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ON THE GENERAL SOLUTION OF A FAMILY OF FUNCTIONAL EQUATIONS WITH TWO PARAMETERS AND ITS APPLICATION

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Abstract: Let $I \subset \mathbb{R}$ be a nonvoid open interval and $(r, q) \in (0, 1)^2$, such that $r \neq q$, $r \neq \frac{1}{2}$ and $q \neq \frac{1}{2}$. In this paper we give all the functions $f, g : I \to \mathbb{R}_+$ such that

$$f\left(\frac{x+y}{2}\right)[r(1-q)g(y)-(1-r)qg(x)] = \frac{r-q}{1-2q}[(1-q)f(x)g(y)-qf(y)g(x)]$$

for all $x, y \in I$.

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1. Introduction

In order to formulate the problem we need to define the notion of *weighted quasi-arithmetic mean*. Let $J \subset \mathbb{R}$ be a nonvoid open interval and denote by $\mathcal{CM}(J)$ the class of continuous and strictly monotone real valued functions defined on the interval J. A function $M : J^2 \to J$ is called a *weighted quasi-arithmetic mean* on J if there exist $0 < p < 1$ and $\varphi \in \mathcal{CM}(J)$ such that

$$M(u,v) = \varphi^{-1}(p\varphi(u) + (1-p)\varphi(v)) =: A_\varphi(u,v;p)$$

for all $u, v \in J$. In this case the number p is said to be the *weight* and the function φ is called the *generating function* of the weighted quasi-arithmetic mean M. If $p = \frac{1}{2}$ in the above equation then M is called a *quasi-arithmetic mean* on J (see [7], [5], [1], [3], [10]). If $\varphi(u) = u$ for all $u \in J$, then we have

$$A(u,v;p) := A_{id}(u,v;p) = pu + (1-p)v \quad (u,v \in J),$$

which is the well-known *weighted arithmetic mean* on J.

Now we can formulate the general problem as follows: When will the nontrivial linear combination of two weighted quasi-arithmetic means defined on the same interval J be a weighted arithmetic mean on J? In other words, determine all $M, N : J^2 \to J$ weighted quasi-arithmetic means and the constants $\mu \neq 0, 1$ and $r \in (0,1)$, such that

$$\mu M(u,v) + (1-\mu)N(u,v) = A(u,v;r)$$

holds for all $u, v \in J$. In detail this equation means the following: determine all the functions $\varphi, \psi \in \mathcal{CM}(J)$ and the constants $(p,q,r) \in (0,1)^3$, $\mu \neq 0, 1$ such that

$$\mu\varphi^{-1}(p\varphi(u)+(1-p)\varphi(v))+(1-\mu)\psi^{-1}(q\psi(u)+(1-q)\psi(v)) = ru+(1-r)v$$

holds for all $u, v \in J$.

If we suppose that $\varphi, \psi \in \mathcal{CM}(J)$ are differentiable on J and $\varphi'(u) > 0$, $\psi'(u) > 0$ for all $u \in J$, and we differentiate the above equation first with respect to u and then with respect to v, then we have

$$\mu\frac{p\varphi'(u)}{\varphi'(A_\varphi(u,v;p))} + (1-\mu)\frac{q\psi'(u)}{\psi'(A_\psi(u,v;q))} = r$$

and

$$\mu\frac{(1-p)\varphi'(v)}{\varphi'(A_\varphi(u,v;p))} + (1-\mu)\frac{(1-q)\psi'(v)}{\psi'(A_\psi(u,v;q))} = 1-r$$

for all $u, v \in J$. Multiplying the first equation by $(1-q)\psi'(v)$, the second equation by $-q\psi'(u)$ and adding the new equations, we have

$$\frac{\mu p(1-q)\varphi'(u)\psi'(v) - \mu q(1-p)\varphi'(v)\psi'(u)}{\varphi'(A_\varphi(u,v;p))} = r(1-q)\psi'(v) - (1-r)q\psi'(u)$$

for all $u, v \in J$. With the notations $f := \varphi' \circ \varphi^{-1}$, $g := \psi' \circ \varphi^{-1}$, $I := \varphi(J)$ for the unknown functions $f, g : I \to \mathbb{R}_+$ and $\varphi(u) = x$ and $\varphi(v) = y$ $(x, y \in I)$, from the above equation we have

$$\begin{aligned} &f(px + (1-p)y)[r(1-q)g(y) - (1-r)qg(x)] = \\ &= \mu[p(1-q)f(x)g(y) - (1-p)qf(y)g(x)] \end{aligned} \tag{1}$$

for all $x, y \in I$. The functional eq. (1) depends on the parameters $(p, q, r) \in (0, 1)^3$ and $\mu \neq 0, 1$ for which, if $x = y$ in (1), by $f(x) > 0$, $g(x) > 0$ we have

$$\mu(p - q) = r - q. \tag{2}$$

The functional eq. (1) was studied in the following special cases:

(i) $p = q = r = \mu = 1/2$ by J. Matkowski ([12]), then by Z. Daróczy and Zs. Páles ([5]) under much weaker conditions.

(ii) $p = q$ (then by (2) $r = q$) by Z. Daróczy and Zs. Páles in [6], [5].

(iii) $\mu = r$ J. Jarczyk and J. Matkowski in [9], and J. Jarczyk ([8]), P. Burai ([2]).

(iv) $\mu = r$ and $p = 1/2$, $q \neq 1/2$ by Z. Daróczy in [3] without any conditions.

In this paper we generalise Z. Daróczy's result from [4], studying the functional eq. (1) in the case $p = 1/2$ and $p \neq q$. Hence, by (2) we have $r \neq q$ and $r \neq \frac{1}{2}$ and

$$\mu = \frac{r - q}{\frac{1}{2} - q} = 2 \cdot \frac{r - q}{1 - 2q}.$$

This means we have to determine all the functions $f, g : I \to \mathbb{R}_+$ ($I \subset \mathbb{R}$ nonvoid open interval) and the constants $(q, r) \in (0, 1)^2$, such that

$$\begin{aligned} &f\left(\frac{x+y}{2}\right)[r(1-q)g(y) - (1-r)qg(x)] = \\ &= \frac{r-q}{1-2q}[(1-q)f(x)g(y) - qf(y)g(x)] \end{aligned} \tag{3}$$

holds for all $x, y \in I$.

2. Main result

Theorem 1. *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $0 < r < 1$, $0 < q < 1$, $r, q \neq 1/2$, $r \neq q$. If the functions $f, g : I \to \mathbb{R}_+$ are solutions of the functional eq.* (3) *then the following cases are possible:*

1) *If $r \neq \frac{q^2}{q^2+(1-q)^2}$ then there exist constants $a, b \in \mathbb{R}_+$ such that*

$$f(x) = a \quad \text{and} \quad g(x) = b \quad \text{for all } x \in I;$$

2) *If $r = \frac{q^2}{q^2+(1-q)^2}$ then there exists an additive function $A\colon \mathbb{R} \to \mathbb{R}$ and real $c_1, c_2 > 0$ such that*

$$g(x) = c_1 e^{A(x)} \quad \text{and} \quad f(x) = c_2 e^{2A(x)} \quad \text{for all } x \in I.$$

Conversely, the functions given in the above cases are solutions of eq. (3).

To prove Th. 1 we need the following lemmas.

Lemma 1. *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $0 < r < 1$, $0 < q < 1$, $r \neq q$, $r, q \neq 1/2$. If the functions $f, g : I \to \mathbb{R}_+$ satisfy the functional eq.* (3) *then*

$$f\left(\frac{x+y}{2}\right)[g(x) + g(y)] = [f(x)g(y) + f(y)g(x)] \tag{4}$$

is true for all $x, y \in I$.

Proof. By interchanging x and y in (3) we have

$$\begin{aligned} f\left(\frac{x+y}{2}\right)[r(1-q)g(x) - (1-r)qg(y)] &= \\ = \frac{r-q}{1-2q}[(1-q)f(y)g(x) - qf(x)g(y)] \end{aligned} \tag{5}$$

for all $x, y \in I$.

We add eqs. (3) and (5), then we have

$$f\left(\frac{x+y}{2}\right)[g(x) + g(y)](r-q) = \frac{r-q}{1-2q}[f(x)g(y) + f(y)g(x)](1-2q)$$

for all $x, y \in I$.

From this equation it follows (4). $\Diamond$

Lemma 2. *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $0 < r < 1$, $0 < q < 1$, $r \neq q$, $r, q \neq 1/2$. If the functions $f, g : I \to \mathbb{R}_+$ satisfy the functional eq.* (3) *then*

$$\begin{aligned} &f(x)g(y)\{q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x)\} = \\ &= f(y)g(x)\{q(1-q)(1-2r)g(x) - [r(1-2q) - q^2(1-2r)]g(y)\} \end{aligned} \tag{6}$$

is true for all $x, y \in I$.

Proof. From (4) by (3) we obtain

$$\frac{f(x)g(y)+f(y)g(x)}{g(x)+g(y)}[r(1-q)g(y)-(1-r)qg(x)] =$$
$$= \frac{r-q}{1-2q}[(1-q)f(x)g(y)-qf(y)g(x)]$$

for all $x, y \in I$. By short computation we obtain (6) for all $x, y \in I$. $\Diamond$

Lemma 3. *Let $I \subset \mathbb{R}$ be a nonvoid open interval and $0 < r < 1$, $0 < q < 1$, $r \neq q$, $r, q \neq 1/2$, $r \neq \frac{q^2}{q^2+(1-q)^2}$. If the functions $f, g : I \to \mathbb{R}_+$ satisfy the functional eq.* (3) *then the following propositions*

$$(7) \qquad q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x) \neq 0$$

and

$$(8) \qquad \frac{f(x)g(y)}{f(y)g(x)} = \frac{q(1-q)(1-2r)g(x) - [r(1-2q) - q^2(1-2r)]g(y)}{q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x)}$$

are true for all $x, y \in I$.

Proof. If $x = y$, then the expression in (7) becomes:

$$g(x)[q(1-q)(1-2r) - r(1-2q) + q^2(1-2r)] = g(x)(q-r) \neq 0,$$

therefore assertion (7) is true.

If $x \neq y$ we assert that

$$q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x) \neq 0.$$

Contrary, we suppose that

$$q(1-q)(1-2r)g(y) - [r(1-2q) - q^2(1-2r)]g(x) = 0$$

and then by $r(1-2q) - q^2(1-2r) \neq 0$, which is equivalent to $r \neq$ $\neq \frac{q^2}{q^2+(1-q)^2}$, we have

$$\frac{g(x)}{g(y)} = \frac{q(1-q)(1-2r)}{r(1-2q) - q^2(1-2r)}.$$

With the above assumption by $q(1-q)(1-2r) \neq 0$, from (6) we have

$$\frac{g(x)}{g(y)} = \frac{r(1-2q) - q^2(1-2r)}{q(1-q)(1-2r)}.$$

From the previous two equations we have

$$[q(1-q)(1-2r)]^2 = [r(1-2q) - q^2(1-2r)]^2,$$

i.e.

$$(q-r)(1-2q)[(1-r)q + (1-q)r] = 0$$

which is impossible. Hence, (7) is true for all $x, y \in I$. From (6) by (7) we have (8) for all $x, y \in I$. $\Diamond$

Lemma 4. *Let $I \subset \mathbb{R}$ be a nonvoid open interval and let $0 < r < 1$, $0 < q < 1$, $r, q \neq 1/2$, $r \neq q$ be fixed numbers such that $r \neq \frac{q^2}{q^2+(1-q)^2}$. If the functions $f, g : I \to \mathbb{R}_+$ with the property $f(y_0) = g(y_0) = 1$ $(y_0 \in I)$ satisfy functional eq.* (3)*, then*

$$[g(x) - g(y)][1 - g(x)][1 - g(y)] = 0 \tag{9}$$

for all $x, y \in I$.

Proof. By Lemma 3 we know that (7) and (8) are true. From (8) with $y = y_0 \in I$ we have

$$f(x) = g(x)\frac{q(1-q)(1-2r)g(x) - [r(1-2q) - q^2(1-2r)]}{q(1-q)(1-2r) - [r(1-2q) - q^2(1-2r)]g(x)}$$

for all $x \in I$. With the notations $\alpha := q(1-q)(1-2r) \neq 0$ and $\beta := := r(1-2q) - q^2(1-2r) \neq 0$ the above equation becomes

$$f(x) = g(x)\frac{\alpha g(x) - \beta}{\alpha - \beta g(x)} \quad \text{for all } x \in I.$$

We substitute this form of f in eq. (8) and we obtain

$$\frac{g(x)\frac{\alpha g(x)-\beta}{\alpha-\beta g(x)}g(y)}{g(y)\frac{\alpha g(y)-\beta}{\alpha-\beta g(y)}g(x)} = \frac{\alpha g(x) - \beta g(y)}{\alpha g(y) - \beta g(x)}$$

for all $x, y \in I$, i.e.

$$[\alpha g(x)-\beta][\alpha-\beta g(y)][\alpha g(y)-\beta g(x)] = [\alpha g(y)-\beta][\alpha-\beta g(x)][\alpha g(x)-\beta g(y)]$$

for all $x, y \in I$. From this equation with the notation

$$F(x, y) := [\alpha g(x) - \beta][\alpha - \beta g(y)][\alpha g(y) - \beta g(x)]$$

we have $F(x, y) = F(y, x)$ for all $x, y \in I$. From this equation with an easy computation and with the notation $A := \alpha\beta^2 + \alpha^2\beta$ it follows

$$Ag(x) - Ag(y) + Ag^2(x)g(y) - Ag(x)g^2(y) + Ag^2(y) - Ag^2(x) = 0$$

for all $x, y \in I$. We can easily observe that

$$A = \alpha\beta^2 + \alpha^2\beta = \alpha\beta(\alpha + \beta) \neq 0,$$

for $\alpha\beta \neq 0$ and $\alpha + \beta = (1 - 2q)[(1 - r)q + (1 - q)r] \neq 0$. Hence

$$[g(x) - g(y)][1 + g(x)g(y) - g(x) - g(y)] = 0.$$

But this is (9) for all $x, y \in I$. $\Diamond$

Proof of Th. 1. (i) First we suppose that the functions $f, g : I \to \mathbb{R}_+$ are solutions of the functional eq. (3) (where $0 < r < 1$, $0 < q < 1$, $r, q \neq 1/2$, $r \neq q$), $r \neq \frac{q^2}{q^2+(1-q)^2}$ and $f(y_0) = g(y_0) = 1$ for $y_0 \in I$. We assert, that in this case $f(x) = g(x) = 1$ for all $x, y \in I$. Contrary, we suppose that there exists $y_1 \in I$ $(y_1 \neq y_0)$, such that

$$g(y_1) = c \neq 1 \quad \text{and} \quad c > 0.$$

With the substitution $y = y_1$ in (9) we have

$$[g(x) - c][1 - g(x)] = 0 \tag{10}$$

for all $x \in I$. We define

$$E := \{x \mid x \in I,\ g(x) = 1\} \neq \emptyset$$

and

$$E^* := \{x \mid x \in I,\ g(x) = c\} \neq \emptyset.$$

By eq. (10) any $x \in I$ is in E or in E^*, i.e. $E \cap E^* = \emptyset$ and $I = E \cup E^*$. By Lemma 3

$$f(x) = g(x)\frac{\alpha g(x) - \beta}{\alpha - \beta g(x)} = \begin{cases} 1 & \text{if } x \in E \\ c\frac{\alpha c - \beta}{\alpha - \beta c} & \text{if } x \in E^*. \end{cases} \tag{11}$$

If $x \in E$ and $y \in E^*$ then by eq. (4) we have

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)c + f(y)}{c+1} = \frac{c + c\frac{\alpha c - \beta}{\alpha - c\beta}}{c+1}.$$

Now, $\frac{x+y}{2} \in E$ or $\frac{x+y}{2} \in E^*$. In the first case we have

$$\frac{c + c\frac{\alpha c - \beta}{\alpha - c\beta}}{c+1} = 1$$

or in the second case

$$\frac{c + c\frac{\alpha c - \beta}{\alpha - c\beta}}{c+1} = c\frac{\alpha c - \beta}{\alpha - c\beta}.$$

In both cases we obtain $c^2 = 1$, i.e. $c = 1$, which is a contradiction.

Then $g(x) = 1$ for all $x \in I$ and by (11) it follows $f(x) = 1$ for all $x \in I$.

(ii) If the pair (f, g) $(f, g : I \to \mathbb{R}_+)$ is a solution of (3) then the pair $\left(\frac{f}{f(y_0)}, \frac{g}{g(y_0)}\right)$ $(y_0 \in I)$ is a solution of (3) too, and $\frac{f(y_0)}{f(y_0)} = 1$, $\frac{g(y_0)}{g(y_0)} = 1$. By (i) we have $\frac{f(x)}{f(y_0)} = 1$, $\frac{g(x)}{g(y_0)} = 1$ for all $x \in I$. With $f(y_0) := a > 0$ and $g(y_0) := b > 0$ we obtain the assertion of Th. 1 for the case $r \neq \frac{q^2}{q^2+(1-q)^2}$.

In the case $r = \frac{q^2}{q^2+(1-q)^2}$, by Lemmas 1 and 2, and with the notations of Lemma 4 (6) becomes

$$f(x)g(y)\alpha g(y) = f(y)g(x)\alpha g(x) \quad \text{for all } x, y \in I.$$

Hence

$$f(x) = cg^2(x),\ c > 0, \quad \text{for all } x \in I. \tag{12}$$

Replacing this form of f in (4) we have

$$g^2\left(\frac{x+y}{2}\right)=g(x)g(y) \quad \text{for all } x,y\in I,$$

consequently, by [10], [11] there exist $A:\mathbb{R}\to\mathbb{R}$ additive function and real $c_1>0$ such that $g(x)=c_1e^{A(x)}$ for all $x\in I$, and by (12), $f(x)=$ $=c_2e^{2A(x)}$, $c_2>0$ for all $x\in I$ and we obtain the assertion of Th. 1.

3. Application

Returning to the generalized problem we need the following definition.

Definition 1. Let $\varphi,\psi\in\mathcal{CM}(J)$. If there exist $a\neq 0$ and b such that

$$\psi(x)=a\varphi(x)+b \quad \text{if } x\in J$$

then we say that φ is equivalent to ψ on J and denote it by $\varphi(x)\sim\psi(x)$ if $x\in J$ or in short $\varphi\sim\psi$ on J.

It is well known that if $0<p<1$ and $\varphi,\psi\in\mathcal{CM}(J)$, then $A_\varphi(x,y;p)=A_\psi(x,y;p)$ for all $x,y\in J$ if and only if $\varphi\sim\psi$ on J.

We define the following sets:

$$T_+(J):=\big\{t\in\mathbb{R}\mid J+t\subset\mathbb{R}_+\big\}$$
$$T_-(J):=\big\{t\in\mathbb{R}\mid -J+t\subset\mathbb{R}_+\big\}.$$

With the help of these notations, set

$$\gamma_t^+(x):=\sqrt{x+t} \text{ if } t\in T_+(J)\ (x\in J)$$
$$\gamma_t^-(x):=\sqrt{-x+t} \text{ if } t\in T_-(J)\ (x\in J).$$

Theorem 2. *Let $J\subset\mathbb{R}$ be a nonvoid open interval and $0<r<1$, $0<q<1$, $r,q\neq\frac{1}{2}$, $r\neq q$. If $\varphi,\psi\in\mathcal{CM}(J)$ solve the functional equation*

$$\begin{aligned}\frac{2(r-q)}{1-2q}\varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right)&+\\ +\left(1-\frac{2(r-q)}{1-2q}\right)\psi^{-1}(q\psi(u)+(1-q)\psi(v))&=ru+(1-r)v\end{aligned}\tag{13}$$

for all $u,v\in J$ and φ,ψ are differentiable on J and $\varphi'(u)>0$, $\psi'(u)>0$ for all $u\in J$ then $\varphi\sim id$ and $\psi\sim id$ on J, furthermore, in the case $r=\frac{q^2}{q^2+(1-q)^2}$ the following cases are also possible:

$$\varphi\sim\log\gamma_t^+,\ \psi\sim\gamma_t^+ \quad \textit{if } t\in T_+(J)$$

or

$$\varphi \sim \log \gamma_t^-, \ \psi \sim \gamma_t^- \quad \textit{if} \ \ t \in T_-(J).$$

Proof. It is enough to solve the functional eq. (13) up to the equivalence of the functions φ and ψ. With the notations $f := \varphi' \circ \varphi^{-1}$, $g := \psi' \circ \varphi^{-1}$, $I := \varphi(J)$ we get that eq. (3) holds. Due to the definition of f, we obtain the differential equation for the function φ:

$$\varphi'(x) = f(\varphi(x)) \quad x \in J. \tag{14}$$

By Th. 1, the case $r \neq \frac{q^2}{q^2+(1-q)^2}$ gives the constant solutions, from which follows that $\varphi \sim id$, $\psi \sim id$. If $r = \frac{q^2}{q^2+(1-q)^2}$ then

$$f(x) = c_2 e^{2A(x)} \text{ and } g(x) = c_1 e^{A(x)} \ \text{ for all } x \in I, \tag{15}$$

where $c_1, c_2 > 0$ and $A : \mathbb{R} \to \mathbb{R}$ is an additive function. Since $\frac{1}{f}$ is a derivative, f has a continuity point and therefore in (15) by [11] $A(x) = = cx$, $x \in \mathbb{R}$, $c \in \mathbb{R}$.

In the case $c = 0$ $\varphi \sim id$ and $\psi \sim id$.

In the case $c \neq 0$ from (14) we have

$$\varphi'(u) = c_2 e^{2c\varphi(u)} \quad \text{ for all } u \in J,$$

from which we deduce that either there exists $t \in T_+(J)$ such that $\varphi \sim \log \gamma_t^+$ on J or there exists $t \in T_-(J)$ such that $\varphi \sim \log \gamma_t^-$ on J.

Due to the definition of g, by (15) we obtain that

$$\psi'(u) = e^{c\varphi(u)} > 0 \quad \text{ for all } u \in J.$$

We know that $\varphi'(u) = c_2 e^{2c\varphi(u)} > 0$, hence $\varphi'(u) = \psi'(u)^2$, $u \in J$, from which we get that either there exists $t \in T_+(J)$ such that $\psi \sim \gamma_t^+$ on J or there exists $t \in T_-(J)$ such that $\psi \sim \gamma_t^-$ on J. $\Diamond$

Remark 1. Let $J := (-\infty, 0)$. Then $T_+(J) = \emptyset$ and $T_-(J) \neq \emptyset$, for example $1 \in T_-(J)$. If the conditions of Th. 2 hold and $r = \frac{q^2}{q^2+(1-q)^2}$, then

$$\varphi(u) \sim \log \sqrt{-u+1} \sim \log(-u+1)$$

and

$$\psi(u) \sim \sqrt{-u+1} \qquad (u \in J)$$

are solutions of the functional eq. (13). Indeed, because of

$$\varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right) = -\sqrt{(-u+1)(-v+1)} + 1,$$

and

$$\psi^{-1}(q\psi(u) + (1-q)\psi(v)) = -(q\sqrt{-u+1} + (1-q)\sqrt{-v+1})^2 + 1,$$

$u, v \in (-\infty, 0)$, we have

$$\mu\big[-\sqrt{(-u+1)(-v+1)}+1\big]+(1-\mu)\big[-q^2(-u+1)-(1-q)^2(-v+1)-$$
$$-2q(1-q)\sqrt{(-u+1)(-v+1)}+1\big]=ru+(1-r)v,$$

which is equivalent to

$$\sqrt{(-u+1)(-v+1)}[-\mu-2q(1-q)(1-\mu)]+\mu+(1-\mu)q^2u-$$
$$-(1-\mu)q^2+(1-\mu)(1-q)^2v-(1-q)^2(1-\mu)+1-\mu=$$
$$=ru+(1-r)v.$$

By $\mu=2\cdot\frac{r-q}{1-2q}$ and $r=\frac{q^2}{q^2+(1-q)^2}$ we get $(1-\mu)q^2=r$ and $(1-\mu)(1-q)^2=$ $=1-r$ and the above equation becomes

$$ru-r+(1-r)v-(1-r)+1=ru+(1-r)v,$$

i.e. φ,ψ solve the functional eq. (13).

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