Mathematica Pannonica
20/1 (2009), 1–8

THE SECOND LEMOINE CIRCLE OF THE TRIANGLE IN AN ISOTROPIC PLANE

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Received: ??August 2008??

MSC 2000: 51 N 25

Keywords: Isotropic plane, standard triangle, second Lemoine circle.

Abstract: The concept of the second Lemoine circle of the triangle in an isotropic plane is defined in this article. Some relationships between the introduced concept and some other elements of the triangle in an isotropic plane are also studied.

1. Standard triangle in an isotropic plane

In an isotropic plane (see e.g. [5] and [6]) the *distance* between the two points $T_i = (x_i, y_i)$ $(i = 1, 2)$ is defined by $T_1T_2 = x_2 - x_1$ and two

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lines with the equations $y = k_i x + l_i$ $(i = 1, 2)$ form the *angle* $k_2 - k_1$. Two points T_1, T_2 with $x_1 = x_2$ are said to be *parallel*; we shall also say they lie on the same *isotropic* line. Two lines with $k_1 = k_2$ are parallel.

A triangle is said to be *admissible* if none of its sides is isotropic. Each admissible triangle ABC can be set by a suitable choice of coordinate system in the standard position, in which its circumscribed circle has the equation $y = x^2$, its vertices are the points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{1}$$

and its sides BC, CA, AB have the equations

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab, \tag{2}$$

where

$$a + b + c = 0 \tag{3}$$

is fulfilled. Then we shall say that ABC is the *standard triangle*. To prove the geometric facts for each admissible triangle it is sufficient to provide a proof for the standard triangle (see [3]).

2. Algebraic relationships in the standard triangle

With the labels

$$p = abc, \quad q = bc + ca + ab \tag{4}$$

a number of useful equalities are proved in [3] as for example

$$q = bc - a^2 = ca - b^2 = ab - c^2, \tag{5}$$

$$q = -(b^2 + bc + c^2) = -(c^2 + ca + a^2) = -(a^2 + ab + b^2). \tag{6}$$

Various symmetric functions of a,b,c can be expressed by means of p and q. We get for example

$$\begin{aligned} a^3 + b^3 + c^3 &= -a^2(b + c) - b^2(c + a) - c^2(a + b) = \\ &= -bc(b + c) - ca(c + a) - ab(a + b) = 3abc = 3p, \\ q^2 &= (bc + ca + ab)^2 = b^2c^2 + c^2a^2 + a^2b^2 + 2abc(a + b + c), \end{aligned}$$

so these equalities are valid

$$a^3 + b^3 + c^3 = 3p, \tag{7}$$

$$b^2c^2 + c^2a^2 + a^2b^2 = q^2. \tag{8}$$

Therefore, we obtain the following

$$\begin{aligned} q^3 &= (b^2c^2 + c^2a^2 + a^2b^2)(bc + ca + ab) = \\ &= b^3c^3 + c^3a^3 + a^3b^3 + abc(bc[b + c] + ca(c + a) + ab(a + b)] = \\ &= b^3c^3 + c^3a^3 + a^3b^3 - 3a^2b^2c^2, \end{aligned}$$

which gives the formula

$$b^3c^3 + c^3a^3 + a^3b^3 = 3p^2 + q^3. \tag{9}$$

There will also be useful to introduce two functions of a, b, c, which are not symmetric, but they are cyclically symmetric. Let it be

$$p_1 = \frac{1}{3}(bc^2 + ca^2 + ab^2), \quad p_2 = \frac{1}{3}(b^2c + c^2a + a^2b). \tag{10}$$

Then we obtain

$$3p_1 + 3p_2 = bc(b + c) + ca(c + a) + ab(a + b) = -3abc = -3p,$$

$$3p_1 - 3p_2 = bc^2 + ca^2 + ab^2 - b^2c - c^2a - a^2b = (b - c)(c - a)(a - b),$$

i.e.

$$p + p_1 + p_2 = 0, \tag{11}$$

$$p_1 - p_2 = \frac{1}{3}(b - c)(c - a)(a - b). \tag{12}$$

Further, because of (9) and (7) it follows

$$\begin{aligned} 9p_1p_2 &= (bc^2 + ca^2 + ab^2)(b^2c + c^2a + a^2b) = \\ &= b^3c^3 + c^3a^3 + a^3b^3 + abc(a^3 + b^3 + c^3) + 3a^2b^2c^2 = \\ &= 3p^2 + q^3 + 3p^2 + 3p^2 = 9p^2 + q^3, \end{aligned}$$

i.e. we obtain the formula

$$p_1p_2 = p^2 + \frac{1}{9}q^3. \tag{13}$$

Now, let us prove the following formula too

$${p_1}^2 + p_1p_2 + {p_2}^2 = p^2 + pp_1 + {p_1}^2 = p^2 + pp_2 + {p_2}^2 = -\frac{1}{9}q^3. \tag{14}$$

Actually, owing to (13) and (11) we obtain

$$q^3 = 9p_1p_2 - 9p^2 = 9p_1p_2 - 9(p_1 + p_2)^2 = -9({p_1}^2 + p_1p_2 + {p_2}^2),$$

and after that from (11) for example we get

$$p^2 + pp_1 + {p_1}^2 = (p_1 + p_2)^2 - (p_1 + p_2)p_1 + {p_1}^2 = {p_1}^2 + p_1p_2 + {p_2}^2.$$

Analogously to formula (13) the following formula

$$pp_1 - {p_2}^2 = pp_2 - {p_1}^2 = \frac{1}{9}\,q^3 \tag{15}$$

is valid too, and this follows because of (11) and (14), since, for example

$$pp_1 - {p_2}^2 = pp_1 - (p + p_1)^2 = -(p^2 + pp_1 + {p_1}^2) = \frac{1}{9}\,q^3.$$

3. The second Lemoine circle in an isotropic plane

Theorem 1. *If the lines through the point $T = (x_o, y_o)$ which are antiparallel to the lines BC, CA, AB with respect to the the pairs of the lines CA, AB; AB, BC; BC, CA meet these pairs of lines in the pairs of the points B_a, C_a; C_b, A_b; A_c, B_c, then these six points have the following abscissas*

$$B_a \ldots \frac{y_o - 2ax_o + ca}{c-a}, \quad C_a \ldots -\frac{y_o - 2ax_o + ab}{a-b},$$
$$C_b \ldots \frac{y_o - 2bx_o + ab}{a-b}, \quad A_b \ldots -\frac{y_o - 2bx_o + bc}{b-c},$$
$$A_c \ldots \frac{y_o - 2cx_o + bc}{b-c}, \quad B_c \ldots -\frac{y_o - 2cx_o + ca}{c-a}.$$

Proof. If the lines, antiparallel to the line BC with respect to the lines CA and AB, have the slope k then $-b-c = -a+k$ i.e. $k = 2a$. Because of that the line through the point T, antiparallel to the line BC, has the equation $y = 2a(x - x_o) + y_o$. From that equation and the equation $y = -bx - ca$ of the line CA we obtain the equation

$$2a(x - x_o) + y_o = -bx - ca$$

for the abscissa of the point B_a. As $2a + b = a - c$ is valid

$$x = \frac{y_o - 2ax_o + ca}{c-a}$$

follows. Abscissa of the point C_a is obtained by the substitution $b \leftrightarrow c$, and the abscissas of the remaining points can be obtained by the cyclic permutations $a \to b \to c \to a$. $\Diamond$

If the point T is the symmedian center K of the triangle ABC, then, with the values $x_o = \frac{3p}{2q}$, $y_o = -\frac{q}{3}$ from [2] the abscissa of the point B_a achieves the form $\frac{p-p_2}{q}$ because of

$$\begin{aligned}
&3q(y_o - 2ax_o + ca) - 3(c-a)(p-p_2) = \\
&= -q^2 - 9ap + 3caq + (a-c)(3abc - b^2c - c^2a - a^2b) = \\
&= -(a^2+ac+c^2)^2 + 9a^2c(a+c) - 3ac(a^2+ac+c^2) + \\
&\quad + (a-c)[-3ac(a+c) - c(a+c)^2 - ac^2 + a^2(a+c)] = 0.
\end{aligned}$$

By the substitution $b \leftrightarrow c$ we get the substitution $p_1 \leftrightarrow p_2$, and therefore the point C_a has the abscissa $\frac{p-p_1}{q}$. Analogous statement is valid for the remaining intersections from Th. 1 in this case. So we get:

Theorem 2. *If the lines through the symmedian center K of an admissible triangle ABC which are antiparallel to the lines BC, CA, AB with* *respect to the pairs of the lines CA, AB; AB, BC; BC, CA meet these pairs of lines in the pairs of the points B_a, C_a; C_b, A_b; A_c, B_c, then the points C_a, A_b, B_c have the abscissa $\frac{p-p_1}{q}$, and the points B_a, C_b, A_c have the abscissa $\frac{p-p_2}{q}$, i.e. these six intersections lie three by three on the two isotropic lines with the equations*

$$x = \frac{p-p_1}{q} \quad \textit{and} \quad x = \frac{p-p_2}{q}.$$

The pair of isotropic lines from Th. 2 is one circle which will be called *second Lemoine circle* of the triangle ABC, by the analogy with the Euclidean case.

According to (11) we get

$$\frac{p-p_1}{q} + \frac{p-p_2}{q} = \frac{3p}{q} = 2 \cdot \frac{3p}{2q},$$

the point K lies on the bisector of the isotropic lines from Th. 2, which is isotropic analogy of the fact from Euclidean geometry, where the symmedian center K is the center of the second Lemoine circle.

Corollary 1. *The symmedian center K of an admissible triangle ABC is the midpoint of the segment between its two sides on the line through the point K which is antiparallel to its third side with respect to these two sides.*

Corollary 2. *Symmedian AK of an admissible triangle ABC is the set of the midpoints of the segments between the lines AB and AC on the lines which are antiparallel to the line BC with respect to the lines AB and AC. Analogous properties have symmedians BK and CK.*

As the point K is the midpoint of the pairs of points B_a, C_a; C_b, A_b; A_c, B_c, then the triangle, which is determined by the lines B_cC_b, C_aA_c, A_bB_a is symmetric with respect to the point K to the triangle which is formed by the lines A_cA_b, B_aB_c, C_bC_a, and these are the lines BC, CA, AB, so the first triangle is symmetric to the triangle ABC with respect to the point K.

The second Lemoine circle of the standard triangle ABC has the equation

$$\left(x - \frac{p-p_1}{q}\right)\left(x - \frac{p-p_2}{q}\right) = 0,$$

which, because of $p - p_1 + p - p_2 = 3p$ and

$$\begin{aligned} 9(p-p_1)(p-p_2) &= 9p^2 - 9p(p_1+p_2) + 9p_1p_2 = \\ &= 9p^2 + 9p^2 + 9p^2 + q^3 = 27p^2 + q^3 \end{aligned}$$

gets this final form

$$x^2 - \tfrac{3p}{q}x + \frac{27p^2 + q^3}{9q^2} = 0. \tag{16}$$

From equation (16) and the equation of the circumscribed circle $y = x^2$ of the triangle ABC we obtain the equation

$$y = \frac{3p}{q}x - \frac{27p^2 + q^3}{9q^2} \tag{17}$$

of the potential axis of these two circles.

Euler circle of the triangle ABC has, according to [1], the equation $y = -2x^2 - q$. From it and equation (16) by the elimination of the part with x^2 for the potential axis of these two circles we achieve the following equation

$$y = -\frac{6p}{q}x + \frac{54p^2 - 7q^3}{9q^2}. \tag{18}$$

The first Lemoine circle of the triangle ABC has according to [4] the equation

$$y = 2x^2 - \frac{3p}{q}x + \frac{27p^2 - 2q^3}{18q^2}.$$

From this equation and equation (16) by the elimination of the part with x^2 we obtain the equation

$$y = \frac{3p}{q}x - \frac{27p^2 + 2q^3}{6q^2} \tag{19}$$

of the potential axis of the first and the second Lemoine circle of the triangle ABC.

The three obtained potential axes have some geometric properties too, which will be proved in the following theorems.

Theorem 3. *The potential axis of the two Lemoine circles of an admissible triangle passes through its symmedian center.*

Proof. Really, for the point $K = \left(\frac{3p}{2q}, -\frac{q}{3}\right)$ we get

$$y - \frac{3p}{q}x = -\frac{q}{3} - \frac{9p^2}{2q^2} = -\frac{27p^2 + 2q^3}{6q^2},$$

and this point satisfies equation (19). $\Diamond$

Theorem 4. *The potential axis of the circumscribed circle and the second Lemoine circle of an admissible triangle passes through the intersections of its sides with antiparallel lines to these sides with respect to the pairs* *of the remaining sides, constructed through the symmedian center of this triangle.*

Proof. The line with the equation

$$y = 2ax - \frac{q}{3} - \frac{3ap}{q} \tag{20}$$

is antiparallel to the line BC with respect to the lines CA and AB and it passes through the symmedian center $K = \left(\frac{3p}{2q}, -\frac{q}{3}\right)$ of the triangle ABC. The point with the coordinates

$$x = \frac{1}{9aq}(q^2 + 9ap - 3bcq), \quad y = -\frac{1}{9q}(q^2 + 9ap + 6bcq)$$

lies on the line (20) and on the line BC with the equation $y = -ax - bc$ because for it we get

$$2ax - y = \frac{1}{9q}(2q^2 + 18ap - 6bcq + q^2 + 9ap + 6bcq) =$$
$$= \frac{1}{9q}(27ap + 3q^2) = \frac{q}{3} + \frac{3ap}{q},$$
$$ax + y = \frac{1}{9q}(q^2 + 9ap - 3bcq - q^2 - 9ap - 6bcq) = -bc.$$

However, this point also lies on the line (17) since this

$$\frac{3p}{q}x - y = \frac{1}{9aq^2}(3pq^2 + 27ap^2 - 9bcpq + aq^3 + 9a^2pq + 6pq^2) =$$
$$= \frac{1}{9aq^2}[9pq^2 + 27ap^2 + aq^3 - 9(bc - a^2)pq] =$$
$$= \frac{1}{9aq^2}(27ap^2 + aq^3) = \frac{27p^2 + q^3}{9q^2}$$

follows. $\Diamond$

Theorem 5. *The potential axis of Euler and the second Lemoine circle of an admissible triangle passes through the intersections of its midlines with antiparallel lines to these midlines constructed through the symmedian center of that triangle.*

Proof. In [3] it is proved that midline B_mC_m of the triangle ABC has the equation

$$y = -ax + \frac{bc}{2} - q. \tag{21}$$

The point with the coordinates

$$x = \frac{1}{18aq}(18ap - 4q^2 + 3bcq), \quad y = -\frac{1}{9q}(7q^2 + 9ap - 3bcq)$$

lies on the lines (20) and (21) because for it we get

$$\begin{aligned} 2ax - y &= \frac{1}{9q}(18ap - 4q^2 + 3bcq + 7q^2 + 9ap - 3bcq) = \\ &= \frac{1}{9q}(27ap + 3q^2) = \frac{q}{3} + \frac{3ap}{q}, \\ ax + y &= \frac{1}{18q}(18ap - 4q^2 + 3bcq - 14q^2 - 18ap + 6bcq) = \\ &= \frac{1}{18q}(9bcq - 18q^2) = \frac{bc}{2} - q, \end{aligned}$$

and it also lies on the potential axis (18) because for it the following equation

$$\begin{aligned} \frac{6p}{q}x + y &= \frac{1}{18aq^2}(108ap^2 - 24pq^2 + 18bcpq - 14aq^3 - 18a^2pq + 6pq^2) = \\ &= \frac{1}{18aq^2}[108ap^2 - 14aq^3 + 18pq(bc - a^2 - q)] = \\ &= \frac{1}{18aq^2}(108ap^2 - 14aq^3) = \frac{54p^2 - 7q^3}{9q^2} \end{aligned}$$

is valid. ♢

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