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ON SOME CATEGORIES ARISING IN THE THEORY OF LOCALLY COMPACT EXTENSIONS

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Abstract: We give a direct proof of the fact that the following three categories are isomorphic: the category of separated local proximity spaces and equicontinuous mappings, the category of LC-proximity spaces and SR-proximally continuous functions, and the category of separated L-supertopological spaces and supertopological mappings. Many basic statements of the theory of Efremovich proximity spaces are generalized for the class of local proximity spaces.

1. Introduction

In 1967, S. Leader ([9]) described the ordered set of all (up to equivalence) locally compact Hausdorff extensions of Tychonoff spaces

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by means of the notion of *separated local proximity* in which the *boundedness* and *proximity* are both primitive terms. Some other descriptions of the locally compact Hausdorff extensions were given by V. Zaharov ([11, 12]) (by means of some special vector lattices of functions), by G. Dimov and D. Doitchinov ([3]) (on the basis of the notion of *supertopological space*), by Dimov ([2]) (using the notion of *LC-proximity*) and recently by G. Dimov and D. Vakarelov ([4]) – a purely proximity-type description by means of the so-called *lc-proximities.* In all these theories, a description (in the language of the regarded structure) of the functions having a continuous extension over corresponding locally compact extensions was given. Therefore, some categories arise: their objects are the corresponding structures, and their morphisms are the functions just mentioned. It is clear that all these categories are isomorphic (indeed, they are all isomorphic (as it follows immediately from the theories listed above) to the category **LCExt** of equivalence classes of locally compact Hausdorff extensions (i.e., the objects of the category **LCExt** are the equivalence classes $[(X,l)]$, where $l : X \to L_{(X,l)}$ is a dense homeomorphic embedding of a Tychonoff space X into a locally compact Hausdorff space $L_{(X,l)}$) with morphisms determined by the continuous functions $f : X \to Y$ having a continuous extension $\hat{f} : L_{(X,l)} \to L_{(Y,l')}$ over the corresponding locally compact Hausdorff extensions (i.e., $\hat{f} \circ l = l' \circ f$)). Here we construct directly these isomorphisms, i.e. the locally compact extensions are not used in our proof. This is done for the categories arising from the descriptions given by Leader ([9]), Dimov and Doitchinov ([3]) and Dimov ([2]). In this way we describe internally the connections between these three structures, i.e. starting with one of them, we build directly the other two. Surprisingly, the proof is not easy (at least that one found by us). It contains some generalizations (for the class of local proximity spaces) of the most of the basic statements of the theory of Efremovich proximity spaces. We hope that the direct descriptions, obtained here, of the transitions from each of these structures to any other of them could be used further. Let us also mention that the direct proof of the isomorphism between the categories arising from the theories of Leader ([9]) and Dimov and Vakarelov ([4]) was given in [4] (in fact, in [4], the existence of such an isomorphism was used for showing that lc-proximities describe the locally compact Hausdorff extensions). We do not discuss here the theory of Zaharov.

The paper is organized as follows. The second section contains all preliminary results and notions. In it we do not describe the theories of locally compact extensions developed in the cited above papers because we have no need of them. We only give the definitions of the structures defined in [9], [3] and [2], and list those statements about them which are used later. In the third section we prove our main theorems – Th. 3.8 and Th. 3.9, where the isomorphisms mentioned above between the corresponding categories are established.

We now fix the notations.

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if f is a morphism of $\mathcal{C}$ with domain X and codomain Y.

If X is a set then by $\mathrm{Exp}(X)$ we denote the power set of X. The set of all natural numbers is denoted by ω. By a "neighborhood" of a point in a topological space we mean a "neighborhood in the sense of Bourbaki", i.e. its interior contains the given point.

For all notions and notations not defined here see [1, 7, 10].

2. Preliminaries

Definition 2.1 (see [10])**.** A *basic proximity* (or, simply, *proximity*) on a set X is a symmetric binary relation δ on $\mathrm{Exp}(X)$ which satisfies the following three conditions:
(P1) $\emptyset\overline{\delta}A$, for every $A \subseteq X$ (where $\overline{\delta}$ is the negation of δ);
(P2) $A\delta A$, for every $A \neq \emptyset$;
(P3) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$.
A basic proximity is called *separated* if it satisfies the condition
(P4) $x\delta y$ implies $x = y$.
When δ is a (separated) basic proximity on a set X then the pair (X, δ) is called a *(separated) proximity space*. We write $A \ll_\delta B$ (or simply $A \ll$ $\ll B$) if $A\overline{\delta}(X \setminus B)$.

If $Y \subseteq X$ then we will denote by δ_Y the restriction of δ to Y.

A function $f : (X_1, \delta_1) \to (X_2, \delta_2)$ between two basic proximity spaces (X_i, δ_i), $i = 1, 2$, is called *proximally continuous* if, for every $A, B \subseteq X_1$, $A\delta_1 B$ implies $f(A)\delta_2 f(B)$.

Let (X, δ) be a proximity space. Then the operator cl_δ on $\mathrm{Exp}(X)$, defined by $cl_\delta(A) = \{x \in X \mid x\delta A\}$, is a Čech closure operator. Hence $\tau_\delta = \{X \setminus A \mid A = cl_\delta(A)\}$ is a topology on X.

Definition 2.2. Let X be a set. A separated proximity δ on X satisfying the axiom

(R) If $x \in X$ and $x \ll A$ then there exists a $B \subseteq X$ such that $x \ll B \ll A$

is called an *R-proximity* and the pair (X, δ) – an *R-proximity space* ([8]); if it satisfies the axiom

(EF) If $A, B \subseteq X$ and $A \ll B$ then there exists a $C \subseteq X$ such that $A \ll C \ll B$

then it is called an *EF-proximity* (or *Efremovich proximity*) and the pair (X, δ) – an *EF-proximity space* ([6]); finally, if δ satisfies the axiom

(LO) For every $A, B \subseteq X$, $cl_\delta(A)\delta cl_\delta(B)$ implies $A\delta B$,

then it is called a *Lodato proximity* (see [10]).

Note that if δ is an R-proximity on a set X then cl_δ is a Kuratowski closure operator ([8]).

Definition 2.3 (see [10])**.** A non-empty family $\mathcal{G}$ of subsets of a set X is called a *grill* in X if it satisfies the following three conditions:

(G1) $\emptyset \notin \mathcal{G}$;

(G2) If $A_1 \cup A_2 \in \mathcal{G}$, then $A_1 \in \mathcal{G}$ or $A_2 \in \mathcal{G}$;

(G3) If $A \in \mathcal{G}$, $B \subseteq X$ and $A \subseteq B$ then $B \in \mathcal{G}$.

Lemma 2.4 (see [10])**.** *Let $\mathcal{G}$ be a grill in X. If $A_0 \in \mathcal{G}$ then there exists an ultrafilter $\mathcal{U}$ such that* (a) $A_0 \in \mathcal{U}$, *and* (b) $\mathcal{U} \subseteq \mathcal{G}$.

Definition 2.5. Let (X, δ) be a proximity space and $\emptyset \neq \sigma \subseteq \mathrm{Exp}(X)$. The family σ is called a *cluster* in (X, δ) if it satisfies the following three conditions:

(CL1) If $A, B \in \sigma$ then $A\delta B$;

(CL2) If $A \subseteq X$ and $A\delta B$ for every $B \in \sigma$ then $A \in \sigma$;

(CL3) If $A \cup B \in \sigma$ then either $A \in \sigma$ or $B \in \sigma$.

Obviously, every cluster in a proximity space (X, δ) is a grill in X.

Theorem 2.6 (see [10])**.** *A family σ of subsets of an EF-proximity space (X, δ) is a cluster iff there exists an ultrafilter $\mathcal{U}$ in X such that*

$$\sigma = \{A \subseteq X \mid A\delta B \textit{ for every } B \in \mathcal{U}\}. \tag{1}$$

If σ is a cluster in (X, δ) and $A_0 \in \sigma$ then there exists an ultrafilter $\mathcal{U}$ in X containing A_0 and satisfying equality (1).

Definition 2.7 (see [10])**.** We say that a family $\mathcal{A}$ of subsets of a proximity space (X, δ) is a *δ-system* if for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{A}$ such that $B \ll A$.

Definition 2.8 (see [10])**.** A *round filter* $\mathcal{F}$ in a proximity space (X,δ) is a filter which is a δ-system. A round filter $\mathcal{F}$ is called an *end* in (X,δ) (or δ-*end*) if it satisfies the following condition:
(E) $A \ll B$ implies that either $(X \setminus A) \in \mathcal{F}$ or $B \in \mathcal{F}$.
The set of all ends in (X,δ) will be denoted by $\Sigma_{\text{end}}(X,\delta)$ or simply by $\Sigma_{\text{end}}(\delta)$.

Proposition 2.9 (see [10])**.** *Every δ-system which has the finite intersection property is contained in a maximal round filter.*

Theorem 2.10 (see [10])**.** *Let δ be an EF-proximity on X. Then $\mathcal{F}$ is a maximal round filter in (X,δ) if and only if $\mathcal{F}$ is an end in (X,δ).*

Proposition 2.11 (see [10])**.** *Let δ be an EF-proximity on X and $A,B \subseteq \subseteq X$. Then $A \ll B$ if and only if every end in X contains either $X \setminus A$ or B.*

Definition 2.12 ([2])**.** Let (X,δ) be an R-proximity space and Σ be a set of round filters in (X,δ) such that:
(SR1) All neighborhood filters of the points of (X,τ_δ) are in Σ, and
(SR2) For $A,B \subseteq X$, $A\delta B$ is equivalent to the existence of an element $\mathcal{F}$ of Σ which does not contain the sets $X \setminus A$ and $X \setminus B$.
Then the pair $\alpha = (\delta,\Sigma)$ is called an *SR-proximity* on the set X and the pair (X,α) – an SR-*proximity space.* If (X,τ) is a topological space and $\alpha = (\delta,\Sigma)$ is an SR-proximity on the set X such that $\tau = \tau_\delta$, then we say that α is an SR-*proximity on the space* X.

A function $f : (X,\alpha_1) \to (Y,\alpha_2)$, where $\alpha_i = (\delta_i,\Sigma_i)$, $i = 1,2$, are SR-proximities, is called *SR-proximally continuous* if for every $\mathcal{F} \in \Sigma_1$ there exists a $\mathcal{G} \in \Sigma_2$ such that $\mathcal{G}$ is contained in the filter in Y generated by the filter-base $f(\mathcal{F})$.

Proposition 2.13 ([2])**.** *The condition $(SR2)$ from Def. 2.12 is equivalent to the following condition:*
(SR2') *For $A,B \subseteq X$, $A\delta B$ if and only if there exists an element $\mathcal{F} \in \Sigma$ such that for every $F \in \mathcal{F}$, $A \cap F \neq \emptyset$ and $B \cap F \neq \emptyset$ hold.*

Proposition 2.14 ([2])**.** *If $f : (X,\alpha_1) \to (Y,\alpha_2)$, where $\alpha_i = (\delta_i,\Sigma_i)$, $i = 1,2$ are SR-proximities, is an SR-proximally continuous function, then $f : (X,\delta_1) \to (Y,\delta_2)$ is a proximally continuous mapping.*

Definition 2.15 ([2])**.** An SR-proximity $\alpha = (\delta,\Sigma)$ on a set X is called an *LC-proximity* on the set X if for every $\mathcal{F} \in \Sigma$ there exists a $U \in \mathcal{F}$ with the following two properties:
(LC1) the restriction δ_U of δ to U is an EF-proximity;

(LC2) if $\mathcal{G} \in \Sigma_{\text{end}}(\delta)$ and $U \in \mathcal{G}$ then $\mathcal{G} \in \Sigma$.
The pair (X, α), where α is an LC-proximity on the set X, is called an *LC-proximity space*.

Definition 2.16 ([9])**.** A non-empty collection $\mathcal{B}$ of subsets of a set X is called a *boundedness* in X if it satisfies the following two conditions:
(B1) $A \in \mathcal{B}$ and $B \subseteq A$ implies $B \in \mathcal{B}$, and
(B2) $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$.

The elements of $\mathcal{B}$ are called *bounded sets.*

Definition 2.17 ([9])**.** A *(separated) local proximity space* is a triple $(X, \beta, \mathcal{B})$, where X is a set, β is a (separated) basic proximity on X, and $\mathcal{B}$ is a boundedness in X, subject to the following axioms:
(LP1) If $A \in \mathcal{B}$, $C \subseteq X$ and $A \ll C$ then there exists a $B \in \mathcal{B}$ such that $A \ll B \ll C$;
(LP2) If $A, B \subseteq X$ and $A\beta C$, then there is a $B \in \mathcal{B}$ such that $B \subseteq C$ and $A\beta B$.

A function $f : X_1 \to X_2$ between two local proximity spaces $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$ is said to be an *equicontinuous mapping* if the following two conditions are fulfilled for any $A, B \subseteq X$:
(EQ1) $A\beta_1 B$ implies $f(A)\beta_2 f(B)$;
(EQ2) $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$.

A filter (resp. cluster) $\mathcal{F}$ in a local proximity space $(X, \beta, \mathcal{B})$ is called *bounded* if $\mathcal{F} \cap \mathcal{B} \neq \emptyset$.

Proposition 2.18 ([9])**.** *Let $(X, \beta, \mathcal{B})$ be a local proximity space. Then:*
(a) *Every finite subset of $(X, \beta, \mathcal{B})$ is bounded;*
(b) *for every $B \in \mathcal{B}$ there exists a $D \in \mathcal{B}$ such that $B\overline{\beta}(X \setminus D)$;*
(c) *β is a Lodato proximity.*

Definition 2.19 ([5])**.** Let X be a set. Suppose that a set $\mathcal{M}$ is given such that $\mathcal{J}(X) \subseteq \mathcal{M} \subseteq \text{Exp}(X)$, where $\mathcal{J}(X)$ is the set of all one-point subsets of X. Then $\Sigma = (\mathcal{M}, \mathcal{V})$ is called a *supertopology* on X (and the pair (X, Σ) – a *supertopological space*), if to every $A \in \mathcal{M}$ there corresponds a filter $\mathcal{V}(A)$ in X such that the following two conditions are satisfied:
(ST1) $A \subseteq U$ for every $A \in \mathcal{M}$ and every $U \in \mathcal{V}(A)$;
(ST2) if $U \in \mathcal{V}(A)$, then there is such a $V \in \mathcal{V}(A)$ that $U \in \mathcal{V}(B)$ provided that $B \in \mathcal{M}$ and $B \subseteq V$.

Note that every supertopology $\Sigma = (\mathcal{M}, \mathcal{V})$ on a set X induces a topology on the set X whose neighborhood filters are precisely the filters

$\{\mathcal{V}(\{x\}) \mid x \in X\}$.

A supertopology $\Sigma = (\mathcal{M}, \mathcal{V})$ on X is said to be *symmetric* if it satisfies the following additional condition:

(STS) If $A, B \in \mathcal{M}$ and $U \cap B \neq \emptyset$ for every $U \in \mathcal{V}(A)$,
then $V \cap A \neq \emptyset$ for every $V \in \mathcal{V}(B)$.

Definition 2.20 ([3])**.** We call a symmetric supertopology $\Sigma = (\mathcal{M}, \mathcal{V})$ on a set X L-*supertopology* (and the pair (X, Σ) – an *L-supertopological space*) if the following conditions are fulfilled:

(LST1) if $A \in \mathcal{M}$ and $B \subseteq A$, then $B \in \mathcal{M}$;

(LST2) if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$;

(LST3) if $A \in \mathcal{M}$, then there is a $U \in \mathcal{V}(A)$ such that $U \in \mathcal{M}$.

Let $(\mathcal{M}_X, \mathcal{V}_X)$ be a supertopology on X and $(\mathcal{M}_Y, \mathcal{V}_Y)$ be a supertopology on Y. A mapping $f : X \to Y$ is called *supertopological* if it satisfies the following conditions:

(STC1) $f(\mathcal{M}_X) \subseteq \mathcal{M}_Y$;

(STC2) $f^{-1}(\mathcal{V}_Y(f(A))) \subseteq \mathcal{V}_X(A)$ for every $A \in \mathcal{M}_X$.

Definition 2.21. An L-supertopology $\Sigma = (\mathcal{M}, \mathcal{V})$ on a set X is called *separated* if for every two different points x and y of X there exists a $V \in \mathcal{V}(\{x\})$ such that $y \notin V$.

3. The results

For proving our main theorems (Th. 3.8 and Th. 3.9), we need some statements about local proximity spaces generalizing some well-known results of the theory of Efremovich proximity spaces.

Notation 3.1 (see [10])**.** Let $\mathcal{F}$ be a family of subsets of a set X. We put

$$\mathcal{F}^* = \{A \subseteq X \mid X \setminus A \notin \mathcal{F}\}.$$

Note that $\mathcal{F}^{**} = \mathcal{F}$, and if $\mathcal{F}$ is a filter then $\mathcal{F} \subseteq \mathcal{F}^*$.

The next proposition generalizes [10, Th. 6.11].

Proposition 3.2. *If $\mathcal{F}$ is a bounded cluster in a local proximity space $(X, \delta, \mathcal{B})$ then $\mathcal{F}^*$ is a bounded end in $(X, \delta, \mathcal{B})$.*

Proof. Following the proof of [10, Th. 6.11], we get that $\mathcal{F}^*$ is a filter satisfying condition (E) (see Def. 2.8).

Let us prove that $\mathcal{F}^*$ is a bounded filter. We have that there exists an $E \in \mathcal{F} \cap \mathcal{B}$. As it follows from Prop. 2.18, there exists an $E' \in \mathcal{B}$ such that $E \ll E'$. Hence either $X \setminus E \in \mathcal{F}^*$ or $E' \in \mathcal{F}^*$. Since $E \in \mathcal{F}$, we get that $X \setminus E \notin \mathcal{F}^*$. Thus $E' \in \mathcal{F}^*$ and therefore $\mathcal{F}^*$ is bounded.

We will now show that $\mathcal{F}^*$ is a δ-system. Let $B \in \mathcal{F}^*$ and $E \not\ll B$ for every $E \in \mathcal{F}$. Then $E\delta(X \setminus B)$ for every $E \in \mathcal{F}$. Hence $X \setminus B \in \mathcal{F}$. This contradiction shows that for every $B \in \mathcal{F}^*$ there exists an $E \subseteq X$ such that $E \ll B$ and $X \setminus E \notin \mathcal{F}^*$. Further, we know that there exists an $E' \in \mathcal{F}^* \cap \mathcal{B}$. Let $A \in \mathcal{F}^*$. We set $A' = A \cap E'$. Then $A' \in \mathcal{F}^* \cap \mathcal{B}$. Hence there exists a $C \subseteq X$ such that $C \ll A'$ and $X \setminus C \notin \mathcal{F}^*$. By (LP1) (see Def. 2.17), there exists a $D \in \mathcal{B}$ such that $C \ll D \ll A'$. Since $\mathcal{F}^*$ satisfies condition (E) and $X \setminus C \notin \mathcal{F}^*$, we get that $D \in \mathcal{F}^*$. From $D \ll A' \subseteq A$ it follows that $D \ll A$. Therefore $\mathcal{F}^*$ is a δ-system. Hence, $\mathcal{F}^*$ is an end. $\Diamond$

The following lemma is a generalization of [10, Lemma 6.8].

Lemma 3.3. *Let $\mathcal{F}$ be a bounded round filter in a local proximity space $(X, \delta, \mathcal{B})$, $A, B \subseteq X$ and $A \ll B$. If $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$, then $\mathcal{F}$ is a subset of some bounded round filter which contains B.*

Proof. Let $\mathcal{G} = \{A \cap F \mid F \in \mathcal{F}\}$ and $\mathcal{G}^0 = \{E \subseteq X \mid \exists C \in \mathcal{G}$ such that $C \ll E\}$. We will prove that $\mathcal{G}^0$ has the required properties. Following the proof of [10, Lemma 6.8], we get that $\mathcal{G}^0$ is a filter finer than $\mathcal{F}$ and $B \in \mathcal{G}^0$. Hence $\mathcal{G}^0$ is a bounded filter. So, we need only to show that $\mathcal{G}^0$ is a δ-system. Let $P \in \mathcal{G}^0$ and $F \cap A \ll P$ for some $F \in \mathcal{F}$. Since $\mathcal{F}$ is bounded, there exists a $C \in \mathcal{F} \cap \mathcal{B}$ such that $C \subseteq F$. Then $A \cap C \ll P$. By condition (LP1) (see Def. 2.17), there exists an $R \in \mathcal{B}$ such that $A \cap C \ll R \ll P$. Thus $R \in \mathcal{G}^0$ and $R \ll P$. Hence $\mathcal{G}^0$ is a δ-system. $\Diamond$

The next proposition generalizes Th. 2.10 (= [10, Th. 6.9]).

Proposition 3.4. *Let $\mathcal{F}$ be a bounded filter in a local proximity space $(X, \delta, \mathcal{B})$. Then $\mathcal{F}$ is a maximal round filter in the proximity space (X, δ) iff $\mathcal{F}$ is an end in (X, δ).*

Proof. Let $\mathcal{F}$ be a maximal round filter in the proximity space (X, δ). Let $A \ll B$ and $B \notin \mathcal{F}$. By Lemma 3.3, there exists an $E \in \mathcal{F}$ such that $E \cap A = \emptyset$. Then $E \subseteq (X \setminus A)$. Hence $X \setminus A \in \mathcal{F}$. Thus $\mathcal{F}$ is an end. The proof in the converse direction is the same as that of [10, Th. 6.7]. $\Diamond$

With the next proposition we generalize [10, Cor. 5.18].

Proposition 3.5. *If E is a bounded subset of a local proximity space $(X, \delta, \mathcal{B})$ then every cluster σ' in the proximity space (E, δ_E) is contained in a unique cluster σ in (X, δ), and*

$$\sigma = \{A \subseteq X : A\delta B, \text{ for every } B \in \sigma'\}.$$

Proof. For every $G \subseteq X$ such that $E \subseteq G$, set

$$\sigma_G = \{C \subseteq G \mid C\delta D \text{ for every } D \in \sigma'\};$$

then, obviously, $\sigma' \subseteq \sigma_G$ and thus $\sigma' \subseteq \sigma$. We will prove that σ is a cluster in (X, δ). Indeed, let $C_1, C_2 \in \sigma$. By Prop. 2.18, there exists an $E_1 \in \mathcal{B}$ such that $E \ll E_1$. Let $D \in \sigma'$. Then $D \subseteq E$ and thus $D \ll E_1$, i.e. $D\overline{\delta}(X \setminus E_1)$. Hence $D\overline{\delta}(C_i \setminus E_1)$, $i = 1, 2$. Since $C_1, C_2 \in \sigma$, and $D \in \sigma'$, we have that $C_i\delta D$, $i = 1, 2$. Then, by (P3), $D\delta(C_i \cap E_1)$, $i = 1, 2$. Set $C_i' = C_i \cap E_1$, $i = 1, 2$. Then $C_i' \in \sigma_{E_1}$, $i = 1, 2$. Since $E_1 \in \mathcal{B}$, δ_{E_1} is an EF-proximity. Thus, by [10, Cor. 5.18], σ_{E_1} is a cluster in (E_1, δ_{E_1}). Therefore $C_1'\delta C_2'$ and hence $C_1\delta C_2$. Let $F \subseteq X$ and $F\delta C$ for every $C \in \sigma$. Since $\sigma' \subseteq \sigma$ we get that $F\delta C$ for every $C \in \sigma'$ and thus $F \in \sigma$. Let $F_1 \cup F_2 \in \sigma$. Then, for every $C \in \sigma'$, $C\overline{\delta}((F_1 \cup F_2) \setminus E_1)$. Since, for every $C \in \sigma'$, $C\delta(F_1 \cup F_2)$, we get that $C\delta((F_1 \cup F_2) \cap E_1)$ for every $C \in \sigma'$. Hence $(F_1 \cup F_2) \cap E_1 \in \sigma_{E_1}$. Then, by [10, Cor. 5.18], $F_1 \cap E_1 \in \sigma_{E_1}$ or $F_2 \cap E_1 \in \sigma_{E_1}$. Therefore $F_1 \in \sigma$ or $F_2 \in \sigma$. So, σ is a cluster. Finally, if σ_1 is a cluster in (X, δ) containing σ' then, obviously, $\sigma_1 \subseteq \sigma$ and thus $\sigma_1 = \sigma$. $\Diamond$

With the next proposition we generalize [10, Th. 5.14].

Proposition 3.6. *Let $(X, \delta, \mathcal{B})$ be a local proximity space, $E \in \mathcal{B}$, $A, B \subseteq$ $\subseteq E$ and $A\delta B$. Then there exists a cluster σ in (X, δ) such that $A, B \in \sigma$.*

Proof. We have that the restriction δ_E of δ to E is an EF-proximity. Then, by [10, Th. 5.14], there exists a cluster σ_E of (E, δ_E) such that $A, B \in \sigma_E$. From Prop. 3.5 it follows that there exists a cluster σ of X such that $\sigma_E \subseteq \sigma$. Then $A, B \in \sigma$. $\Diamond$

Notation 3.7. Let $\mathcal{C}_1$ be the category of LC-proximity spaces and SR-proximally continuous mappings between them, $\mathcal{C}_2$ be the category of separated local proximity spaces and equicontinuous mappings between them, and $\mathcal{C}_3$ be the category of separated L-supertopological spaces and supertopological mappings between them.

Theorem 3.8. *The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic.*

Proof. We will construct two covariant functors

$$F : \mathcal{C}_1 \to \mathcal{C}_2 \text{ and } G : \mathcal{C}_2 \to \mathcal{C}_1$$

such that $G \circ F = id_{\mathcal{C}_1}$ and $F \circ G = id_{\mathcal{C}_2}$.

Step 1. Construction of the functor $F:\mathcal{C}_1\to\mathcal{C}_2$ on the objects of the category $\mathcal{C}_1$.

Let $(X,\alpha)\in|\mathcal{C}_1|$ and $\alpha=(\delta,\Sigma)$. Set

$$\begin{aligned}\mathcal{B}'=\big\{B\subseteq X\mid \exists\,\mathcal{F}\in\Sigma \text{ and } \exists\, U,V,W\in\mathcal{F} \text{ such that}\\ B\subseteq W\ll V\ll U \text{ and } U \text{ satisfies conditions}\\ \text{(LC1), (LC2) of Def. 2.15}\big\}.\end{aligned}\tag{2}$$

Now let

$$F(X,\alpha)=(X,\delta,\mathcal{B}),$$

where $\mathcal{B}$ is the family of all finite unions of the elements of $\mathcal{B}'$. We will prove that $(X,\delta,\mathcal{B})$ is a local proximity space. It is obvious that $\mathcal{B}$ is a boundedness. Further, by Def. 2.12, δ is a separated proximity. We shall show that condition (LP1) of Def. 2.17 is fulfilled. Let $B\in\mathcal{B}$ and $B\ll D$. We can assume w.l.o.g. that $B\in\mathcal{B}'$. Then there exist $\mathcal{F}\in\Sigma$ and $U,V,W\in\mathcal{F}$ such that $B\subseteq W\ll V\ll U$ and U satisfies conditions (LC1), (LC2). It follows that $B\ll V$. Thus $B\ll(V\cap D)$, i.e. $B\overline{\delta}(X\setminus\setminus(V\cap D))$. Now the equality $X\setminus(V\cap D)=(X\setminus U)\cup(U\setminus(V\cap D))$ implies that $B\overline{\delta}(U\setminus(V\cap D))$, i.e. $B\ll_{\delta_U}(V\cap D)$. Since δ_U is an EF-proximity, we get that there exists a $C\subseteq U$ such that $B\ll_{\delta_U}C\ll_{\delta_U}(V\cap D)$. So, $B\overline{\delta}(X\setminus U)$ and $B\overline{\delta}(U\setminus C)$. Then $B\overline{\delta}((X\setminus U)\cup(U\setminus C))$, i.e. $B\overline{\delta}(X\setminus C)$. Thus $B\ll C$. Analogously, $C\ll(V\cap D)\subseteq D$. Hence $B\ll C\ll D$. It remains to show that $C\in\mathcal{B}'$. By (LC1), there exists a $W_1\subseteq U$ such that $C\ll_{\delta_U}W_1\ll_{\delta_U}V$. This implies that $C\ll W_1\ll V$ because $C\ll V\ll\ll U$. Set $W'=W_1\cup W$. Then $W'\in\mathcal{F}$ and $C\subseteq W'\ll V\ll U$. Thus $C\in\mathcal{B}'\subseteq\mathcal{B}$. So, condition (LP1) from Def. 2.17 is fulfilled. For checking condition (LP2) from the same definition, let $A\delta B$. By Prop. 2.13, there exists an $\mathcal{F}\in\Sigma$ such that for every $F\in\mathcal{F}$, $A\cap F\neq\emptyset$ and $B\cap F\neq\emptyset$. Since α is an LC-proximity and $\mathcal{F}$ is a round filter, we get that there exists a $C\in\mathcal{B}\cap\mathcal{F}$. Then $C\cap F\in\mathcal{F}$ for every $F\in\mathcal{F}$ and thus $B\cap C\cap\cap F\neq\emptyset$ for every $F\in\mathcal{F}$. Hence $B\cap C\in\mathcal{B}$ and, by (SR2'), $A\delta(B\cap C)$. Therefore, $(X,\delta,\mathcal{B})$ is a local proximity space.

Step 2. Construction of the functor $G:\mathcal{C}_2\to\mathcal{C}_1$.

Let $(X,\delta,\mathcal{B})\in|\mathcal{C}_2|$. Set

$$\begin{aligned}G(X,\delta,\mathcal{B})=(X,\alpha),\ \text{where } \alpha=(\delta,\Sigma) \text{ and}\\ \Sigma=\big\{\mathcal{F}\in\Sigma_{\text{end}}\mid\ \mathcal{B}\cap\mathcal{F}\neq\emptyset\big\}.\end{aligned}\tag{3}$$

We will prove that α is an LC-proximity. Let $x\in X$ and $x\ll A$. By Prop. 2.18(a), $\{x\}\in\mathcal{B}$. Then (LP1) implies that there exists a $B\in\in\mathcal{B}$ such that $\{x\}\ll B\ll A$. Thus δ is an R-proximity. We will prove that α is an SR-proximity. Using Prop. 2.18(c), one can easily show that condition (SR1) (see Def. 2.12) is fulfilled. We will prove that condition (SR2') (see Prop. 2.13) is satisfied; this will imply, according to Prop. 2.13, that condition (SR2) from Def. 2.12 is fulfilled. Let $A, B \subseteq X$ and there exists an $\mathcal{F} \in \Sigma$ such that $F \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$ for every $F \in \mathcal{F}$. Suppose that $A\overline{\delta}B$, i.e. $A \ll (X \setminus B)$. Since the filter $\mathcal{F}$ is an end, we get that either $X \setminus A \in \mathcal{F}$ or $X \setminus B \in \mathcal{F}$, a contradiction. So, $A\delta B$. Conversely, let $A\delta B$. We will prove that there exists an $\mathcal{F} \in \Sigma$ such that for every $F \in \mathcal{F}$, $F \cap A \neq \emptyset$ and $F \cap B \neq \emptyset$. By condition (LP2) (see Def. 2.17), there exist $C, D \in \mathcal{B}$ such that $(A \cap C)\delta(B \cap D)$. Set $E = C \cup D$, $A' = A \cap E$ and $B' = B \cap E$. Then $E \in \mathcal{B}$ and thus, by Prop. 2.18(b), there exists some $E_1 \in \mathcal{B}$ such that $E \ll E_1$. Now, condition (LP1) (see Def. 2.17) implies that there exists a $P \in \mathcal{B}$ such that $E \ll P \ll E_1$. Since the restriction δ_{E_1} of δ to E_1 is an Efremovich proximity, Prop. 2.11 yields that there exists an end $\mathcal{F}_{E_1}$ in (E_1, δ_{E_1}) such that $A' \cap F \neq \emptyset$ and $B' \cap F \neq \emptyset$ for every $F \in \mathcal{F}_{E_1}$. Obviously, $A' \ll_\delta P$ implies that $A' \ll_{\delta_{E_1}} P$ and hence either $E_1 \setminus A' \in \mathcal{F}_{E_1}$ or $P \in \mathcal{F}_{E_1}$. Since $A' \cap (E_1 \setminus A') = \emptyset$, we get that $P \in \mathcal{F}_{E_1}$. Set $\mathcal{F}_P = \{P \cap F \mid F \in \in \mathcal{F}_{E_1}\}$. Obviously, $\mathcal{F}_P$ is a filter-base of $\mathcal{F}_{E_1}$. We will prove that $\mathcal{F}_P$ is a filter-base of a round filter in (X, δ). Let $F \in \mathcal{F}_P \subseteq \mathcal{F}_{E_1}$. There exists a $G \in \mathcal{F}_{E_1}$ such that $G \ll_{\delta_{E_1}} F$, i.e. $G\overline{\delta}(E_1 \setminus F)$. We have that $G \subseteq F \subseteq P \ll_\delta E_1$. Therefore $G\overline{\delta}(X \setminus E_1)$. Thus $G\overline{\delta}((X \setminus E_1) \cup (E_1 \setminus F))$, i.e. $G \ll F$. Hence $\mathcal{F}_P$ is a filter-base of a round filter $\mathcal{F}$ in (X, δ) and $\mathcal{F}_{E_1} \subseteq \mathcal{F}$. We will prove that $\mathcal{F}$ is an end. Let $A_1, B_1 \subseteq X$ and $A_1 \ll B_1$, i.e. $A_1\overline{\delta}(X \setminus B_1)$. Then $(A_1 \cap E_1)\overline{\delta}((X \setminus B_1) \cap E_1)$. The next three cases are possible:

1. $A_1 \cap E_1 = \emptyset$. Then $E_1 \subseteq (X \setminus A_1)$. Hence $(X \setminus A_1) \in \mathcal{F}$.
2. $(X \setminus B_1) \cap E_1 = \emptyset$. Then $E_1 \subseteq B_1$ so $B_1 \in \mathcal{F}$.
3. $A_1 \cap E_1 \neq \emptyset$ and $(X \setminus B_1) \cap E_1 \neq \emptyset$. Obviously, $(A_1 \cap E_1)\overline{\delta}(E_1 \setminus B_1)$ is equivalent to $(A_1 \cap E_1) \ll_{\delta_{E_1}} (B_1 \cap E_1)$. Then $E_1 \setminus A_1 \in \mathcal{F}_{E_1}$ or $(B_1 \cap \cap E_1) \in \mathcal{F}_{E_1}$. Hence $(X \setminus A_1) \in \mathcal{F}$ or $B_1 \in \mathcal{F}$.

In all cases we have that $X \setminus A_1 \in \mathcal{F}$ or $B \in \mathcal{F}$. Thus $\mathcal{F}$ is an end in (X, δ). From $E_1 \in \mathcal{F} \cap \mathcal{B}_1$ it follows that $\mathcal{F} \in \Sigma$. Since $\mathcal{F}_P$ is a filter-base for $\mathcal{F}$ and $\mathcal{F}_P \subseteq \mathcal{F}_{E_1}$, we get that, for every $F \in \mathcal{F}$, $A' \cap F \neq \emptyset$ and $B' \cap F \neq \emptyset$, and hence $A \cap F \neq \emptyset$ and $B \cap F \neq \emptyset$. So, condition (SR2') is fulfilled. Therefore, (X, α) is an SR-proximity space.

We will now prove that α is an LC-proximity. Let $\mathcal{F} \in \Sigma$. Then there exists a $U \in \mathcal{F} \cap \mathcal{B}$. Obviously, the restriction δ_U of δ to U is an

Efremovich proximity. From this and from the definition of Σ it follows that $\alpha = (\delta, \Sigma)$ is an LC-proximity.

Let

$(X_1, \beta_1, \mathcal{B}_1), (X_2, \beta_2, \mathcal{B}_2) \in |\mathcal{C}_2|$, and $f \in \mathcal{C}_2((X_1, \beta_1, \mathcal{B}_1), (X_2, \beta_2, \mathcal{B}_2))$.
We will prove that $f \in \mathcal{C}_1(G(X_1, \beta_1, \mathcal{B}_1), G(X_2, \beta_2, \mathcal{B}_2))$. For simplicity, we shall write "$\ll_i$" instead of "$\ll_{\beta_i}$", $i = 1, 2$.

Let $\mathcal{F}_1$ be a bounded end in $(X_1, \beta_1, \mathcal{B}_1)$. Set

$$\mathcal{F}_2 = \big\{A \subseteq X_2 \mid \exists\, V \in \mathcal{B}_2 \text{ such that } V \ll_2 A \text{ and } \forall\, F \in \mathcal{F}_1,\ V \cap f(F) \neq \emptyset\big\}. \tag{4}$$

We will prove that $\mathcal{F}_2 \in \Sigma_2$ and $\mathcal{F}_2$ is contained in the filter with the filter-base $f(\mathcal{F}_1)$. Indeed, there exists a $C \in \mathcal{F}_1 \cap \mathcal{B}_1$; then $f(C) \in \mathcal{B}_2$ and $f(C) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. Since $f(C) \ll_2 X_2$, it follows that $X_2 \in \mathcal{F}_2$. So, $\mathcal{F}_2 \neq \emptyset$. It is obvious that $\emptyset \notin \mathcal{F}_2$ and that $\mathcal{F}_2$ is closed under supersets. Let $A_1, A_2 \in \mathcal{F}_2$. Then, by (4), there exist $V_1, V_2 \in \in \mathcal{B}_2$ such that $V_i \ll_2 A_i$ and $V_i \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$, $i = 1, 2$. Now, condition (LP1) (see Def. 2.17) implies that there exist $W_1, W_2 \in \in \mathcal{B}_2$ such that $V_i \ll_2 W_i \ll_2 A_i$, $i = 1, 2$. Then $W_1 \cap W_2 \ll_2 A_1 \cap A_2$ and $W_i \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$, $i = 1, 2$. Suppose that there exists an $F_0 \in \mathcal{F}_1$ such that $(W_1 \cap W_2) \cap f(F_0) = \emptyset$. Then $f(F_0) \subseteq X_2 \setminus (W_1 \cap W_2)$. Hence $(X_2 \setminus (W_1 \cap W_2)) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$, i.e. $((X_2 \setminus W_1) \cup \cup (X_2 \setminus W_2)) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. From this we get that either $(X_2 \setminus W_1) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$, or $(X_2 \setminus W_2) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. Indeed, suppose that there exist $F_1, F_2 \in \mathcal{F}_1$ such that $(X_2 \setminus W_1) \cap f(F_1) = \emptyset$ and $(X_2 \setminus W_2) \cap f(F_2) = \emptyset$; then $F = F_1 \cap F_2 \in \mathcal{F}_1$ and $((X_2 \setminus W_1) \cup (X_2 \setminus W_2)) \cap f(F) = \emptyset$, a contradiction. We can assume w.l.o.g. that $(X_2 \setminus W_1) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$, i.e. $f(F) \setminus W_1 \neq \emptyset$ for every $F \in \mathcal{F}_1$. We have that $V_1 \ll_2 W_1$. Hence, by (EQ1) (see Def. 2.17), $f^{-1}(V_1) \ll_1 f^{-1}(W_1)$. Since $\mathcal{F}_1$ is an end, it follows that either $X_1 \setminus \setminus f^{-1}(V_1) \in \mathcal{F}_1$ or $f^{-1}(W_1) \in \mathcal{F}_1$. If $F_1 = f^{-1}(W_1) \in \mathcal{F}_1$ then $f(F_1) \subseteq W_1$ and $f(F_1) \setminus W_1 = \emptyset$, so $f^{-1}(W_1) \notin \notin \mathcal{F}_1$. Hence $F_2 = X_1 \setminus f^{-1}(V_1) \in \in \mathcal{F}_1$. Then $f(F_2) = f(X_1) \setminus V_1 \subseteq X_2 \setminus V_1$ and thus $f(F_2) \cap V_1 = \emptyset$, a contradiction. Therefore, $W_1 \cap W_2 \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. Hence $A_1 \cap A_2 \in \mathcal{F}_2$. Thus $\mathcal{F}_2$ is a filter.

We will now prove that $\mathcal{F}_2$ is a bounded round filter. Let $U \in \mathcal{F}_2$. Then there exists a $V \in \mathcal{B}_2$ such that $V \ll_2 U$ and $V \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. By condition (LP1) from Def. 2.17, there exists a $W \in \mathcal{B}_2$ such that $V \ll_2 W \ll_2 U$. Then $W \in \mathcal{F}_2 \cap \mathcal{B}_2$ and $W \ll_2 U$. Thus $\mathcal{F}_2$ is a bounded round filter.

We will prove that $\mathcal{F}_2$ is contained in the filter with the filter-base $f(\mathcal{F}_1)$. Let $U \in \mathcal{F}_2$. Then there exists a $V \in \mathcal{B}_2$ such that $V \ll_2 U$ and $V \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. Hence $f^{-1}(V) \ll_1 f^{-1}(U)$. This implies that either $X_1 \setminus f^{-1}(V) \in \mathcal{F}_1$ or $f^{-1}(U) \in \mathcal{F}_1$. Supposing that that $F_1 = X_1 \setminus f^{-1}(V) \in \mathcal{F}_1$, we obtain that $f(F_1) \cap V = \emptyset$. This contradiction shows that $f^{-1}(U) \in \mathcal{F}_1$. Then $f(f^{-1}(U)) \in f(\mathcal{F}_1)$ and $f(f^{-1}(U)) \subseteq U$. Thus $\mathcal{F}_2$ is contained in the filter, generated by the filter-base $f(\mathcal{F}_1)$.

Now we will prove that $\mathcal{F}_2$ is an end. Let $A, B \subseteq X_2$ and $A \ll_2 B$. There are two possible cases:

1. $A \in \mathcal{B}_2$. Then there exists a $C \in \mathcal{B}_2$ such that $A \ll_2 C \ll_2 B$. Suppose that $B \notin \mathcal{F}_2$. Then there exists an $F_0 \in \mathcal{F}_1$ such that $C \cap f(F_0) = \emptyset$. Hence $f(F_0) \subseteq X_2 \setminus C$. We can assume w.l.o.g. that $f(F_0) \in \mathcal{B}_2$. Since $X_2 \setminus C \ll_2 X_2 \setminus A$, we get that $f(F_0) \ll_2 X_2 \setminus A$. Obviously, $f(F_0) \cap f(F) \neq \emptyset$ for every $F \in \mathcal{F}_1$. Hence $X_2 \setminus A \in \mathcal{F}_2$. So, we proved that either $B \in \mathcal{F}_2$ or $X_2 \setminus A \in \mathcal{F}_2$.

2. $A \notin \mathcal{B}_2$. There exists a $C \in \mathcal{F}_2 \cap \mathcal{B}_2$. Obviously, $C \cap A \ll_2 B$ and $C \cap A \in \mathcal{B}_2$. Then, by the previous case, we get that either $B \in \mathcal{F}_2$ or $(X_2 \setminus (C \cap A)) \in \mathcal{F}_2$. Let $(X_2 \setminus (C \cap A)) \in \mathcal{F}_2$. Then $(X_2 \setminus (C \cap A)) \cap C = C \setminus A \in \mathcal{F}_2$. Since $C \setminus A \subseteq X_2 \setminus A$, it follows that $X_2 \setminus A \in \mathcal{F}_2$. So, we proved that either $B \in \mathcal{F}_2$ or $X_2 \setminus A \in \mathcal{F}_2$.

Hence $\mathcal{F}_2$ is a bounded end in $(X_2, \beta_2, \mathcal{B}_2)$ contained in the filter generated by the filter-base $f(\mathcal{F}_1)$. So f is an SR-proximally continuous mapping. Set $G(f) = f$.

Step 3. Proof of the equality $G \circ F = id_{\mathcal{C}_1}$ on the objects of the category $\mathcal{C}_1$.

Let $(X, \alpha) \in |\mathcal{C}_1|$, where $\alpha = (\delta, \Sigma)$. Then $G(F(X, \alpha)) = G(X, \delta, \mathcal{B}) = (X, \alpha_1)$, where $\alpha_1 = (\delta, \Sigma_1)$.

We will prove that $\Sigma = \Sigma_1$. Let $\mathcal{F} \in \Sigma$. Since α is an LC-proximity, there exists a $U \in \mathcal{F}$ such that the restriction δ_U of δ to U is an Efremovich proximity and, moreover, if $\mathcal{G} \in \Sigma_{\text{end}}(\delta)$ and $U \in \mathcal{G}$ then $\mathcal{G} \in \Sigma$. Since $\mathcal{F}$ is a round filter, there exist $V, W \in \mathcal{F}$ such that $W \ll V \ll U$. Then, obviously, $W \in \mathcal{B} \cap \mathcal{F}$. Hence $\mathcal{F} \in \Sigma_1$. So, $\Sigma \subseteq \Sigma_1$. Conversely, let $\mathcal{G} \in \Sigma_1$. Then $\mathcal{G}$ is an end and there exists a $B \in \mathcal{G} \cap \mathcal{B}$. We have that $B = \bigcup\{B_i \in \mathcal{B}' \mid i = 1, \ldots, n\}$ for some $n \in \omega, n \geq 1$ (see (2) for $\mathcal{B}'$). For every $i = 1, \ldots, n$, there exist $\mathcal{F}_i \in \Sigma$ and $W_i, V_i, U_i \in \mathcal{F}_i$ such that $B_i \subseteq W_i \ll V_i \ll U_i$ and U_i satisfies conditions (LC1) and (LC2) from Def. 2.15. If there exists an $i_0 \in \{1, \ldots, n\}$ such that $B_{i_0} \in \mathcal{G}$, then

$U_{i_0} \in \mathcal{G}$ and hence $\mathcal{G} \in \Sigma$. Let now $B_i \notin \mathcal{G}$ for every $i = 1, \ldots, n$. Then there exists an $j_0 \in \{1, \ldots, n\}$ such that $X \setminus B_{j_0} \notin \mathcal{G}$. (Indeed, suppose that $X \setminus B_i \in \mathcal{G}$ for every $i \in \{1, \ldots, n\}$. Then $\bigcap_{i=1}^{n}(X \setminus B_i) = X \setminus \big(\bigcup_{i=1}^{n} B_i\big) = X \setminus B \in \mathcal{G}$. Since $B \in \mathcal{G}$, we get a contradiction.) The filter $\mathcal{G}$ is an end and $B_{j_0} \ll U_{j_0}$. Hence either $X \setminus B_{j_0} \in \mathcal{G}$ or $U_{j_0} \in \mathcal{G}$. Thus $U_{j_0} \in \mathcal{G}$ and therefore $\mathcal{G} \in \Sigma$. So, $\Sigma_1 \subseteq \Sigma$. We have proved that $\Sigma = \Sigma_1$.

Step 4. Proof of the equality $F \circ G = id_{\mathcal{C}_2}$ on the objects of the category $\mathcal{C}_2$.

Let $(X, \delta, \mathcal{B}) \in |\mathcal{C}_2|$. Then $G(X, \delta, \mathcal{B}) = (X, \alpha)$, where $\alpha = (\delta, \Sigma)$, and $F(X, \alpha) = (X, \delta, \mathcal{B}_1)$. We have to prove that $\mathcal{B} = \mathcal{B}_1$. Let $B \in \mathcal{B}$ and $B \neq \emptyset$. By Prop. 2.18(b), there exists a $B_1 \in \mathcal{B}$ such that $B \ll B_1$. Using (LP1) (see Def. 2.17), we construct by induction a family $\{A_n \mid n \in \omega, n \geq 1\}$ of subsets of X such that $B \ll \ldots \ll A_n \ll A_{n-1} \ll \ldots \ll A_1 \ll B_1$. Obviously, $\{A_n\}$ is a δ-system and has the finite intersection property. Then Prop. 2.9 implies that there exists a maximal round filter $\mathcal{F}$ in (X, δ) such that $\{A_n \mid n \in \omega, n \geq 1\} \subseteq \mathcal{F}$. Thus $B_1 \in \mathcal{F}$, i.e. $\mathcal{F}$ is a bounded maximal round filter in (X, δ). Hence, by Prop. 3.4, $\mathcal{F}$ is a bounded end in (X, δ). Therefore $\mathcal{F} \in \Sigma$. Since $A_1 \subseteq B_1$, we get that $A_1 \in \mathcal{B}$ and hence the restriction δ_{A_1} of δ to A_1 is an Efremovich proximity. Thus A_1 satisfies conditions (LC1) and (LC2) from Def. 2.15. Finally, from $B \subseteq A_3 \ll A_2 \ll A_1$ we get that $B \in \mathcal{B}_1$. Hence $\mathcal{B} \subseteq \mathcal{B}_1$. Let now $C \in \mathcal{B}_1$. We can assume w.l.o.g. that C has the following property: there exist an $\mathcal{F} \in \Sigma$ and $V_1, U_1 \in \mathcal{F}$ such that $C \ll V_1 \ll U_1$, $C \in \mathcal{F}$ and U_1 satisfies conditions (LC1), (LC2) from Def. 2.15. By (LP1), there exist $V, U \in \mathcal{B}_1$ such that $C \ll V \ll U \ll V_1$. Since $C \in \mathcal{F}$, we get that $V, U \in \mathcal{F}$. From $U \subseteq U_1$ it follows that U satisfies conditions (LC1), (LC2) from Def. 2.15. Suppose that $C \notin \mathcal{B}$. Then $C \in \mathcal{B}_1 \setminus \mathcal{B}$. Set $\mathcal{G} = \{B \subseteq U \mid B \in \mathcal{B}_1 \setminus \mathcal{B}\}$ and $\mathcal{G}' = \{A \subseteq U \mid$ there exists a $B \in \mathcal{G}$ such that $B \subseteq A\}$. We will prove that $\mathcal{G}'$ is a grill in U. It is obvious that $C \in \mathcal{G}'$, $\mathcal{G}' \cap \mathcal{B} = \emptyset$ and $\emptyset \notin \mathcal{G}'$. Let $A = A_1 \cup A_2 \in \mathcal{G}'$. Then there exists a $B \in \mathcal{G}$ such that $B \subseteq A_1 \cup A_2$. Put $B_i = A_i \cap B$, $i = 1, 2$. Then $B_i \in \mathcal{B}_1$, $i = 1, 2$. Suppose that $B_i \notin \mathcal{G}$ for $i = 1, 2$. Then $B_1, B_2 \in \mathcal{B}$, so that $B_1 \cup B_2 = B \in \mathcal{B}$. This contradiction shows that either $B_1 \in \mathcal{G}$ or $B_2 \in \mathcal{G}$. Then either $A_1 \in \mathcal{G}'$ or $A_2 \in \mathcal{G}'$. Hence $\mathcal{G}'$ is a grill in U. Then from Lemma 2.4 it follows that there exists an ultrafilter $\mathcal{L}$ in U such that $C \in \mathcal{L}$ and $\mathcal{L} \subseteq \mathcal{G}'$. Set $\sigma = \{A \subseteq U \mid A\delta B$ for every $B \in \mathcal{L}\}$. Then, by Th. 2.6, σ is a cluster in (U,δ_U), $\mathcal{L}\subseteq\sigma$ and $C\in\sigma$. Set $\sigma'=\{A\subseteq X\mid A\delta B \text{ for every } B\in\sigma\}$. Then Prop. 3.5 implies that σ' is a cluster in (X,δ). Since $C\in\sigma'\cap\mathcal{B}_1$, we get that σ' is a bounded cluster in $(X,\delta,\mathcal{B}_1)$. Then, by Prop. 3.2, $\mathcal{F}'=(\sigma')^*=\{E\subseteq X\mid X\setminus E\notin\sigma'\}$ is a bounded end in $(X,\delta,\mathcal{B}_1)$. Let $(X,\alpha_1)=G(X,\delta,\mathcal{B}_1)$. Then $(X,\alpha_1)=G(F(X,\alpha))$. Hence, by Step 3, $\alpha=\alpha_1$. So, $\alpha_1=(\delta,\Sigma)$. Therefore, an end in (X,δ) is bounded with respect to $\mathcal{B}_1$ if and only if it is bounded with respect to $\mathcal{B}$. Thus $\mathcal{F}'$ is a bounded end in $(X,\delta,\mathcal{B})$. Since $\mathcal{F}'\subseteq(\mathcal{F}')^*=\sigma'$, we get that σ' is a bounded cluster in $(X,\delta,\mathcal{B})$, i.e. there exists a $B_0\in\sigma'\cap\mathcal{B}$. Let's show that $\mathcal{L}\cap\mathcal{G}$ is a filter-base of $\mathcal{L}$. Indeed, if $L\in\mathcal{L}$, then $L\cap C\in\mathcal{B}_1\cap\mathcal{L}$ and $L\cap C\notin\mathcal{B}$. Hence $L\cap C\in\mathcal{G}$. From $L\cap C\subseteq L$ it follows that $\mathcal{L}\cap\mathcal{G}$ is a filter-base of $\mathcal{L}$. We have that $B_1\setminus B\neq\emptyset$ for every $B_1\in\mathcal{G}$ and for every $B\subseteq U$ such that $B\in\mathcal{B}$, i.e. $B_1\cap(X\setminus B)\neq\emptyset$ for every $B\in\mathcal{B}_U$, where $\mathcal{B}_U=\{B\in\mathcal{B}\mid B\subseteq U\}$. Then $(U\setminus B)\cap L\neq\emptyset$ for every $B\in\mathcal{B}_U$ and for every $L\in\mathcal{L}\cap\mathcal{G}$. Since $\mathcal{L}\cap\mathcal{G}$ is a filter-base of $\mathcal{L}$, we get that $U\setminus B\in\mathcal{L}$ for every $B\in\mathcal{B}_U$. Now, from $C\in\mathcal{L}$ it follows that $(U\setminus B)\cap C\in\mathcal{L}$ for every $B\in\mathcal{B}_U$. Since $B_0\in\sigma'\cap\mathcal{B}$ and $\mathcal{L}\subseteq\sigma\subseteq\sigma'$, we get that $B_0\delta((U\setminus B)\cap C)$, for every $B\in\mathcal{B}_U$. We have that $C\overline{\delta}(X\setminus V)$. Thus $(B_0\cap V)\delta((U\setminus B)\cap C)$, for every $B\in\mathcal{B}_U$. Then $(B_0\cap V)\delta(U\setminus B)$, for every $B\in\mathcal{B}_U$, i.e.

$$B_0\cap V \not\ll_{\delta_U} B \text{ for every } B\in\mathcal{B}_U. \tag{5}$$

From $(B_0\cap V)\subseteq V\ll U$ and $B_0\cap V\in\mathcal{B}$ it follows that there exists an $M\in\mathcal{B}_U$ such that $(B_0\cap V)\ll M\ll U$, and then $B_0\cap V\ll_{\delta_U} M$. This contradicts (5). Hence $C\in\mathcal{B}$. Thus $\mathcal{B}_1\subseteq\mathcal{B}$. Therefore $\mathcal{B}=\mathcal{B}_1$.

Step 5. Definition of the functor F on the morphisms of the category $\mathcal{C}_1$.

Let $(X_i,\alpha_i)\in|\mathcal{C}_1|$, $\alpha_i=(\delta_i,\Sigma_i)$, $i=1,2$, and $f\in\mathcal{C}_1((X_1,\alpha_1),(X_2,\alpha_2))$.

We will prove that $f\in\mathcal{C}_2(F(X_1,\alpha_1),F(X_2,\alpha_2))$. Let $F(X_i,\alpha_i)= =(X_i,\delta_i,\mathcal{B}_i)$, $i=1,2$. From Prop. 2.14 we have that $f:(X_1,\delta_1)\to \to(X_2,\delta_2)$ is a proximally continuous mapping.

It remains to show that if $B\in\mathcal{B}_1$ then $f(B)\in\mathcal{B}_2$. Let $W\in\mathcal{B}_1$. Suppose that $f(W)\notin\mathcal{B}_2$.

As it follows from the proof of Step 3, for $i=1,2$, Σ_i coincides with the set of all bounded ends in the local proximity space $(X_i,\delta_i,\mathcal{B}_i)$.

From Prop. 2.18(b) it follows that there exists a $V\in\mathcal{B}_1$ such that $W\ll_1 V$. Set

$$\begin{aligned}\Omega=\big\{E\in\mathcal{B}_2\mid(\exists\mathcal{F}\in\Sigma_1)(\exists B\in\mathcal{B}_2)((B\ll_2 E)\wedge\\ \wedge((\forall\, H\in\mathcal{F})((H\cap V\neq\emptyset)\wedge(B\cap f(H)\neq\emptyset))))\big\}.\end{aligned} \tag{6}$$

Note that:

1) if $E \in \Omega$ and $E' \in \mathcal{B}_2$ then $E \cup E' \in \Omega$; and

2) if $E \in \Omega$ then there exists an $E' \in \Omega$ such that $E \ll_2 E'$.

From $f(W) \notin \mathcal{B}_2$ it follows that $f(W)$ is not a subset of any element of Ω. Hence, for every $E \in \Omega$, $f(W) \setminus E \neq \emptyset$. Thus

$$W \cap (X_1 \setminus f^{-1}(E)) \neq \emptyset, \text{ for every } E \in \Omega. \tag{7}$$

Set $\mathcal{D}' = \{X_1 \setminus f^{-1}(E) \mid E \in \Omega\}$. Then

1. $D \neq \emptyset$, for every $D \in \mathcal{D}'$;
2. if $D_1, D_2 \in \mathcal{D}'$ then $D_1 \cap D_2 \in \mathcal{D}'$;
3. $\mathcal{D}'$ is a δ_1-system (indeed, let $D \in \mathcal{D}'$; then there exists an $E \in \Omega$ such that $D = X_1 \setminus f^{-1}(E)$; there exists an $E' \in \Omega$ such that $E \ll_2 E'$; then $f^{-1}(E) \ll_1 f^{-1}(E')$; hence $X_1 \setminus f^{-1}(E') \ll_1 X_1 \setminus f^{-1}(E)$ and $D' = X_1 \setminus f^{-1}(E') \in \mathcal{D}'$).

From $W \in \mathcal{B}_1$ and $W \ll_1 V$ it follows that for every $n \in \omega$, $n \geq 1$, there exists a $W_n \in \mathcal{B}_1$ such that $W \ll_1 \ldots \ll_1 W_n \ll_1 W_{n-1} \ll_1 \ldots \ll_1 \ll_1 W_1 \ll_1 V$. Set $\mathcal{D}'' = \{W_n \mid n = 1, 2, \ldots\}$ and $\mathcal{D} = \{D' \cap D'' \mid D' \in \in \mathcal{D}' \cup \{X_1\},\ D'' \in \mathcal{D}'' \cup \{X_1\}\}$. Then, obviously, $\mathcal{D}' \cup \mathcal{D}'' \subseteq \mathcal{D}$, $\mathcal{D}$ is a δ_1-system, its elements are non-empty sets and it is closed under finite intersections. Hence $\mathcal{D}$ is a filter-base. Let $\mathcal{F}'$ be the filter generated by the filter-base $\mathcal{D}$. Then $\mathcal{F}'$ is a bounded round filter in $(X_1, \delta_1, \mathcal{B}_1)$. There exists a maximal bounded round filter $\mathcal{F}$ containing $\mathcal{F}'$. Then Prop. 3.4 implies that $\mathcal{F}$ is a bounded end in $(X_1, \delta_1, \mathcal{B}_1)$. Hence $\mathcal{F} \in \Sigma_1$. We have that $\mathcal{D}'' \subseteq \mathcal{F}$ and therefore $W_1 \in \mathcal{F}$; hence $V \in \mathcal{F}$. Thus $V \cap H \neq \emptyset$, for every $H \in \mathcal{F}$. Since f is an SR-proximally continuous function, there exists a $\mathcal{G} \in \Sigma_2$ contained in the filter generated by the filter-base $f(\mathcal{F})$. The filter $\mathcal{G}$ is a bounded end in $(X_2, \delta_2, \mathcal{B}_2)$. Thus there exists an $E \in \in \mathcal{G} \cap \mathcal{B}_2$ and a $B \in \mathcal{G}$ such that $B \ll_2 E$. There exists an $H_0 \in \mathcal{F}$ such that $f(H_0) \subseteq B$; therefore $B \cap f(H) \neq \emptyset$, for every $H \in \mathcal{F}$. Hence $E \in \Omega$ and $H_0 \subseteq f^{-1}(B) \subseteq f^{-1}(E)$. Thus $f^{-1}(E) \in \mathcal{F}$. From $\mathcal{D}' \subseteq \mathcal{F}' \subseteq \mathcal{F}$ it follows that $X_1 \setminus f^{-1}(E) \in \mathcal{F}$. This contradiction shows that $f(W) \in \mathcal{B}_2$.

Thus f is an equicontinuous mapping. Put $F(f) = f$.

Step 6. Proof of the equalities $F \circ G = id_{\mathcal{C}_2}$ and $G \circ F = id_{\mathcal{C}_1}$ on the morphisms of the categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

This follows trivially from the definitions of F and G on the morphisms.

Hence the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic. $\Diamond$

Theorem 3.9. *The categories $\mathcal{C}_2$ and $\mathcal{C}_3$ are isomorphic.*

Proof. *Step 1. Construction of the functor $F : \mathcal{C}_3 \to \mathcal{C}_2$.*

Let $(X, \Sigma) \in |\mathcal{C}_3|$ and $\Sigma = (\mathcal{M}, \mathcal{V})$. Put

$$F(X, \Sigma) = (X, \delta, \mathcal{M}),$$

where δ is a proximity of X defined as follows: if $A, B \subseteq X$ then

$$\begin{aligned} &A\delta B \text{ iff } \exists A', B' \in \mathcal{M} \text{ such that } A' \subseteq A, \\ &B' \subseteq B \text{ and } A' \cap V \neq \emptyset, \forall\, V \in \mathcal{V}(B'). \end{aligned} \tag{8}$$

Note that if $B \in \mathcal{M}$ then

$$\begin{aligned} &A\delta B \text{ iff } \exists A' \in \mathcal{M} \text{ such that } A' \subseteq A \\ &\text{and } V \cap A' \neq \emptyset, \text{ for every } V \in \mathcal{V}(B). \end{aligned} \tag{9}$$

Indeed, it is obvious that the right side of (9) implies the left one. Conversely, if $A\delta B$ and $V \in \mathcal{V}(B)$ then there exists such a $W \in \mathcal{V}(B)$ that $V \in \mathcal{V}(C)$ provided that $C \in \mathcal{M}$ and $C \subseteq W$. There exist $A' \subseteq A$, $B' \subseteq B$ such that $A', B' \in \mathcal{M}$ and $V' \cap A' \neq \emptyset$, for every $V' \in \mathcal{V}(B')$. Since $B' \in \mathcal{M}$ and $B' \subseteq B \subseteq W$, we get that $V \in \mathcal{V}(B')$. Hence $V \cap A' \neq \emptyset$. So, (9) is proved.

We will now show that $(X, \delta, \mathcal{M})$ is a local proximity space. Obviously, conditions (LST1) and (LST2) (see Def. 2.20) imply that $\mathcal{M}$ is a boundedness. Let's prove that δ is a basic proximity. As it follows from condition (STS) (see Def. 2.19), δ is a symmetric relation. It is obvious that $\emptyset \overline{\delta} A$ for every $A \subseteq X$, and $A\delta A$ for every non-empty $A \subseteq X$ (indeed, if $A \neq \emptyset$ then set $A' = \{x\}$, where x is some point of A). Let $A, B, C \subseteq X$ and $A\delta(B \cup C)$. Then there exist $A', D \in \mathcal{M}$ such that $A' \subseteq A$, $D \subseteq (B \cup C)$ and $V \cap D \neq \emptyset$, for every $V \in \mathcal{V}(A')$. Let $D_1 = D \cap B$ and $D_2 = D \cap C$. Obviously, $D_1, D_2 \in \mathcal{M}$. Suppose that $A\overline{\delta}B$ and $A\overline{\delta}C$. Then there exist $V_1, V_2 \in \mathcal{V}(A')$ such that $V_1 \cap D_1 = \emptyset$ and $V_2 \cap D_2 = \emptyset$. Thus we have $V = V_1 \cap V_2 \in \mathcal{V}(A')$ and $V \cap D = \emptyset$. This contradiction shows that either $A\delta B$ or $A\delta C$. Conversely, let $A, B, C \subseteq X$ and either $A\delta B$ or $A\delta C$. Let, e.g., $A\delta B$. Then there exist $A', B' \in \mathcal{M}$ such that $A' \subseteq A$, $B' \subseteq B$ and $B' \cap V \neq \emptyset$ for every $V \in \mathcal{V}(A')$. Since $B' \subseteq B \cup C$, we get that $A\delta(B \cup C)$. Hence, $A\delta(B \cup C)$ iff either $A\delta B$ or $A\delta C$. So, δ is a basic proximity.

Let's prove a fact that will be used later:

$$\text{if } B \in \mathcal{M} \text{ then } V \in \mathcal{V}(B) \text{ iff } B \ll V. \tag{10}$$

Indeed, let $V \in \mathcal{V}(B)$. Suppose that $B\delta(X \setminus V)$. Then, by (9), there exists a $U \subseteq X \setminus V$ such that $U \in \mathcal{M}$ and $U \cap V' \neq \emptyset$ for every $V' \in \mathcal{V}(B)$. Then, in particular, $U \cap V \neq \emptyset$, a contradiction. Hence $B \ll V$.

Conversely, let $B \in \mathcal{M}$ and $B \ll V$. If $V = X$ then, clearly, $V \in \mathcal{V}(B)$. So, let $V \neq X$. Then $X \setminus V \neq \emptyset$. There exists a $V' \in \mathcal{M}$ such that $V' \subseteq \subseteq X \setminus V$. Now, $B\overline{\delta}(X \setminus V)$ and (9) imply that there exists a $U' \in \mathcal{V}(B)$ such that $U' \cap V' = \emptyset$. As it follows from (LST1) and (LST3) (see Def. 2.20), we can suppose w.l.o.g. that $U' \in \mathcal{M}$. The next two cases are possible:

1. $U' \cap (X \setminus V) = \emptyset$. Then $U' \subseteq V$. Since $\mathcal{V}(B)$ is a filter, it follows that $V \in \mathcal{V}(B)$.

2. $U' \cap (X \setminus V) \neq \emptyset$. Set $D = U' \cap (X \setminus V)$. We have that $D \in \mathcal{M}$ and $D \subseteq (X \setminus V)$. Since $B \ll V$, there exists a $U'' \in \mathcal{V}(B)$ such that $D \cap U'' = \emptyset$ (by (9)). Then $U = U' \cap U'' \in \mathcal{V}(B)$ and $U \cap (X \setminus V) = \emptyset$. Thus $U \subseteq V$ and hence $V \in \mathcal{V}(B)$.

So, (10) is established.

Since Σ is a separated L-supertopology, we get that δ is a separated proximity. It remains to show that $(X, \delta, \mathcal{M})$ satisfies conditions (LP1) and (LP2) of Def. 2.17. The conditions (LP2) is obviously fulfilled. We will prove that condition (LP1) takes place. Let $B \ll D$ and $B \in \mathcal{B}$. Then, by (10), $D \in \mathcal{V}(B)$. Hence, setting

$$\mathcal{V}_D(B) = \{V \cap D \mid V \in \mathcal{V}(B)\},$$

we get that $\mathcal{V}_D(B) \subseteq \mathcal{V}(B)$. By (LST3), there exists a $U_0 \in \mathcal{V}_D(B) \cap \mathcal{M}$. From (ST2) (see Def. 2.19) it follows that there exists a $V \in \mathcal{V}(B)$ such that $U_0 \in \mathcal{V}(A)$ provided that $A \subseteq V$ and $A \in \mathcal{M}$. By (LST3), we can suppose that $V \in \mathcal{M}$. Then $U_0 \in \mathcal{V}(V)$. Since $V \in \mathcal{V}(B)$, (10) implies that $B \ll V$. Suppose that $V\delta(X \setminus D)$. Then, by (9), there exists a $D' \subseteq (X \setminus D)$ such that $D' \in \mathcal{M}$ and $D' \cap U_0 \neq \emptyset$. Since $U_0 \in \mathcal{V}_D(B)$, it follows that $U_0 = A \cap D$ for some $A \in \mathcal{V}(B)$. Thus $D' \cap (A \cap D) \neq \emptyset$, i.e. $D' \cap D \neq \emptyset$. This contradiction shows that $V \ll D$. Thus $B \ll V \ll D$. Hence, $(X, \delta, \mathcal{M})$ is a local proximity space.

Let $f : (X_1, (\mathcal{M}_1, \mathcal{V}_1)) \to (X_2, (\mathcal{M}_2, \mathcal{V}_2))$ be a supertopological mapping. We will prove that $f : F(X_1, (\mathcal{M}_1, \mathcal{V}_1)) \to F(X_2, (\mathcal{M}_2, \mathcal{V}_2))$ is an equicontinuous mapping. Let $F(X_i, (\mathcal{M}_i, \mathcal{V}_i)) = (X_i, \beta_i, \mathcal{M}_i)$, $i = = 1, 2$. Obviously, the condition (EQ2) from Def. 2.17 follows from the condition (STC1) (see Def. 2.19). Let $A, B \subseteq X_1$ and $A\beta_1 B$. Then there exist $C, D \in \mathcal{M}_1$ such that $(A \cap C)\beta_1(B \cap D)$; thus, by (9),

$$U \cap (B \cap D) \neq \emptyset \text{ for every } U \in \mathcal{V}_1(A \cap C). \tag{11}$$

Suppose that $f(A)\overline{\beta_2}f(B)$. Then $f(A \cap C)\overline{\beta_2}f(B \cap D)$. Hence, by (9), there exists a $V \in \mathcal{V}_2(f(A \cap C))$ such that $V \cap f(B \cap D) = \emptyset$. Therefore $f^{-1}(V) \cap (B \cap D) = \emptyset$. Since f is supertopological, we get that

$f^{-1}(V) \in \mathcal{V}_1(A \cap C)$. This contradicts (11). Hence $f(A)\beta_2 f(B)$. So, $f : F(X_1, (\mathcal{M}_1, \mathcal{V}_1)) \to F(X_2, (\mathcal{M}_2, \mathcal{V}_2))$ is an equicontinuous mapping. We put $F(f) = f$.

Step 2. Construction of the functor $G : \mathcal{C}_2 \to \mathcal{C}_3$.

Let $(X, \delta, \mathcal{B}) \in |\mathcal{C}_2|$. Set

$$G(X, \delta, \mathcal{B}) = (X, (\mathcal{M}, \mathcal{V})),$$

where

$$\mathcal{M} = \mathcal{B} \text{ and } \mathcal{V} = \{\mathcal{V}(A) = \{B \subseteq X \mid A \ll B\} \mid A \in \mathcal{M}\}. \tag{12}$$

We will prove that $G(X, \delta, \mathcal{B}) \in |\mathcal{C}_3|$. Obviously, condition (ST1) from Def. 2.19 is satisfied. We will show that condition (ST2) takes place. Let $A \in \mathcal{M}$ and $U \in \mathcal{V}(A)$, i.e. $A \ll U$. Then condition (LP1) (see Def. 2.17) implies that there exists a $C \in \mathcal{B}$ such that $A \ll C \ll U$. Thus $C \in \mathcal{V}(A)$. Let $B \subseteq C$. Then $B \in \mathcal{M}$ and $B \ll U$. Hence $U \in \mathcal{V}(B)$, for every $B \subseteq C$. So, $\Sigma = (\mathcal{M}, \mathcal{V})$ is a supertopology. Now we will prove that $\Sigma = (\mathcal{M}, \mathcal{V})$ is a symmetric supertopology. Let $A, B \in \mathcal{M}$ and $U \cap A \neq \emptyset$, for every $U \in \mathcal{V}(B)$. Suppose that there exists a $V \in \mathcal{V}(A)$ such that $V \cap B = \emptyset$. Then $A\overline{\delta}(X \setminus V)$ and $B \subseteq (X \setminus V)$. Thus $A\overline{\delta}B$, i.e. $(X \setminus A) \in \mathcal{V}(B)$. This contradiction shows that $B \cap V \neq \emptyset$, for every $V \in \mathcal{V}(A)$. So, $\Sigma = (\mathcal{M}, \mathcal{V})$ is a symmetric supertopology. Further, conditions (LST1) and (LST2) (see Def. 2.20) are obviously satisfied. We will prove that condition (LST3) is also fulfilled. Let $A \in \mathcal{M}$ and $V \in \mathcal{V}(A)$, i.e. $A \ll V$. Then condition (LP1) (see Def. 2.17) implies that there exists a $C \in \mathcal{M}$ such that $A \ll C \ll V$. Thus $C \in \mathcal{V}(A) \cap \mathcal{M}$. Hence, $\Sigma = (\mathcal{B}, \mathcal{V})$ is an L-supertopology. Obviously, it is separated.

Let $f \in \mathcal{C}_2((X_1, \beta_1, \mathcal{B}_1), (X_2, \beta_2, \mathcal{B}_2))$. We will prove that

$$f \in \mathcal{C}_3(G(X_1, \beta_1, \mathcal{B}_1), G(X_2, \beta_2, \mathcal{B}_2)).$$

Let $B \in \mathcal{M}_1 = \mathcal{B}_1$. Then $f(B) \in \mathcal{B}_2 = \mathcal{M}_2$. Hence $f(\mathcal{M}_1) \subseteq \subseteq \mathcal{M}_2$. Let $A \in \mathcal{M}_1$ and $V \in f^{-1}(\mathcal{V}_2(f(A)))$. Then there exists a $U \in \mathcal{V}_2(f(A))$ such that $V = f^{-1}(U)$. Since $f(A) \ll_2 U$, it follows that $f^{-1}(f(A)) \ll_1 f^{-1}(U)$. Then $A \ll_1 V$, i.e. $V \in \mathcal{V}_1(A)$. Therefore $f^{-1}(\mathcal{V}_2(f(A))) \subseteq \mathcal{V}_1(A)$. Hence, f is a supertopological mapping. We put $G(f) = f$.

Step 3. Proof of the equality $G \circ F = id_{\mathcal{C}_3}$.

Let $(X, \Sigma) \in |\mathcal{C}_3|$, where $\Sigma = (\mathcal{M}, \mathcal{V})$. Then $G(F(X, \Sigma)) = = G(X, \delta, \mathcal{M}) = (X, \Sigma_1)$, where $\Sigma_1 = (\mathcal{M}, \mathcal{V}_1)$. We have to prove that $\mathcal{V} = \mathcal{V}_1$. Let $A \in \mathcal{M}$ and $V \in \mathcal{V}(A)$. Then, by (10), $A \ll V$ and thus $V \in \mathcal{V}_1(A)$. Conversely, let $V \in \mathcal{V}_1(A)$. Then $A \ll V$. Thus, by (10),

$V \in \mathcal{V}(A)$. Hence $\mathcal{V}(A) = \mathcal{V}_1(A)$, for every $A \in \mathcal{M}$. So, $G(F(X,\Sigma)) = (X,\Sigma)$.

Step 4. Proof of the equality $F \circ G = id_{\mathcal{C}_2}$.

Let $(X,\delta,\mathcal{B}) \in |\mathcal{C}_2|$. Then $F(G(X,\delta,\mathcal{B})) = F(X,\Sigma) = (X,\delta_1,\mathcal{B})$, where $\Sigma = (\mathcal{B},\mathcal{V})$. We have to prove that $\delta = \delta_1$. Let $A \ll_\delta B$, i.e. $A\overline{\delta}(X \setminus B)$. Then $C\overline{\delta}(X \setminus B)$, for every $C \subseteq A$. Hence $B \in \mathcal{V}(C)$, for every $C \in \mathcal{M}$ such that $C \subseteq A$. Suppose that $A\delta_1(X \setminus B)$. Then there exist a $C_1 \subseteq A$ and a $B_1 \subseteq (X \setminus B)$ such that $B_1, C_1 \in \mathcal{M}$ and $C' \cap B_1 \neq \emptyset$ for every $C' \in \mathcal{V}(C_1)$. We have that $B \in \mathcal{V}(C_1)$. Hence $B \cap B_1 \neq \emptyset$. This contradiction shows that $A \ll_{\delta_1} B$. Conversely, let $A \ll_{\delta_1} B$. Then, by (10), $B \in \mathcal{V}(A)$. This means, however, that $A \ll_\delta B$. Hence $(X,\delta,\mathcal{B}) = (X,\delta_1,\mathcal{B})$.

Therefore, the categories $\mathcal{C}_2$ and $\mathcal{C}_3$ are isomorphic. $\Diamond$

Corollary 3.10 *For every (separated) L-supertopology $\Sigma = (\mathcal{M},\mathcal{V})$ on a set X, the induced topology $\mathcal{T}$ on X is a (Hausdorff) completely regular topology.*

Proof. By Th. 3.9, there exists $(X,\delta,\mathcal{B}) \in |\mathcal{C}_2|$ such that $(X,\Sigma) = G(X,\delta,\mathcal{B})$. Hence, by (12), for every $x \in X$, $\mathcal{V}(x) = \{C \subseteq X \mid \{x\} \ll C\}$. Obviously, for every $A \subseteq X$, $\mathrm{int}_{(X,\tau_\delta)}(A) = \{x \in X \mid \{x\} \ll A\}$. Therefore, $\tau_\delta = \mathcal{T}$. Now, all follows from [9, 2.1]. $\Diamond$

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