Mathematica Pannonica
20/1 (2009), 1–11

SOME THEOREMS ON THE PRIME DIVISORS OF INTEGERS

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Dedicated to my friend, János Fehér on his seventieth anniversary

Received: February 2009??

MSC 2000: 11 N 37, 11 N 60

Keywords: Sum of prime divisors, number of prime divisors.

Abstract: The distribution of $\kappa(n)$ = sum of distinct prime divisors $\mod t_k$ is investigated over the set of integers having k distinct prime divisors.

1. Introduction

Let $\mathcal{P}$ be the set of primes, p with and without suffixes always denote prime numbers. Let $p(n)$ be the smallest and $P(n)$ be the largest prime divisors of n. Let

$$\omega(n) = \sum_{p|n} 1; \quad \kappa(n) = \sum_{p|n} p; \quad \varrho(n) := \frac{\kappa(n)}{\omega(n)}.$$

Let $\mathcal{P}_k = \{n \mid \omega(n) = k\}$. For the sake of simplicity we shall write $x_1 = \log x$, $x_2 = \log x_1$, $x_{r+1} = \log x_r$ $(r = 2, 3, \ldots)$.

Let $R(x) = \#\{n \leq x \mid \varrho(n) = \text{integer}\}$.

W. Banks and his coauthors proved in [1] that

$$c_1 < \frac{R(x)x_2}{x} < c_2 \quad \text{if} \quad x > c_3$$

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Financially supported by OTKA T46993.

with some positive constants c_1, c_2, c_3.

In [4] I proved that

$$R(x) = (1 + o_x(1))c \cdot \frac{x}{x_2} \quad (x \to \infty) \tag{1.1}$$

with a suitable constant $c > 0$.

I obtained it quite easily by using our method developed in a joint paper with J.-M. De Koninck [2]. We used this method in [3] as well.

We shall prove much more than (1.1) (Th. 1). Namely we can determine the asymptotic of

$$\#\{n \le x, \quad \omega(n) = k, \quad \kappa(n) = l \pmod{t_k}\} \tag{1.2}$$

where $1 \le t_k \le cx_2$, $l \pmod{t_k}$ arbitrary, and $k \in J_x = \left[x_2 - x_2^{3/4}, x_2 + x_2^{3/4}\right]$.

We note that our theorem remains valid for every k located in an interval larger than J_x. Furthermore, we can give the asymptotic of the numbers in (1.2) after substituting $\kappa(n)$ by $\kappa_r(n)$ $(r = 2, 3, \ldots)$, where $\kappa_r(n) = \sum_{p|n} p^r$, or with $\kappa_P(n) = \sum_{p|n} P(p)$, where $P \in \mathbb{Z}[x]$.

2. Lemmata

2.1. Let $e(\alpha) := e^{2\pi i \alpha}$ for real number α.

Lemma 1. *Let*

$$c_R(n) := \sum_{\substack{h=1\\(h,R)=1}}^{R} e\left(\frac{hn}{R}\right)$$

be the Ramanujan sum. Then

$$c_R(n) = \frac{\mu(t)\varphi(R)}{\varphi(t)}, \quad t = \frac{R}{(R,n)}.$$

Proof. See G. Tenenbaum [5], p. 35. ◊

Lemma 2. *Let $\mathbb{Z}_q^*$ be the set of reduced residue classes* mod q*, $\lambda_{q,h}(s)$ be the number of solutions of $l_1 + \cdots + l_h \equiv s \bmod q$, where l_ν run over all possible values of $\mathbb{Z}_q^*$, independently. Then*

$$\lambda_{q,h}(s) = \frac{1}{q} \sum_{a=0}^{q-1} e\left(\frac{-sa}{q}\right) c_q(a)^h. \tag{2.1}$$

(1) *If* $q = p_1^{a_1} \dots p_\nu^{a_\nu}$ *is odd, then*

$$\left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{1}{q} \right| \le \frac{c}{q} \sum_{j=1}^{\nu} \frac{1}{\varphi(p_j)^{h-1}}. \tag{2.2}$$

(2) *If* $q = \text{even} = 2^{a_0} p_1^{a_1} \dots p_\nu^{a_\nu}$, p_j *are odd, then*
2a) *in the case* $h + s \equiv 1 \pmod 2$ *we have* $\lambda_{q,h}(s) = 0$,
2b) *in the case* $h + s \equiv 0 \pmod 2$ *we obtain*

$$\left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{2}{q} \right| \le \frac{c}{q} \sum_{j=1}^{\nu} \frac{1}{\varphi(p_j)^{h-1}}, \tag{2.3}$$

c is an absolute, positive constant.
Proof. (2.1) is clear. Let $q =$ odd. Separating $a = 0$ in (2.1) we have

$$\left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{1}{q} \left(\frac{c_q(0)}{\varphi(q)} \right)^h \right| \le \frac{1}{q} \sum_{a=1}^{q-1} \left| \frac{c_q(a)}{\varphi(q)} \right|^h. \tag{2.4}$$

Since $c_q(0) = \varphi(q)$, and

$$(U :=) \sum_{a=1}^{q} \left| \frac{c_q(a)}{\varphi(q)} \right|^h = \prod_{j=1}^{\nu} \sum_{b=1}^{p_j^{a_j}} \left| \frac{c_{p_j{}^{a_j}}(b)}{\varphi(p_j^{a_j})} \right|^h$$

from Lemma 1 we obtain that

$$\sum_{b=1}^{p_j^{a_j}} \left| \frac{c_{p_j{}^{a_j}}(b)}{\varphi(p_j^{a_j})} \right|^h = 1 + \sum_{\substack{l=1 \\ (l,p_j)=1}}^{p_j-1} \frac{|\mu(l)|}{|\varphi(p_j)|^h} \le 1 + \frac{1}{|\varphi(p_j)|^{h-1}}.$$

The right-hand side of (2.4) equals to $\frac{U-1}{q} \le \frac{1}{q} \left\{ \prod \left(1 + \frac{1}{|\varphi(p_j)|^{h-1}} \right) - 1 \right\}$, whence (2.2) is obvious.

The assertion for the case 2a) is clear.

Let us consider 2b). Observe that in (2.1) for $a = q/2$ we have $c_q\left(\frac{q}{2}\right) = \frac{\mu(2)\varphi(q)}{\varphi(2)} = -\varphi(q)$, thus $e\left(\frac{-sa}{q}\right) c_q\left(\frac{q}{2}\right)^h = (-1)^{h-s}\varphi(q)^h$. Separating $a = 0$ and $a = q/2$ in (2.1), we obtain

$$\left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{2}{q} \right| \le \frac{1}{q} \sum_{\substack{a \pmod q \\ a \ne 0, q/2}} \left| \frac{c_q(a)}{\varphi(q)} \right|^h.$$

We can repeat the argument used earlier, and obtain (2.3) directly. $\Diamond$

2.2. Let
$$\pi(x,k,l)=\sum_{\substack{p\le x\\ p\equiv l \pmod k}} 1.$$

Lemma 3 (Siegel–Walfisz)**.** *Let c, B be arbitrary positive constants. Then for $\frac{x}{x_1^c}\le y\le x$, $k\le x_1^B$, $(k,l)=1$, we have*

$$\pi(x+y,k,l)-\pi(x,k,l)=\frac{li(x+y)-lix}{\varphi(k)}\big(1+\mathcal{O}(\exp(-c_1\sqrt{x_1}))\big) \tag{2.5}$$

uniformly in k,l, c_1 is an absolute positive constant.

Let $\pi_r(x)=\#\{n\le x\mid \omega(n)=r\}$. According to Hardy and Ramanujan we have

$$\pi_r(x)\le c_1\frac{x}{x_1}\frac{(x_2+c)^{r-1}}{(r-1)!}\quad (x\ge e), \tag{2.6}$$

$c,c_1>0$ are absolute constants.

Lemma 4. *Let $U_r(x,W)$ be the number of those $n\le x$ with $\omega(n)=r$ for which $p^2\mid n$ and $p>W$. Then*

$$U_r(x,W)\le c_1\frac{x}{x_1}\frac{(x_2+c)^{r-2}}{(r-2)!}\frac{1}{W\log W}+\mathcal{O}(x^{3/4}), \tag{2.7}$$

if $2\le W\le x^{1/4}$, say.

Proof. If $p^2\mid n$, $n\le x$, $\omega(n)=r$, $p>W$, then $n=p^\alpha m$, $\omega(m)=r-1$, $\alpha\ge 2$, $m\le x/p_\alpha$, then the number of m is less than
$$c\left(\sum_{\substack{p>W\\ \alpha\ge 2\\ p^\alpha\le\sqrt{x}}}\frac{1}{p^\alpha}\right)\frac{x}{x_1}\frac{(x_2+c)^{r-2}}{(r-2)!}+x\sum_{p\ge\sqrt{x}}1/p^\alpha.$$
Hence (2.7) is clear. $\Diamond$

Lemma 5. *Let $G_L(x)$ be the number of those integers $n\le x$ which have two prime divisors p_1 and p_2 satisfying $L<p_1<p_2<4p_1$. Then*

$$G_L(x)\ll\frac{x}{\log L}. \tag{2.8}$$

Let $G_{L,r}(x)$ be the number of those $n\le x$ with $\omega(n)=r$ for which $p_1p_2\mid n$ holds with prime numbers p_1,p_2 such that $L<p_1<p_2<4p_1$. Assume that $r\ge 3$. Then

$$G_{L,r}(x) \ll \frac{x}{x_1}\frac{x_2^{r-3}}{(r-3)!}\frac{1}{(\log L)} + \frac{x}{x_1}. \tag{2.9}$$

Proof. We have

$$G_L(x) \le \sum_{L<p_1<p_2<4p_1} \left[\frac{x}{p_1 p_2}\right] \le x \sum_{L<p_1\le\sqrt{x}} \frac{1}{p_1} \sum_{p_1<p_2<4p_1} \frac{1}{p_2}$$
$$\ll x \sum_{p_1>L} \frac{1}{p_1 \log p_1} \ll x/\log L.$$

Thus (2.8) is true.

We have

$$G_{L,r}(x) \le \sum_{\substack{L<p_1\le p_2\le 4p_1\\ \alpha,\beta}} \pi_{r-2}\left(\frac{x}{p_1^\alpha p_2^\beta}\right). \tag{2.10}$$

The contribution of those $p_1^\alpha p_2^\beta$ for which $\alpha \ge 2$ and $p_1^\alpha > x^{1/4}$ or $\beta \ge 2$ and $p_2^\beta > x^{1/4}$ is less than

$$\ll \sum_{p_1^\alpha > x^{1/4}} \frac{x}{p_1^\alpha} \sum_{p_1<p_2<4p_1} \frac{1}{p_2} \ll x \sum \frac{1}{p_1^\alpha \log L} \ll x^{0,9}.$$

The contribution of those $p_1 p_2$ for which $p_1 > x^{1/4}$ is less than $G_{x^{1/4}}(x) \ll \ll \frac{x}{x_1}$. Finally, if $p_1^\alpha \le x^{1/4}$, $p_2^\beta \le x^{1/4}$ then

$$\pi_{r-2}\left(\frac{x}{p_1^\alpha p_2^\beta}\right) \le \frac{cx}{p_1^\alpha p_2^\beta}\frac{1}{x_1}\frac{x_2^{r-3}}{(r-3)!}.$$

From these inequalities (2.9) follows. $\Diamond$

2.3. Let B and c_0 be large positive constants,

$$\mathcal{L} := \{l_j : j = 0, 1, 2, \ldots\}, \tag{2.11}$$

where

$$l_0 = \exp(x_2^B), \quad l_{j+1} = l_j + \frac{l_j}{(\log l_j)^{c_0}}. \tag{2.12}$$

Let $I(l_j) = [l_j, l_{j+1})$, $\beta(l_j) = li(l_{j+1}) - li(l_j)$. If $u \in \mathcal{L}$, $u = l_\nu$, then let $\Delta u := l_{\nu+1} - l_\nu$, and so $I(u) = [u, u + \Delta u]$.

Let $Y = [\sqrt{x}, x]$. We shall consider such h-tuples $(u_1, \ldots, u_h)$ for which

$$(2.13)\qquad (l_0 \le)\ u_1 < \cdots < u_h, \quad u_\nu \in \mathcal{L} \quad (\nu = 1, \ldots, h).$$

We say that $(u_1, \ldots, u_h)$ is

a) feasible if $u_1 \cdots u_h \le Y$,

b) well spaced if $u_{j+1} \ge 2u_j \quad (j = 1, \ldots, h-1)$,

c) completely suitable, if $(u_1 + \Delta u_1) \cdots (u_h + \Delta u_h) \le Y$.

Let

$$\mathcal{M}_h(l_0, Y) = \{m = p_1 \cdots p_h \le Y, \quad l_0 \le p_1 < \cdots < p_h\},$$

and let

$$(2.14)\qquad M_h(l_0, Y) := \#(\mathcal{M}_h(l_0, Y)).$$

Let us assume that $h \le cx_2$.

Adapting the method of Sathe and A. Selberg, we can deduce that

$$(2.15)\qquad M_h(l_0, Y) = (1 + o_Y(1)) \frac{Y}{\log Y} \cdot \frac{x_2^{h-1}}{(h-1)!} \prod_{p < l_0} (1 - 1/p).$$

We shall count those elements $m = p_1 \cdots p_h \in \mathcal{M}_h(l_0, Y)$ for which at least one of the following assertion is true:

α) there exists such an i for which $p_{i+1} < 4p_i$,

β) $p_{i+1} > 4p_i$ $(i = 1, \ldots, h-1)$, and if $u_1, \ldots u_h \in \mathcal{L}$ are defined by $p_i \in I(u_i)$, then

$$(u_1 + \Delta u_1) \cdots (u_h + \Delta u_h) > Y.$$

From Lemma 5 we obtain that no more than

$$(2.16)\qquad c \frac{Y}{\log Y} \frac{x_2^{h-3}}{(h-3)!} \frac{1}{\log l_0} + \frac{Y}{x_1}$$

integers exist, for which α) holds.

Assume that $p_i \in I(u_i)$ $(i = 1, \ldots, h)$, $u_{i+1} \ge 2u_i$ $(i = 1, \ldots, h-1)$,

$$u_1 \cdots u_h \le p_1 \cdots p_h \le (u_1 + \Delta u_1) \cdots (u_h + \Delta u_h).$$

Since the right-hand side is bigger than Y, therefore

$$\prod_{\nu=1}^{h} u_\nu = \prod_{\nu=1}^{h} (u_\nu + \Delta u_\nu) \cdot \prod_{\nu=1}^{h} \frac{1}{1 + \frac{\Delta u_\nu}{u_\nu}} > Y \exp\left\{ -\frac{1}{2} \sum_{\nu=1}^{h} \frac{\Delta u_\nu}{u_\nu} \right\},$$

and

$$\sum_{\nu=1}^{h} \frac{\Delta u_\nu}{u_\nu} \le \sum_{\nu=0}^{h-1} \frac{1}{(\log 2^\nu l_0)^{c_0}} \ll \frac{hx_2}{x_2^{Bc_0}}.$$

Thus $p_1 \cdots p_h \in [Y_1, Y]$, where $Y_1 = Y \exp\left(-c \frac{hx_2}{x_2^{Bc_0}}\right)$.

Consequently, the number of elements $m \in \mathcal{M}_h(l_0, Y)$ belonging to $\beta)$, is no more than

$$\pi_h(Y) - \pi_h(Y_1) \ll (Y - Y_1) \frac{1}{x_1} \frac{x_2^{h-1}}{(h-1)!} \ll \ll Y \cdot \frac{1}{x_1} \frac{x_2^{h-Bc_0+1}}{(h-1)!}, \quad \text{if} \quad h \ll x_2. \tag{2.17}$$

This can be deduced from the asymptotic formula for $\pi_h(x)$ (see e.g. [5]). In [6] a short interval version of the asymptotic of $\pi_h(x)$ has been proved.

Assume now that $(u_1, \ldots, u_h)$ is feasible, well-spaced, and completely suitable. Let

$$E_h(u_1, \ldots, u_h) = \#\big\{p_1 \cdots p_h \mid p_\nu \in I(u_\nu), \quad \nu = 1, \ldots, h\big\}. \tag{2.18}$$

Let

$$\beta(u) = li(u + \Delta u) - li\ u, \quad \text{if} \quad u \in \mathcal{L}. \tag{2.19}$$

In [3] we proved

Lemma 6. *If $(u_1, \ldots, u_h)$ is a well-spaced, feasible h-tuple, then*

$$E_h(u_1, \ldots, u_h) = \prod_{\nu=1}^{h} \beta(u_\nu) \left(1 + \mathcal{O}\left(e^{-c_3 x_2^{B/2}}\right)\right), \tag{2.20}$$

the constant implied by $\mathcal{O}$ is absolute.

2.4. Let $1 \le R \le cx_2$, and classify the primes $p > l_0 \mod R$. It is known that

$$\pi(u + \Delta u, R, t) - \pi(u, R, t) = \frac{1}{\varphi(R)} \beta(u) \left(1 + \mathcal{O}\left(e^{-c(\log u)^{1/2}}\right)\right)$$

if $(t, R) = 1$.

Let $H = H_R$ be defined on the set of primes $p > l_0$ by

$$H(p) \equiv p \pmod R, \quad H(p) \in [0, R-1].$$

Let $\alpha = t_1 \cdots t_h$ be a word over the alphabet

$$\mathcal{E}_R = \{t \mid t \in [0, R-1], \quad (t, R) = 1\}.$$

We say that $H(p_1 \cdots p_h) = \alpha$, if $p_1 < \cdots < p_h, \quad H(p_j) = t_j$.

Let

(2.21)
$$E_h^{(R)}(u_1,...,u_h \mid \alpha) := \#\bigl\{p_1\cdots p_h \mid p_j \in I(u_j),\ H(p_j)=t_j,\ j=1,\dots,h\bigr\}.$$

By the observation used in [3] we obtain also

(2.22)
$$E_h^{(R)}(u_1,\dots,u_h \mid \alpha) = \frac{1}{\varphi(R)^h} E_h(u_1,\dots,u_h)\bigl(1+\mathcal{O}(\exp(-c(\log l_0)^{1/2}))\bigr).$$

Let $T_R(\alpha) := \sum_{j=1}^{h} t_j \pmod R$, for $\alpha = t_1 t_2 \cdots t_h$.

From Lemma 2 we deduce that

$$(2.23)\qquad \sum\nolimits_{\substack{\alpha \\ T_R(\alpha)\equiv s \pmod R}} E_h^{(R)}(u_1,\dots,u_h \mid \alpha) =$$

$$= \frac{\lambda_{R,h}(s)}{\varphi(R)^h} E_h^{(R)}(u_1,\dots,u_h)\left(1+\mathcal{O}(\exp(-c(\log l_0)^{1/2}))\right) =$$
$$= \frac{\delta_R(h+s)}{R} E_h^{(R)}(u_1,\dots,u_h)\left(1+\mathcal{O}(\exp(-c(\log l_0)^{1/2}))\right) +$$
$$+\mathcal{O}\left(\frac{1}{R\cdot 2^{h-1}} E_h^{(R)}(u_1,\dots,u_h)\right),$$

where $\delta_R(m)=1$ if $R=$ odd, while for $R=$ even $\delta_R(m)=2$ if $m\equiv 0 \pmod 2$, and $\delta_r(m)=0$, if $m\equiv 1 \pmod 2$. Hence, by (2.15), (2.16), (2.23) we obtain that

$$(2.24)\quad M_h(l_0,Y,R,s) := \#\bigl\{\nu = p_1\dots p_h \le Y \mid l_0 \le p_1 < \dots < p_h,$$
$$T_R(H(p_1)\dots H(p_h)) \equiv s \pmod R\bigr\} =$$
$$= \frac{\delta_R(h+s)}{R} M_h(l_0,Y) + \mathcal{O}\Bigl(\exp\bigl(-c(\log l_0)^{1/2}\bigr) + \frac{1}{2^{h-1}}\Bigr) M_h(l_0,Y) +$$
$$+\mathcal{O}\left(\frac{Y}{x_1}\Bigl(\frac{x_2^{h-3}}{(h-3)!}\cdot\frac{1}{\log l_0}+1\Bigr)\right).$$

3. Formulation and proof of the theorem

Let us write every $n \le x$ as $(n =)\, A(n, l_0)B(n, l_0)$, where

$$A(n, l_0) = \prod_{\substack{p^\alpha || n \\ p < l_0}} p^\alpha; \quad B(n, l_0) = \frac{n}{A(n, l_0)}.$$

Let $k \in J_x$, $1 \le t_k \le cx_2$. We classify the integers $n \in \mathcal{P}_k$, $n \le x$ according to $A(n, l_0)$.

Let $\mathcal{P}_{k,m}(x)$ be the set of those $n \in \mathcal{P}_k$, $n \le x$, for which $A(n, l_0) = m$, and $\mathcal{P}'_{k,m}(x)$ be that subset of $\mathcal{P}_{k,m}(x)$ which consists of those $n = m\nu \in \mathcal{P}_{k,m}$ for which ν is square-free. From Lemma 4 we obtain that

$$\#\big(\cup(\mathcal{P}_{k,m}(x) \setminus \mathcal{P}'_{k,m}(x))\big) \ll \frac{\pi_k(x)}{l_0 \log l_0}, \tag{3.1}$$

where we sum over all m satisfying $A(m, l_0) = m$.

Starting from the well-known estimate

$$\psi(x, y) := \#\{n \le x \mid P(n) \le y\} \ll x \exp\left(-\frac{x_1}{2 \log y}\right)$$

(see for instance Tenenbaum [5]) we can deduce that

$$\#\big\{n \le x \mid A(n, l_0) > \exp\big(x_2^{B+1}\big)\big\} \ll \frac{x}{x_2^{2B}}. \tag{3.2}$$

We omit the details.

Furthermore, for a suitable constant $b > 0$,

$$\#\big\{n \le x \mid \omega(A(n, l_0)) > bx_3\big\} \ll \frac{x}{x_2^{2B}} \tag{3.3}$$

holds.

The proof is simple. The left-hand side of (3.3) is less than

$$\frac{1}{2^{bx_3}} \sum_{n \le x} \tau(A(n, l_0)) \le \frac{x}{2^{bx_3}} \sum_{P(d) \le l_0} \frac{\tau(d)}{d} \ll \frac{x}{2^{bx_3}} \prod_{p < l_0} \Big(1 + \frac{2}{p} + \frac{3}{p^2} + \cdots\Big) \ll$$

$$\ll \frac{x}{2^{bx_3}} \exp(\log\log l_0) \ll \frac{x}{x_2^{2B}}, \quad \text{if} \quad b > \frac{3B}{\log 2}.$$

Let

$$B(x, k, t_k, s) := \#\big\{n \le x \mid n \in \mathcal{P}_k, \quad \kappa(n) \equiv s \pmod{t_k}\big\}. \tag{3.4}$$

Theorem. *Let* $J_x = \left[x_2 - x_2^{3/4}, x_2 + x_2^{3/4}\right]$, $k \in J_x$, t_k *be an integer* $1 \le t_k \le cx_2$, c *an arbitrary constant. Then*

$$\frac{B(x,k,t_k,s)}{\pi_k(x)} = \frac{\mu(t_k)}{t_k}(1+o_x(1)) \tag{3.5}$$

holds uniformly in $k \in J_x$, *and* t_k.

Proof. (3.5) is an easy consequence of our previous inequalities and lemmas.

Let m be fixed, $P(m) < l_0$, and consider all those $n = m\nu \le x$ for which ν is square-free, $p(\nu) \ge l_0$, $\omega(n) = k$, $\kappa(n) \equiv s \pmod{t_k}$. In the notations of (2.24) the following relation holds.

$$M_{k-\omega(m)}\left(l_0, \frac{x}{m}, t_k, s-\kappa(m)\right) = \tag{3.6}$$

$$= (1+o_x(1))\frac{\delta_{tk}(k+s-(\kappa(m)+\omega(m)))}{t_k} M_{k-\omega(m)}\left(l_0, \frac{x}{m}\right).$$

Let $\varrho(l_0) = \prod_{p<l_0}(1-1/p)$. From (2.15) we deduce that

$$M_{k-\omega(m)}\left(l_0, \frac{x}{m}\right) = (1+o_x(1))\varrho(l_0)\cdot\frac{1}{m}\cdot\pi_k(x) \tag{3.7}$$

if

$$m \ll \exp(x_2^{B+1}), \quad \omega(m) \le bx_3, \quad P(m) < l_0.$$

Furthermore, if $t_k = \text{odd}$, then $\delta_{t_k}(\nu) = 1$ for every ν, if $t_k = \text{even}$, then $\kappa(m) + \omega(m) \equiv 0 \pmod 2$, if m is odd, and $\kappa(m)+\omega(m) \equiv 1 \pmod 2$, if m is even, consequently

$$\delta_{t_k}(k+s) - (\kappa(m)+\omega(m)) = \begin{cases} \delta_{t_k}(k+s) & \text{if} \quad m = \text{odd}, \\ \delta_{t_k}(k+s-1) & \text{if} \quad m = \text{even}. \end{cases}$$

Let t_k be odd. From (3.6), (3.7), and from

$$\sum{}^{*} \frac{1}{m} = (1+o_x(1))\prod_{p<l_0}\left(1+\frac{1}{p}+\cdots\right) = (1+o_x(1))\frac{1}{\varrho(l_0)} \tag{3.8}$$

we obtain (3.5) for $t_k = \text{odd}$. On the left-hand side we sum over m under (3.8).

Let $t_k = \text{even}$. If $k+s \equiv 0 \pmod 2$, then we have to sum over odd m satisfying (3.8):

(3.9)

$$\sum_{\text{odd}}^{*} \frac{1}{m} = (1+o_x(1)) = \prod_{3\le p<l_0}\left(1+\frac{1}{p}+\cdots\right) = \cdot(1+o_x(1))\cdot\frac{1}{2}\cdot\frac{1}{\varrho(l_0)}.$$

If $k+s\equiv 1 \pmod 2$, then we have to sum over the even m. Since (3.8)–(3.9) equals to $\sum_{m=\text{even}}^{*}\frac{1}{m}$, therefore it is $(1+o_x(1))\frac{1}{2}\cdot\frac{1}{\varrho(l_0)}$, also.

The proof of the theorem is complete. ♢

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