

for almost all $(x,y) \in \mathbb{R}^2$.

There exist such x_0 and y_0 so that with the substitutions $x = x_0$ and $y = y_0$, respectively, we get from eq. (5) that

$$f(x) = \frac{1}{g(y_0)} \tilde{h}(ax+by_0) \tilde{k}(cx+dy_0)$$

holds for almost all $y \in \mathbb{R}$, and

$$g(y) = \frac{1}{f(x_0)} \tilde{h}(ax_0+by) \tilde{k}(cx_0+dy)$$

holds for almost all $x \in \mathbb{R}$. Since $\tilde{h}, \tilde{k}$ are continuous, therefore there exist unique continuous functions $\tilde{f}$ and $\tilde{g}$, defined by the right-hand side of the last two equality, which are almost everywhere equal to f and g on $\mathbb{R}$, and if we replace f and g by $\tilde{f}$ and $\tilde{g}$, respectively, then the functional equation

$$\tilde{f}(x) \tilde{g}(y) = \tilde{h}(ax+by) \tilde{k}(cx+dy) \tag{6}$$

is satisfied almost everywhere on $\mathbb{R}^2$.

Both sides of (6) define continuous functions on $\mathbb{R}^2$, which are equal to each other on a dense subset of $\mathbb{R}^2$, therefore we obtain that (6) is satisfied everywhere on $\mathbb{R}^2$.

Further $f = \tilde{f}$, $g = \tilde{g}$, $h = \tilde{h}$, $k = \tilde{k}$ almost everywhere on $\mathbb{R}$, respectively. $\Diamond$

Hence, by the help of the measurable (continuous) solutions of eq. (1) satisfied everywhere, we can give the solutions of the almost everywhere satisfied functional equation.

Theorem 2. *Suppose that the measurable functions $f, g, h, k : \mathbb{R} \to \mathbb{R}_+$ satisfy functional eq.* (1) *almost everywhere, then*

$$\begin{aligned} f(x) &= \alpha_1 \exp\left[a_1 x + b_1 x^2\right] && a.e.\ x \in \mathbb{R}, \\ g(x) &= \alpha_2 \exp\left[a_2 x - \frac{bd}{ac} b_1 x^2\right] && a.e.\ x \in \mathbb{R}, \\ h(x) &= \beta_1 \alpha_1 \alpha_2 \exp\left[\frac{a_1 d - a_2 c}{\Delta} x + \frac{d}{a} b_1 x^2\right] && a.e.\ x \in \mathbb{R}, \\ k(x) &= \beta_2 \alpha_1 \alpha_2 \exp\left[\frac{a_2 a - a_1 b}{\Delta} x - \frac{b}{c} b_1 x^2\right] && a.e.\ x \in \mathbb{R}, \end{aligned}$$

where $a_1, a_2, b_1 \in \mathbb{R}$ are arbitrary constants and $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, 2)$ are arbitrary constants, satisfying $\alpha_1 \beta_1 \alpha_2 \beta_2 = 1$.

Proof. The measurable (continuous) solutions of eq. (1) satisfied everywhere (see [2] and [6]) and the previous lemma immediately gives our statement. ♢

3. A characterization of normal distributions

A well-known characterization of the family of normal distributions is the following: The independent random variables X and Y have normal distributions if and only if $X + Y$ and $X - Y$ are independent (see [7]).

A possible characterization of univariate distributions is based on the following general Transformation Theorem. (See [3].)

Theorem 3. *Let $X = (X_1, \dots, X_n)$ be an absolutely continuous random variable with density function $f : \mathbb{R}^N \to \mathbb{R}$, which is zero outside of a region $\Omega_x \subset \mathbb{R}^N$. Let $\psi : \Omega_x \to \Omega_y \subset \mathbb{R}^n$ be a one-to-one transformation onto Ω_y and denote ψ^{-1} its inverse transformation.*

If the Jacobi determinant $J(y) = \det\left(\frac{\partial \psi^{-1}(y)}{\partial y}\right)$ exists, is continuous and does not change sign in Ω_y, then the random variable $Y = \psi(X)$ is absolutely continuous with density function g such that

$$g(y) = \begin{cases} f\left(\psi^{-1}(y)\right) |J(y)| & \text{if} \quad y \in \Omega_y \text{ a.e.}, \\ 0 & \text{if} \quad y \in \mathbb{R}^N \setminus \Omega_y. \end{cases}$$

We shall use the transformation

$$\psi(x,y) = (x+y, x-y)\,. \tag{7}$$

The function ψ defined in (7) is bijective, $\psi^{-1}(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$, and the Jacobi determinant of ψ^{-1} is of the form

$$J(u,v) = -\frac{1}{2} \quad (u,v \in \mathbb{R})\,.$$

Obviously, J is continuous and does not change sign on $\mathbb{R}^2$.

Let X, Y be absolutely continuous and independent random variables with range in $\mathbb{R}_+$. Let us denote the densities by f_X, f_Y respectively. Then, by the Transformation Theorem, the random variable

$$(U,V) = \psi(X,Y) = (X+Y, X-Y)$$

is absolutely continuous with density function g defined by

$$g(u,v) := \frac{1}{2} f_X\left(\frac{u+v}{2}\right) f_Y\left(\frac{u-v}{2}\right) \tag{8}$$

for almost all $(u,v) \in \mathbb{R}^2$.

It is easy to see that if the independent random variables X and Y have normal distribution, then U and V are independent.

The converse question can be formulated as follows: Assume that X and Y are independent and the random vector $(U,V) = \psi(X,Y)$ has independent components. Is it true in this case that X, Y have normal distribution?

If U and V are independent with density functions f_U, f_V respectively, then from (8) we get the functional equation

$$f_U(u) f_V(v) = \frac{1}{2} f_X\left(\frac{u+v}{2}\right) f_Y\left(\frac{u-v}{2}\right) \qquad \text{a.e. } (u,v) \in \mathbb{R}^2 \tag{9}$$

for unknown density functions $f_X, f_Y, f_U, f_V : \mathbb{R} \to \mathbb{R}_+$.

With the substitution $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$ we get from (9) the equation

$$f_X(x) f_Y(y) = 2 f_U(x+y) f_V(x-y)$$

for almost all $(x,y) \in \mathbb{R}^2$, and with the notations

$$f(t) = f_X(t)\,, \quad g(t) = f_Y(t)\,,$$
$$h(t) = 2 f_U(t)\,, \quad k(t) = f_V(t)$$

we get the equation

$$f(x)\,g(y) = h(x+y)\,k(x-y)$$

for almost all $(x,y) \in \mathbb{R}^2$, which is a special case of equation (1) with constants $a = b = c = 1$ and $d = -1$.

From Th. 2, we get that

$$f_X(x) = \alpha_1 \exp\left[a_1 x + b_1 x^2\right] \qquad \text{a.e. } x \in \mathbb{R},$$

and

$$f_Y(x) = \alpha_2 \exp\left[a_2 x + b_1 x^2\right] \qquad \text{a.e. } x \in \mathbb{R},$$

where $\alpha_1, \alpha_2, a_1, a_2, b_1 \in \mathbb{R}$ are arbitrary constants.

Since f_X and f_Y are density functions, we get that X and Y are normal distributions with densities

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma^2}} \qquad \text{a.e. } x \in \mathbb{R},$$

and

$$f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma^2}} \qquad \text{a.e. } x \in \mathbb{R},$$

respectively.

From Th. 2, we also get that

$$f_U(x) = \frac{1}{2}\beta_1\alpha_1\alpha_2 \exp\left[\frac{1}{2}(a_1+a_2)x - b_1 x^2\right] \qquad \text{a.e. } x \in \mathbb{R},$$

and

$$f_V(x) = \beta_2\alpha_1\alpha_2 \exp\left[\frac{1}{2}(a_1-a_2)x - b_1 x^2\right] \qquad \text{a.e. } x \in \mathbb{R},$$

where $a_1, a_2, b_1 \in \mathbb{R}$ are arbitrary constants and $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, 2)$ are arbitrary constants, satisfying $\alpha_1\beta_1\alpha_2\beta_2 = 1$.

Hence if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are independent normal random variables, then their sum is normally distributed with $U = X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$; their difference is normally distributed with $V = X - Y \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

4. Linear regressions with conditionals in location families

Let (X, Y) be an absolutely continuous bivariate random variable, whose joint, marginal and conditional density functions are denoted by $f_{(X,Y)}$, f_X, f_Y, $f_{X|Y}, f_{Y|X}$ respectively. One can write $f_{(X,Y)}$ in two different ways and obtain the functional equation

$$f_{(X,Y)}(x, y) = f_{X|Y}(x, y) f_Y(y) = f_{Y|X}(x, y) f_X(x)$$

for a.e. $(x, y) \in \mathbb{R}^2$.

It could be a difficult problem to characterize distributions whose conditionals belong to given location families.

In his paper ([8]), Narumi was the first who studied this question. Based on the analysis of Narumi, in their book ([1]) Arnold, Castillo and Sarabia considered among others all possible distributions with given regression functions $E(X|Y = y) = a(y)$ and $E(Y|X = x) = b(x)$ with conditionals in location families.

Thus the conditional densities were required to be of the form (with the implicit inclusion of the assumption that conditional variances are constant)

$$f_{X|Y}(x, y) = g_1(x - a(y)) \tag{10}$$

and

$$f_{Y|X}(x, y) = g_2(y - b(x)). \tag{11}$$

For certain choices of the functions $a(y)$ and $b(x)$, it is possible to determine the nature of the joint distribution associated with (10) and (11).

It is natural to inquire about the case in which $a(y)$ and $b(x)$ are linear (see Sec. 7.4 in [1]). In that case, we will have

$$f_Y(y) g_1(x - a_1 y - a_2) = f_X(x) g_2(y - b_1 x - b_2). \tag{12}$$

Narumi solved (12) by taking logarithms of both sides and differentiating, assuming the existence of derivatives up to the third order. It is quite natural to assume only the measurability of the unknown functions and that eq. (12) holds for almost all pairs (x, y).

Eq. (12) can be rewritten in terms of functions

$$\bar{g}_1(t) = g_1(t - a_2), \qquad \bar{g}_2(t) = g_2(t - b_2).$$

Hence we get

$$f_Y(y)\,\bar{g}_1(x-a_1y)=f_X(x)\,\bar{g}_2(y-b_1x) \tag{13}$$

for almost every $(x,y)\in\mathbb{R}^2$. Write y instead of $y-b_1x$ we get the equation

$$f_X(x)\,\bar{g}_2(y)=f_Y(y+b_1x)\,\bar{g}_1((1-a_1b_1)\,x-a_1y) \tag{14}$$

for almost every $(x,y)\in\mathbb{R}^2$. Eq. (14) with the notations

$$f(t)=f_X(t),\qquad g(t)=\bar{g}_2(t),$$
$$h(t)=f_Y(t),\qquad k(t)=\bar{g}_1(t),$$

yields

$$f(x)\,g(y)=h(ax+by)\,k(cx+dy)\qquad\text{a.e.}\quad(x,y)\in\mathbb{R}^2,$$

with constants

$$a=b_1,\quad b=1,\quad c=1-a_1b_1,\quad d=-a_1.$$

Here $\Delta=-1$.

By the help of Th. 2 we have the solution of equation (14) and hence the solution of equation (13) and (12), i.e.

$$f_X(x)=\alpha_1\exp\left[A_1x+B_1x^2\right]\qquad\text{a.e. } x\in\mathbb{R},$$

$$\bar{g}_2(x)=\alpha_2\exp\left[A_2x+\frac{a_1B_1}{b_1(1-a_1b_2)}x^2\right]\qquad\text{a.e. } x\in\mathbb{R},$$

$$f_Y(x)=\beta_1\alpha_1\alpha_2\exp\left[(A_1a_1+A_2(1-a_1b_1))\,x-\frac{a_1B_1}{b_1}x^2\right]\qquad\text{a.e. } x\in\mathbb{R},$$

$$\bar{g}_1(x)=\beta_2\alpha_1\alpha_2\exp\left[(A_1-A_2b_1)\,x-\frac{B_1}{1-a_1b_1}x^2\right]\qquad\text{a.e. } x\in\mathbb{R},$$

moreover for a.e. $x\in\mathbb{R}$

$$g_1(x)=\beta_2\alpha_1\alpha_2\exp\left[(A_1-A_2b_1)(x+a_2)-\frac{B_1}{1-a_1b_1}(x+a_2)^2\right],$$

$$g_2(x)=\alpha_2\exp\left[A_2(x+b_2)+\frac{a_1B_1}{b_1(1-a_1b_2)}(x+b_2)^2\right].$$

After substituting the corresponding functions to these formula

$$f_{(X,Y)}(x,y)=f_Y(y)\,\bar{g}_1(x-a_1y)=f_X(x)\,\bar{g}_2(y-b_1x)$$

we can calculate the joint density function

$$f_{(X,Y)}(x,y)=\alpha_1\alpha_2\exp\left[(A_1-A_2b_1)\,x+A_2y-\frac{a_1B_1}{1-a_1b_1}\left(\frac{x^2}{a_1}-2xy+\frac{y^2}{b_1}\right)\right]$$

for a.e. $x\in\mathbb{R}$.

As in [1], we also can conclude that either X and Y are independent or (X,Y) must have a classical bivariate normal distribution.

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