

# EXISTENCE AND UNIQUENESS IN COURNOT MODELS WITH COST EX- TERNALITIES

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**Abstract:** In this paper we examine single product Cournot oligopolies, without product differentiation, under the assumption that the cost of each firm depends on its own output and also on the output of the rest of the industry. The competition of the firms on the secondary market for manpower, capital, energy, and so forth as well as the spillover effect of the R&D investments of the firms can be taken into account with this more general cost structure. The existence of a unique Nash–Cournot equilibrium is proved under realistic conditions. Our result is a straightforward generalization of the well-known existence and uniqueness theorem of concave oligopolies.

## 1. Introduction

Cournot oligopolies are the most frequently discussed models in the literature of mathematical economics. Based on the pioneering work

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of [4] many researchers have examined the properties of the different variants and extensions of the classical Cournot model. Models with product differentiation, multi-product models, labor-managed firms, rent-seeking games to mention only a few have been introduced and investigated within the Cournot framework. A comprehensive summary of the earlier literature is given in [8]. Multi-product models and some extended models are discussed in [9], where some further applications to water resources and the international fishery are introduced. The interdependence of the firms is considered in these models through the inverse demand functions, which depend on the total output of all firms. However very few attempts have been made to analyse the interdependence of firms via their cost structures. The introduction of oligopsonies by [3] served this purpose, when the competition of the firms on the labor and capital markets was also included in their profit functions. In this paper we adopt a different approach by assuming that the cost of each firm depends on its own output and also on the output of the rest of the industry. This framework is a realistic way to model the spillover effect of various externalities, such as of the R&D investments of the competitors. Such models have been already introduced and the asymptotic properties of the equilibria examined for duopolies and symmetric firms, see in particular [7], and [2]. However in these works no general result was presented for the existence and uniqueness of the equilibria in the general  $n$ -person nonsymmetric case. The existence, under suitable concavity conditions, can be established by using fixed-point theorems, however the uniqueness of the equilibrium cannot be guaranteed and no practical method has been suggested to find the equilibria. In this paper we will use an idea originally suggested in [10], where the  $n$ -dimensional equilibrium problem was reduced to the solution of a one-dimensional monotonic equation leading to the existence of a unique solution and to a simple computational procedure. Another way of departing from the usual concavity assumptions of single-product oligopolies was initiated by [5], when more realistic cost functions were used, and the payoff functions were not concave. These results have also been discussed in [6]. The structure of the paper is as follows. In Sec. 2 we set up the Cournot model with cost externalities, and prove the existence of a unique equilibrium. Sec. 3 concludes the paper.

## 2. The mathematic model and equilibrium analysis

Consider an  $n$ -firm single-product oligopoly without product differentiation and with cost externalities. Let  $x_k$  denote the output of firm  $k$ ,  $Q_k = \sum_{l \neq k} x_l$  the output of the rest of the industry for firm  $k$ , and  $Q = \sum_{l=1}^n x_l$  the total output of the industry. It is assumed that the price function is  $f(Q)$ , and the cost function of firm  $k$  depends on both  $x_k$  and  $Q_k$ , so we write  $C_k(x_k, Q_k)$  to denote the cost function. If  $L_k$  is the capacity limit of firm  $k$ , then an  $n$ -person noncooperative game is obtained, where the firms are the players, the set of strategies of player  $k$  is the compact interval  $[0, L_k]$  and the payoff function of this player is

$$(1) \quad \varphi_k(x_k, Q_k) = x_k f(x_k + Q_k) - C_k(x_k, Q_k).$$

Assume that the functions  $f$  and  $C_k$ ,  $k = 1, 2, \dots, n$ , are twice continuously differentiable, and furthermore that

- (A)  $f'(x_k + Q_k) < 0$ ;
- (B)  $f'(x_k + Q_k) + x_k f''(x_k + Q_k) \leq 0$ ;
- (C)  $f'(x_k + Q_k) - C''_{kxx}(x_k, Q_k) < 0$

for all feasible values of  $x_k$  and  $Q_k$ , where  $C'_{kx}$  and  $C''_{kxx}$  denote the first and second order partial derivatives of  $C_k$  with respect to  $x_k$ .

Condition (A) means that the price function is strictly decreasing. Condition (B) is satisfied if  $f$  is concave or slightly convex, and condition (C) holds if  $C_k$  is convex or slightly concave in  $x_k$ . If  $f$  is linear, then  $f'' = 0$ , so condition (B) is implied by (A), and (C) requires that for all  $k$ ,  $C''_{kxx}$  is bounded from below by a negative lower bound.

Notice first that

$$(2) \quad \frac{\partial \varphi_k}{\partial x_k}(x_k, Q_k) = f(x_k + Q_k) + x_k f'(x_k + Q_k) - C'_{kx}(x_k, Q_k)$$

and

$$\frac{\partial^2 \varphi_k}{\partial x_k^2}(x_k, Q_k) = 2f'(x_k + Q_k) + x_k f''(x_k + Q_k) - C''_{kxx}(x_k, Q_k) < 0,$$

so with fixed values of  $Q_k$ , the payoff function  $\varphi_k$  is strictly concave in  $x_k$ . Therefore for any feasible values of  $Q_k$ , there is a unique best response

of firm  $k$  given by

$$(3) \quad R_k(Q_k) = \begin{cases} 0, & \text{if } f(Q_k) - C'_{kx}(0, Q_k) \leq 0, \\ L_k, & \text{if } f(L_k + Q_k) + L_k f'(L_k + Q_k) - C'_{kx}(L_k, Q_k) \geq 0, \\ z_k, & \text{otherwise,} \end{cases}$$

where  $z_k$  is the unique solution of the equation

$$(4) \quad f(z_k + Q_k) + z_k f'(z_k + Q_k) - C'_{kx}(z_k, Q_k) = 0$$

inside the interval  $(0, L_k)$ . Notice that equation (4) has a unique solution since the left-hand side is strictly decreasing in  $z_k$ , its value is positive at  $z_k = 0$  and negative at  $z_k = L_k$ . By implicit differentiation of (4) with respect to  $Q_k$  we have

$$f' \cdot (R'_k + 1) + R'_k \cdot f' + z_k f'' \cdot (R'_k + 1) - C''_{kxx} R'_k - C''_{kxQ} = 0$$

implying that

$$(5) \quad R'_k = -\frac{f' + z_k f'' - C''_{kxQ}}{2f' + z_k f'' - C''_{kxx}},$$

where  $C''_{kxQ}$  is the mixed second order partial derivative of  $C_k$ . The denominator of (5) is negative under assumptions (A)–(C), and the value of  $R'_k$  belongs to interval  $(-1, 0]$  if in addition we assume that

(D)  $f'(x_k + Q_k) + x_k f''(x_k + Q_k) \leq C''_{kxQ}(x_k, Q_k) < C''_{kxx}(x_k, Q_k) - f'(x_k + Q_k)$  for all  $k$  and all feasible values of  $x_k$  and  $Q_k$ .

The left-hand side of this inequality is nonpositive and the right-hand side is positive by assumptions (B) and (C). Therefore this condition requires that the mixed second order partial derivatives of the cost functions must not have large absolute values.

We can also rewrite the best responses as functions of the total output  $Q$  of the industry, using  $\tilde{R}_k(Q)$  to denote these modified best response functions. In this case we have

$$(6) \quad \tilde{R}_k(Q) = \begin{cases} 0, & \text{if } f(Q) - C'_{kx}(0, Q) \leq 0, \\ Q, & \text{if } Q < L_k \text{ and } f(Q) + Q f'(Q) - C'_{kx}(Q, 0) \geq 0, \\ L_k, & \text{if } Q \geq L_k \text{ and} \\ & f(Q) + L_k f'(Q) - C'_{kx}(L_k, Q - L_k) \geq 0, \\ z_k^*, & \text{otherwise,} \end{cases}$$

where  $z_k^*$  is the unique solution of the equation

$$(7) \quad f(Q) + z_k f'(Q) - C'_{kx}(z_k, Q - z_k) = 0$$

inside the interval  $(0, \min\{Q, L_k\})$ . Notice that the left-hand side of (7) is strictly decreasing in  $z_k$ , since its derivative with respect to  $z_k$  is

$$f'(Q) - C''_{kxx}(z_k, Q - z_k) + C''_{kxQ}(z_k, Q - z_k) < 0,$$

where the last inequality follows because of assumption (D). In addition, in the last case of (6), the left-hand side is positive at  $z_k = 0$  and negative at  $z_k = \min\{Q, L_k\}$ .

We will next examine the properties of the modified best response function  $\tilde{R}_k(Q)$ . To this end we introduce the notation

$$(8) \quad g_k(x_k, Q) = f(Q) + x_k f'(Q) - C'_{kx}(x_k, Q - x_k)$$

for all feasible  $x_k$  and  $Q$ . We note that

$$\frac{\partial g_k}{\partial x_k} = f' - C''_{kxx} + C''_{kxQ} < 0,$$

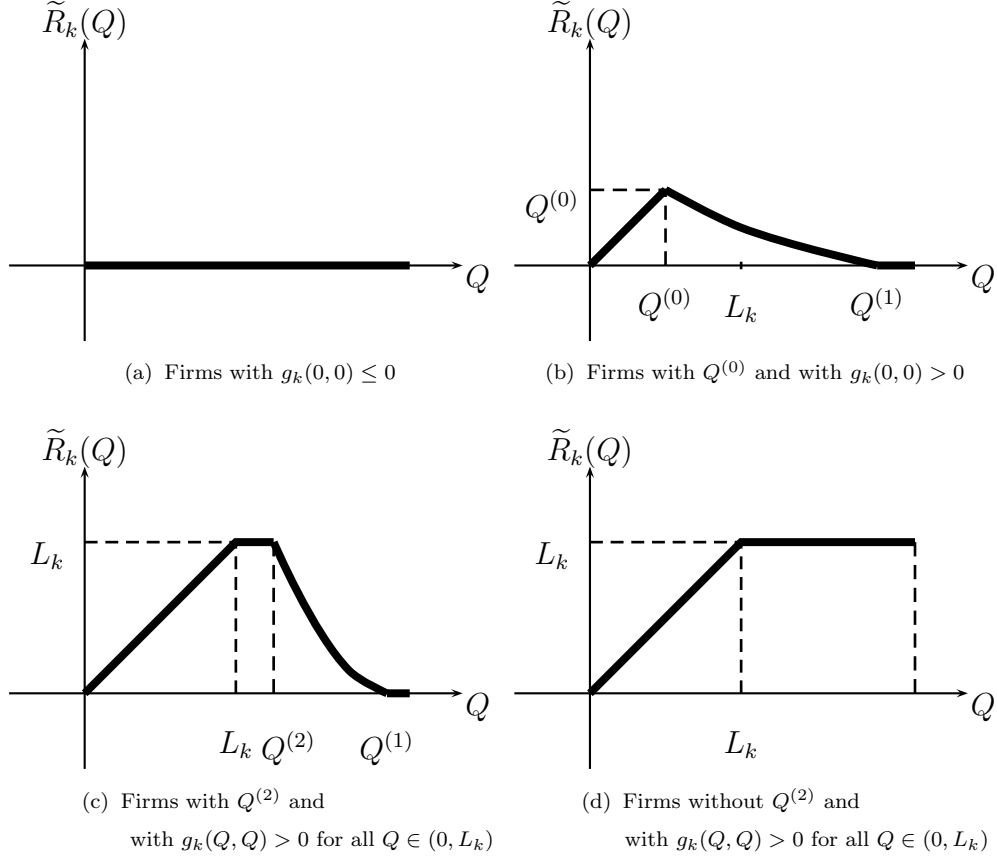
by the right-hand inequality of assumption (D) and that

$$\frac{\partial g_k}{\partial Q} = f' + x_k f'' - C''_{kxQ} \leq 0$$

by the left-hand inequality of assumption (D), hence the function  $g_k$  is strictly decreasing in  $x_k$  and is nonincreasing in  $Q$ .

Assume first that for a firm  $k$ ,  $g_k(0, 0) \leq 0$ . Then for all  $Q \geq 0$ ,  $g_k(0, Q) \leq 0$ , so the first case of (6) occurs for all  $Q \geq 0$ , therefore  $\tilde{R}_k(Q) \equiv 0$ . Since these firms have zero output, they have no effect on the payoffs of other firms (see Fig. 1, panel (a)). Therefore in the subsequent analysis these firms can be omitted. So we assume that for all  $k$ ,  $g_k(0, 0) > 0$ . Then  $g_k(Q, Q) > 0$  with sufficiently small  $Q > 0$ . If there is a  $Q^{(0)} \in (0, L_k)$  such that  $g_k(Q^{(0)}, Q^{(0)}) = 0$ , then the monotonicity of  $g_k(x_k, Q)$  implies that the third case of (6) can never occur with  $Q \geq L_k$ . If  $Q \leq Q^{(0)}$ , then the second case of (6) applies, so  $\tilde{R}_k(Q) = Q$ . If  $Q > Q^{(0)}$  and  $g_k(0, Q) \leq 0$ , then  $\tilde{R}_k(Q) = 0$ , otherwise it is the unique solution of equation (7). The monotonicity of  $g_k(x_k, Q)$  also implies that  $g_k(0, Q) \leq 0$  either cannot occur, or it does for all  $Q \geq Q^{(1)}$ , where  $Q^{(1)}$  denotes the smallest solution of equation  $g_k(0, Q) = 0$ . Clearly,  $Q^{(1)} > Q^{(0)}$ . (See Fig. 1, panel (b).)

Assume next that  $g_k(Q, Q) > 0$  for all  $Q \in (0, L_k)$ . Then  $g_k(L_k, L_k) \geq 0$  implying that  $\tilde{R}_k(L_k) = L_k$ . We have now two possibilities. If there is a feasible  $Q$  such that  $g_k(L_k, Q) = 0$ , then let  $Q^{(2)}$  denote the largest solution of the equation  $g_k(L_k, Q) = 0$ . Then for all  $Q \in [L_k, Q^{(2)}]$ ,  $\tilde{R}_k(Q) = L_k$ . Clearly for all  $Q > Q^{(2)}$ , the first or the last case of (6) occurs, therefore  $\tilde{R}_k(Q)$  is either zero or the solution of equation (7).

Figure 1: The graph of  $\tilde{R}_k(Q)$ 

Notice that  $Q^{(1)} > Q^{(2)}$  in this case. (See Fig. 1, panel (c).) Otherwise  $\tilde{R}_k(Q) = L_k$  for all  $Q > L_k$  (see Fig. 1, panel (d)). We can finally show that the solution of (7) is nonincreasing in  $Q$ . Assume that it is not, that is, there are  $Q < \bar{Q}$  such that

$$z_k = \tilde{R}_k(Q) < \tilde{R}_k(\bar{Q}) = \bar{z}_k.$$

Then

$$0 = g_k(z_k, Q) \geq g_k(z_k, \bar{Q}) > g_k(\bar{z}_k, \bar{Q}) = 0,$$

which is clearly a contradiction. Fig. 1 shows the different possibilities of the graph of  $\tilde{R}_k(Q)$ . Note that in these cases, the horizontal segments might be omitted.

We are now ready to prove our main result.

**Theorem 1.** *Under conditions (A)–(D) there is a unique Nash equilibrium.*

**Proof.** Assume first that for all firms,  $g_k(0, 0) \leq 0$ . Then clearly  $\bar{x}_k = 0$  for all  $k$  is the only equilibrium. Assume next that  $g_k(0, 0) > 0$  for only one firm. Then  $\bar{x}_l = 0$  for all  $l \neq k$ , and  $\bar{x}_k = Q$  is the solution of the maximization problem

$$(9) \quad \max_{0 \leq Q \leq L_k} \{Qf(Q) - C_k(Q, 0)\}.$$

Since the objective function is strictly concave in  $Q$ , there is a unique optimal solution:

$$(10) \quad \bar{x}_k = Q = \begin{cases} L_k, & \text{if } L_k f'(L_k) + f(L_k) - C'_{kx}(L_k, 0) \geq 0, \\ \bar{z}_k, & \text{otherwise,} \end{cases}$$

where  $\bar{z}_k$  is the unique solution of the equation

$$(11) \quad z_k f'(z_k) + f(z_k) - C'_{kx}(z_k, 0) = 0.$$

Notice that the left-hand side is strictly decreasing in  $z_k$ , it is positive at  $z_k = 0$  and negative at  $z_k = L_k$ .

Assume next that  $g_k(0, 0) > 0$  for at least two firms. We will finally show that the equation

$$(12) \quad \sum_{k=1}^n \tilde{R}_k(Q) - Q = 0$$

has exactly one solution, so the equilibrium is therefore unique. Let  $h(Q)$  denote the left-hand side of (12). With sufficiently small  $Q > 0$ ,  $\tilde{R}_k(Q) = Q$  for at least two firms, so with small  $Q > 0$ ,  $h(Q) > 0$ . If  $Q = \sum_{k=1}^n L_k$ , then  $h(Q) \leq 0$ , since for all  $k$  and  $Q$  we have  $\tilde{R}_k(Q) \leq L_k$ .

Since  $\tilde{R}_k(Q)$  is continuous for all  $k$ , there is at least one positive solution of equation (12). The uniqueness of the solution can be proved in the following way. Assume that  $(0 <) Q^* < Q^{**}$  are two solutions of equation (12). Notice that  $\tilde{R}_k(Q)$  is nonincreasing in  $Q$  everywhere except on the initial segment, where  $\tilde{R}_k(Q) = Q$ . In addition,

$$\sum_{k=1}^n \tilde{R}_k(Q^*) < \sum_{k=1}^n \tilde{R}_k(Q^{**})$$

is possible only if for at least one  $k$ ,  $\tilde{R}_k(Q^*) = Q^*$ . Then equation (12) implies that for all other firms  $\tilde{R}_l(Q^*) = 0$ , and for the same firms

$\tilde{R}_l(Q^{**}) = 0$  as well, since  $Q^{**} > Q^*$ . This would lead to two possible equilibria when only one firm produces positive amount. Since we have already demonstrated that the maximization problem (9) cannot have multiple solutions, this is impossible.  $\diamond$

### 3. Conclusions

We have established the existence and uniqueness of the equilibria in single-product Cournot models without product differentiation under the assumption that the cost of each firm depends on its own output as well as on the output of the rest of the industry. The competition of the firms on the secondary market, and the spillover effect of the R&D investments of the competitors can be realistically modelled in this way. In the special case when the cost of each firm depends on its own output, then the mixed partial derivative  $C''_{kxQ}$  is identically zero. Then assumption (D) is identical with (B) and (C), so it can be omitted. So in this case our conditions are reduced to the usual conditions of concave oligopolies (see Ch. 2 of [1]).

Assume that the cost of firm  $k$  is  $C_k(x_k, Q_k) = x_k M_k(Q_k)$ , so the fixed cost is zero and the marginal cost of firm  $k$  depends on  $Q_k$ . In this case  $C''_{kxQ}(x_k, Q_k) = M'_k(Q_k)$  and  $C''_{kxx}$  is identically zero. Therefore condition (C) reduces to (A) and condition (D) becomes

$$f'(x_k + Q_k) + x_k f''(x_k + Q_k) \leq M'_k(Q_k) < -f'(x_k + Q_k)$$

for all feasible  $x_k$  and  $Q_k$ .

Since the left-hand side is nonpositive and the right-hand side is positive, this condition does not restrict the sign of  $M'_k$ , it only requires that  $|M'_k(Q_k)|$  be sufficiently small.

It would be interesting to investigate the existence and uniqueness of the equilibrium with nonconcave payoff functions by generalizing the results given in [5]. We leave this issue to future research.

Many results on the local asymptotic stability of the equilibria under both discrete and continuous time scales for the classical Cournot model are based on the assumption that the derivative of the best response function always belongs to the interval  $(-1, 0]$ . Since this is true also with externalities under conditions (A)–(D), all relevant results known for the classical Cournot model remain true in the more general case with cost externalities (see [1]). The equilibrium under the adaptive adjustment process is always locally asymptotically stable with



continuous time scales, and is locally asymptotically stable with discrete time scales if the speeds of adjustments are sufficiently small.

We leave for future research the global asymptotic properties of the general  $n$ -firm case.

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