Mathematica Pannonica
19/2 (2008), 197–217

REPRESENTING QUASI-UNIFORM SPACES AS THE PRIMES OF ORDERED SPACES

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Received: December 2007

Keywords: Uniformity, quasi-uniformity, maximal point representation, prime element representation, hyperspaces.

Abstract: Inspired by recent results on representing topological spaces as sets of maximal elements in ordered structures, we use hyperspace constructions to show how some classes of quasi-uniform spaces, and, more generally bitopolical spaces, can be characterized as the sets of prime elements of ordered structures.

0. Introduction

Much progress has been made in recent years in characterizing various classes of spaces as the sets of maximal points of one or another variety of ordered structure. Such considerations have their motivation in computer science, but they have taken on a life of their own as topological questions.

A survey of these results shows that completeness, in many of its variations, is frequently a theme. Since the theory of uniform spaces includes a notion of completeness that generalizes the complete metric spaces, it seems only natural that some sort of maximal point results should exist for that theory.

The natural ordered structures for such a study are the quasi-

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uniform spaces, both because the set of maximal points would inherit a quasi-uniform (and possibly uniform) structure and because the intersection of a T_0 quasi-uniformity is a partial order relation. If, out of a desire for generality, we allow all the spaces of interest to be quasi-uniform but not always uniform, then we see that the set of maximal points is no longer the right set to look at, since it is not general enough, but that the set of prime elements will take its place.

There is already a prime element theorem in the literature. From the standard reference book for domain theory, [6], we can take the following theorem. We state all the relevant definitions here since many of them will be used later. We remind the reader that, in general, no separation axioms should be assumed.

Definition. A closed subset of a space is *irreducible* if it is not the union of two proper closed subsets. A space is *quasisober* if every irreducible subset is the closure of a point, and it is *sober* if it is T_0 and quasisober.

Definition. A space is *locally compact* if every neighborhood of every point contains a compact neighborhood of the point.

Definition. A lattice is *distributive* if it satisfies the distributive laws, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition. On any partially ordered set $(X, \leq)$ define the "way below" relation $\ll$ by $x \ll y$ iff for any directed subset D of X, if $\sup D$ exists and $y \leq \sup D$ then $D \cap \uparrow x \neq \phi$. The poset $(X, \leq)$ is *continuous* if for any $x \in X$, the set $\{y \in X \mid y \ll x\}$ is directed and its supremum is x. The term *continuous lattice* refers to a complete lattice which is continuous as a poset.

Definition. For any semilattice $(X, \leq)$, an element p is *prime* if for any x, $y \in X$, $\inf\{x, y\} \leq p$ implies that either $x \leq p$ or $y \leq p$. Define $\mathrm{Prime}(X, \leq)$ to be the set of all primes. $\mathrm{Spec}(X, \leq)$ is $\mathrm{Prime}(X, \leq)$ minus the greatest element of $(X, \leq)$, if there is one.

Definition. The *hull-kernel topology* on $\mathrm{Spec}(X, \leq)$ is the topology generated by sets of the form $\mathrm{Spec}(X, \leq) - \uparrow x$ for all $x \in X$.

The following theorem is a combination of Thms. V-4.4, V-5.5, and V-5.6 from [6].

Theorem 0.1. *A topological space is sober if and only if it is homeomorphic to* $\mathrm{Spec}\, L$ *with the hull-kernel topology for some complete lattice* L*. A topological space is sober and locally compact if and only if it is homeomorphic to* $\mathrm{Spec}\, L$ *with the hull-kernel topology for some distributive* *continuous lattice L.*

We concentrate in this paper on characterizations whose proofs make use of hyperspaces. Indeed, the last theorem can be classified as a hyperspace theorem – while the standard proof considers the lattice of open sets under inclusion, it is obviously equivalent to consider the lattice of closed sets under reverse inclusion. Our results may not be as elegant as the last theorem, but they are steps in a direction that we believe will lead to even better things.

Definition. Given a quasi-uniform space $(X,\mathcal{U})$, let $\exp(X,\mathcal{U}^{-1})$ be the set of non-empty $\mathcal{U}^{-1}$-closed subsets of X and define a quasi-uniformity $\mathcal{H}(\mathcal{U})$ on $\exp(X,\mathcal{U}^{-1})$ in the following way: If R is a relation on X let $H(R)$ be the set of pairs $(A,B) \in \exp(X,\mathcal{U}^{-1}) \times \exp(X,\mathcal{U}^{-1})$ such that $B \subseteq R[A]$. Let $\mathcal{H}(\mathcal{U})$ be the quasiuniformity generated by the basis consisting of relations $H(U)$ for $U \in \mathcal{U}$. Then the *hyperspace* of $(X,\mathcal{U})$ is $(\exp(X,\mathcal{U}^{-1}),\mathcal{H}(\mathcal{U}))$.

We note that the mapping from X to $\exp(X,\mathcal{U}^{-1})$ which takes each point to its $\mathcal{U}^{-1}$-closure is an embedding of quasi-uniform spaces if and only if $(X,\mathcal{U})$ is T_0.

It is crucial to point out that this is not the quasi-uniform hyperspace of Levine and Stager [12]. Their motive was to have a construction which, in the case that the base space was a uniformity, yielded the usual hyperspace uniformity. We have no such desire. Instead it is important here that $\mathcal{H}(\mathcal{U})$ is usually not a uniformity even when $\mathcal{U}$ is. This is a consequence of our desire here that the partial order given by $\cap\mathcal{H}(\mathcal{U})$ is just reverse inclusion on $\exp(X,\mathcal{U}^{-1})$, and that is not, in general, a property of the Levine–Stager hyperspace.

This definition is somewhat at odds with how we have defined the quasi-uniform hyperspace in earlier papers, but there is an underlying consistency. In [2], out of a (possibly misplaced) desire for generality, we introduced bi-quasi-uniform spaces $(X,\mathcal{U},\mathcal{U}^*)$, where $\mathcal{U}$ and $\mathcal{U}^*$ are quasi-uniformities on X. We proposed there that such a space should have a bi-quasi-uniform hyperspace $(\exp(X,\mathcal{U}),\mathcal{H}(\mathcal{U}^{-1})^{-1},\mathcal{H}(\mathcal{U}^*))$, where $\exp(X,\mathcal{U})$ is the set of non-empty $\mathcal{U}$-closed subsets of X (except that the H operator was defined differently there, so the inverses were in different places).

When the bi-quasi-uniform space $(X,\mathcal{U},\mathcal{U}^*)$ satisfied $\mathcal{U}^* = \mathcal{U}^{-1}$, we called that space self-adjoint, and noted that self-adjoint spaces had self-adjoint hyperspaces, i.e., the hyperspace of $(X,\mathcal{U},\mathcal{U}^{-1})$ is

$(\exp(X,\mathcal{U}),\mathcal{H}(\mathcal{U}^{-1})^{-1},\mathcal{H}(\mathcal{U}^{-1}))$.

Here our "hyperspace" is, in a way, the "dual of the hyperspace of the dual" from the other paper. Given a quasi-uniform space $(X,\mathcal{U})$, expand it to a bi-quasi-uniform space $(X,\mathcal{U},\mathcal{U}^{-1})$. The dual of that is $(X,\mathcal{U}^{-1},\mathcal{U})$, and the hyperspace of that is $(\exp(X,\mathcal{U}^{-1}),\mathcal{H}(\mathcal{U})^{-1},\mathcal{H}(\mathcal{U}))$. The dual of the hyperspace of the dual is then $(\exp(X,\mathcal{U}^{-1}),\mathcal{H}(\mathcal{U}),\mathcal{H}(\mathcal{U})^{-1})$, which is the bispace version of our hyperspace $(\exp(X,\mathcal{U}^{-1}),\mathcal{H}(\mathcal{U}))$ in the definition above.

Our guiding principle in the discussion of bitopological spaces $(X,\mathcal{T},\mathcal{T}^*)$ is that there is always a preorder on the set X that is somehow involved in the discussion, and that the members of $\mathcal{T}$ should always be closed upward in the preorder and the members of $\mathcal{T}^*$ should always be closed downward in the preorder. Likewise, we expect that a bi-quasi-uniform space $(X,\mathcal{U},\mathcal{U}^*)$ should be discussed in a context where there is a preorder on X, and that the associated bitopological space $(X,\mathcal{T}(\mathcal{U}),\mathcal{T}(\mathcal{U}^*))$ should satisfy the same property with respect to this preorder. A special case of this is when the space is self-adjoint, i.e., $(X,\mathcal{U},\mathcal{U}^{-1})$, and the preorder $\leq$, regarded as a subset of $X\times X$, is $\cap\mathcal{U}$.

This special case is consistent with both of the hyperspaces we have just discussed. In [2], the most significant partial order on the hyperspace $(\exp(X,\mathcal{U}),\mathcal{H}(\mathcal{U}^{-1})^{-1},\mathcal{H}(\mathcal{U}^{-1}))$ was inclusion, which, indeed is the intersection of $\mathcal{H}(\mathcal{U}^{-1})^{-1}$. In this paper, the emphasis shifts to the partial order of reverse inclusion, and that order on the set of points $\exp(X,\mathcal{U}^{-1})$ is, indeed, $\cap\mathcal{H}(\mathcal{U})$. Because of this shift of emphasis, it is appropriate that the hyperspace we deal with here is the dual of the hyperspace of the dual from the other paper.

We also use a hyperspace which is purely topological construction. Two topologies can be defined on the hyperspace which depend only on the two topologies of the base space. For greatest generality, we will put all the results in the language of bitopological spaces, noting that a quasi-uniformity generates a bitopological structure because there is a front topology and there is a back topology.

Definition. For a bitopological space $(X,\mathcal{T},\mathcal{T}^*)$, let $\exp(X,\mathcal{T}^*)$ be the collection of non-empty, $\mathcal{T}^*$-closed subsets of X. For subsets $A_1, A_2, ..., A_n$ of X, $\langle A_1,A_2,\ldots,A_n\rangle$ will denote the collection of all members F of $\exp(X,\mathcal{T}^*)$ such that $F\subseteq\cup_{i=1}^n A_i$ and for each $i=1,2,\ldots,n$, $F\cap A_i\neq\phi$. The topology $U(\mathcal{T})$ on $\exp(X,\mathcal{T}^*)$, the upper Vietoris topology, is generated by the basis consisting of all $\langle O\rangle$ for $O\in\mathcal{T}$, and the topology $L(\mathcal{T}^*)$

on $\exp(X,\mathcal{T}^*)$, the lower Vietoris topology, is generated by the subbasis consisting of all $\langle X,O\rangle$ for $O\in\mathcal{T}^*$. Then $(\exp(X,\mathcal{T}^*),U(\mathcal{T}),L(\mathcal{T}^*))$ is the *hyperspace* of $(X,\mathcal{T},\mathcal{T}^*)$.

As with our quasi-uniform hyperspace defined above, this definition is the dual of the hyperspace of the dual from our other papers on this topic.

Definition. A bitopological space $(X,\mathcal{T},\mathcal{T}^*)$ is R_0 if for any $x\in X$ and any $O\in\mathcal{T}$, if $x\in O$ then $c^*\{x\}\subseteq O$.

Definition. For any bitopological property P that a space $(X,\mathcal{T},\mathcal{T}^*)$ could have, we say the space $(X,\mathcal{T},\mathcal{T}^*)$ has property P^* when the dual space $(X,\mathcal{T}^*,\mathcal{T})$ has property P.

We note that the mapping from X to $\exp(X,\mathcal{T}^*)$ which takes each point to its $\mathcal{T}^*$-closure is an embedding of quasi-uniform spaces if and only if $(X,\mathcal{T},\mathcal{T}^*)$ is T_0 and R_0^*.

Definition. For a topology $\mathcal{T}$ on a set X, the specialization order $\leq$ on X is defined by $x\leq y$ iff $x\in c\{y\}$. A bitopological space $(X,\mathcal{T},\mathcal{T}^*)$ has two such orders, $\leq$ and $\leq^*$ for $\mathcal{T}$ and $\mathcal{T}^*$, respectively.

Note that a bitopological space $(X,\mathcal{T},\mathcal{T}^*)$ is R_0 iff $y\leq^* x$ implies $x\leq y$, for any x, $y\in X$. Since the dual statement is true, too, it follows that a bispace is R_0 and R_0^* iff the two orders are reverses of each other.

Two properties of orderings will be used throughout this paper.

Definition. A preorder $(X,\leq)$ is *bounded-complete* if every subset with an upper bound has a least upper bound, or, alternatively, if every non-empty subset has a greatest lower bound. $(X,\leq)$ is *directed-complete* if every directed subset has a least upper bound.

Most of our terminology not dealing with bitopological spaces is consistent with [6]. Most of our bitopological terminology was used in the papers [1], [2], [3], and those papers, in turn, show the mixed influence of [4], [5], [7], and [10].

1. Some maximal element and prime element Applications of the hyperspace

Our first result considers how uniform spaces can be maximal point sets of some ordered structures. It is natural to make the larger structures be quasi-uniform spaces and to set the order to be given by the intersection of the quasi-uniformity.

Proposition 1.1. *For a uniform space to be T_2 it is necessary and sufficient for it to be unimorphic to the set of maximal points of a T_0 quasi-uniform space $(X,\mathcal{U})$, with order $\leq$ generated by the quasi-uniformity, such that* (1) *for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, if x is maximal then $V[x] \subseteq U^{-1}[x]$.*

Furthermore, any combination of the following properties of $(X,\mathcal{U})$ could be added above and the equivalence would still hold.

(2) *$(X, \leq)$ is bounded-complete.*

(3) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$,* $\inf(V[x]) \in U[x]$.

(4) *Every $x \in X$ is bounded above by a maximal element.*

(5) *For any $U \in \mathcal{U}$ and any subset $S \subseteq X$ and any $x \in X$, if x is maximal and* $\inf S \leq x$ *then $x \in U[S]$.*

Proof. Necessity. Suppose that $(Y, \mathcal{V})$ is a T_2 uniform space. Let $(X,\mathcal{U})$ be $(\exp(Y, \mathcal{V}), \mathcal{H}(\mathcal{V}))$. (Since there is only one topology on Y it will not be necessary to specify whether we are using $\mathcal{V}$ or $\mathcal{V}^{-1}$.) $(X,\mathcal{U})$ will be T_0 and the order $\cap\mathcal{U}$ will be reverse inclusion. The mapping $f : Y \to X$ which takes each x to $\{x\}$ will embed $(Y, \mathcal{V})$ unimorphically into $(X,\mathcal{U})$, and its image will be $\mathrm{Max}(X)$.

The properties (1)–(5) will be satisfied.

Sufficiency. Let $(X,\mathcal{U})$ be T_0 and satisfy (1), and let $\leq$ be $\cap\mathcal{U}$. By (1), $\mathrm{Max}(X)$ with the subspace quasi-uniformity will be a uniform space. Since $(X,\mathcal{U})$ is T_0 then $\mathrm{Max}(X)$ is Hausdorff. ◊

Directed-completeness is often required for the computer science applications of partial orders. If we are going to employ, as our ordered structures, the quasi-uniform hyperspaces described in the introduction, we cannot expect directed-completeness to hold in our ordered structures unless we assume $\mathcal{U}^{-1}$-compactness in the base space. However, because of the computer science motivations for this topic, some sort of completeness, akin to directed-completeness, is desirable. The natural property that suggests itself, in view of the presence of a quasi-uniformity, is that Cauchy directed sets should have suprema.

Definition. For a quasi-uniform space $(X,\mathcal{U})$ and a set $D \subseteq X$, where D is upward-directed by the preorder $\leq$ given by $\cap\mathcal{U}$, we say that D is *Cauchy* if for any $U \in \mathcal{U}$ there is a $d \in D$ such that for any $d' \in D$ we have $d \in U[d']$.

Note that a Cauchy directed set as defined here is both a *right K-Cauchy net*, as in [11], and a *bi-Cauchy net*, in that it is Cauchy for the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$. Since we will only be considering the Cauchy property for these directed subsets of a space, we do not need to distinguish between various Cauchy conditions, for quasi-uniform spaces, that exist in the literature.

Definition. A uniform space is *supercomplete* if its uniform hyperspace is complete.

Definition. For a quasi-uniform space $(X,\mathcal{U})$, a filter $\mathcal{F}$ on X is *stable* if for any $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that for any $F' \in \mathcal{F}$, $F \subseteq U[F']$.

The property of supercompleteness was treated at length by Isbell in [8] and [9]. Stable filters were utilized there for the uniform setting, and the definition was extended to the quasi-uniform spaces in [11]. An important characterization of supercompleteness is that it is equivalent to every stable filter having a cluster point. This was stated in the discussion at the very end of [8].

Our next result, possibly our main one, is a maximal point characterization of the Hausdorff, supercomplete uniform spaces.

Proposition 1.2. *For a uniform space to be T_2 and supercomplete it is necessary and sufficient for it to be unimorphic to the set of maximal points of a T_0 quasi-uniform space $(X,\mathcal{U})$, with order $\leq$ generated by the quasi-uniformity, such that*

(1) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, if x is maximal then $V[x] \subseteq U^{-1}[x]$.*

(2) *$(X, \leq)$ is bounded-complete.*

(3) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, $\inf(V[x]) \in U[x]$.*

(4) *Any Cauchy upward-directed subset of X has a supremum in X.*

(5) *Every $x \in X$ is bounded above by a maximal element.*

(6) *For any $U \in \mathcal{U}$ and any subset $S \subseteq X$ and any $x \in X$, if x is maximal and* $\inf S \leq x$ *then $x \in U[S]$.*

Furthermore, property (4) *could be augmented by saying that the supremum of the Cauchy upward-directed set is a $\mathcal{U}$-limit of the set, and the equivalence will still hold.*

Proof. Necessity. Suppose that $(Y,\mathcal{V})$ is a supercomplete, T_2 uniform space. We proceed as in the last proof to let $(X,\mathcal{U})$ be $(\exp(Y,\mathcal{V}), \mathcal{H}(\mathcal{V}))$.

If D is a Cauchy upward-directed subset of X then it will be a filter base for a stable filter on Y. By supercompleteness, this filter will have a non-empty set of cluster points and this set of cluster points will equal $\cap D$, since the members of D are closed. So (4) is satisfied.

It further follows from supercompleteness that any $\mathcal{V}$-uniform neighborhood of $\cap D$ contains a member of D. This makes $\cap D$ a $\mathcal{U}$ limit of D. So the modified form of (4) also follows.

Sufficiency. Let $(X,\mathcal{U})$ have the stated properties. By (1), $\mathrm{Max}(X)$ with the subspace quasi-uniformity will be a uniform space. Since $(X,\mathcal{U})$ is T_0 then $\mathrm{Max}(X)$ is Hausdorff. It remains to show that $\mathrm{Max}(X)$ is supercomplete.

Let $\mathcal{F}$ be a stable filter on $\mathrm{Max}(X)$. Then, by (2), we can define a directed set $D \subseteq X$ by $D = \{\inf A \mid A \in \mathcal{F}\}$. We show that D is Cauchy.

Given any $U \in \mathcal{U}$, choose a V as in property (3) and then choose a $W \in \mathcal{U}$ with $W \circ W \subseteq V$. Since $\mathcal{F}$ is stable, there is an $A \in \mathcal{F}$ such that for any $B \in \mathcal{F}$, $A \subseteq W[B]$. Let $a = \inf A$ and we claim that for any $x \in D$, $a \in U[x]$.

To this end, given $x \in D$, $x = \inf B$ for some $B \in \mathcal{F}$. $A \subseteq W[B]$, and, by definition of $\leq$, $B \subseteq W[x]$. So $A \subseteq W \circ W[x] \subseteq V[x]$. Therefore $\inf(V[x]) \leq a$, and since $\inf(V[x]) \in U[x]$, so is $a \in U[x]$.

So D is Cauchy. By (4), $\sup D$ exists in X. By (5), there is a maximal element m above $\sup D$. We note that for any $A \in \mathcal{F}$, $\inf A \leq$ $\leq m$. So, by (6), $m \in U[A]$ for any $A \in \mathcal{F}$ and any $U \in \mathcal{U}$. So m is a cluster point for $\mathcal{F}$. $\Diamond$

We suspect that the characterization given in Prop. 1.2 is not yet in its best form. The proposition suggests, though, that supercompleteness may be an important property for applications of maximal point ideas to uniform spaces.

Since topological supercompleteness is the same as paracompactness, we have the following corollary.

Corollary 1.3. *A topological space is T_3, paracompact iff it is homeomorphic to the set of maximal points of a T_0 space admitting a quasi-uniformity $(X,\mathcal{U})$, with order $\leq$ generated by the quasi-uniformity, such that*

(1) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, if x is maximal then $V[x] \subseteq U^{-1}[x]$.*

(2) *$(X,\leq)$ is bounded-complete.*

(3) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, $\inf(V[x]) \in U[x]$.*

(4) *Any Cauchy upward-directed subset of X has a supremum in X.*

(5) *Every $x \in X$ is bounded above by a maximal element.*

(6) *For any* $U \in \mathcal{U}$ *and any subset* $S \subseteq X$ *and any* $x \in X$*, if* x *is maximal and* $\inf S \leq x$ *then* $x \in U[S]$.

In metric spaces, supercompleteness is equivalent to completeness. This gives us another corollary.

Corollary 1.4. *A topological space is completely metrizable iff it is homeomorphic to the set of maximal points of a* T_0 *space admitting a quasi-uniformity* $(X,\mathcal{U})$*, with order* $\leq$ *generated by the quasi-uniformity, such that*

(1) *For any* $U \in \mathcal{U}$ *there is a* $V \in \mathcal{U}$ *such that for any* $x \in X$*, if* x *is maximal then* $V[x] \subseteq U^{-1}[x]$.

(2) $(X, \leq)$ *is bounded-complete.*

(3) *For any* $U \in \mathcal{U}$ *there is a* $V \in \mathcal{U}$ *such that for any* $x \in X$*,* $\inf(V[x]) \in U[x]$.

(4) *Any Cauchy upward-directed subset of* X *has a supremum in* X.

(5) *Every* $x \in X$ *is bounded above by a maximal element.*

(6) *For any* $U \in \mathcal{U}$ *and any subset* $S \subseteq X$ *and any* $x \in X$*, if* x *is maximal and* $\inf S \leq x$ *then* $x \in U[S]$.

(7) $\mathcal{U}$ *has a countable basis.*

We wish to transform Prop. 1.2 to characterize some class of quasi-uniform spaces, such that the intersection of that class with the uniform spaces is the supercomplete uniform spaces. A very nice generalization of supercompleteness to the quasi-uniform setting is given in [11], where it is shown that the Levine–Stager hyperspace of a quasi-uniform space $(X,\mathcal{U})$ is right K-complete iff every $\mathcal{U}$-stable filter has a $\mathcal{U}$-cluster point.

Unfortunately, we have failed to find any way to adapt that generalization to the problem at hand. The obstacle is that the wrong kind of cluster points are asserted to exist. The supremum of a $\mathcal{U}$-Cauchy directed set (under reverse inclusion) of $\mathcal{U}^{-1}$-closed sets should be its set of $\mathcal{U}^{-1}$-cluster points.

Definition. A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is *round* if for any $F \in \mathcal{F}$ there is an $F' \in \mathcal{F}$ and a $U \in \mathcal{U}$ such that $U[F'] \subseteq F$.

So, we seek a new generalization of supercompleteness. One property that characterizes the supercomplete uniform spaces is that every stable, round filter is the set of uniform neighborhoods of some (necessarily non-empty) set. We propose the following definition.

Definition. A quasi-uniform space $(X,\mathcal{U})$ is *supercomplete* if every stable, round filter is the set of $\mathcal{U}$-uniform neighborhoods of some set. Note that the set in question must be non-empty and must be the set of $\mathcal{U}^{-1}$-cluster points of the filter.

Now we have the following prime element version of Prop. 1.2.

Proposition 1.5. *For a quasi-uniform space to be T_0, sober*, and supercomplete it is necessary and sufficient for it to be unimorphic to the set of primes of a T_0 quasi-uniform space $(X,\mathcal{U})$, with order $\leq$ generated by the quasi-uniformity, such that*

(1) *$\mathcal{U}^{-1}$ induces the hull-kernel topology on* $\mathrm{Prime}(X,\leq)$.

(2) *$(X,\leq)$ is bounded-complete.*

(3) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, $\inf(V[x]) \in U[x]$.*

(4) *Any Cauchy upward-directed subset of X has a supremum in X, which is the $\mathcal{U}$-limit of the set.*

(5) *For any $U \in \mathcal{U}$ and any subset $S \subseteq X$ and any $x \in X$, if x is prime and* $\inf S \leq x$ *then $x \in U[S]$.*

(6) *For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for any $x \in X$, $V[x] \cap \mathrm{Prime}(X,\leq) \subseteq U[(\uparrow x) \cap \mathrm{Prime}(X,\leq)]$.*

Proof. Necessity. Suppose that $(Y,\mathcal{V})$ is a T_0, sober*, supercomplete quasi-uniform space. We let $(X,\mathcal{U})$ be $(\exp(Y,\mathcal{V}^{-1}),\mathcal{H}(\mathcal{V}))$. The mapping $f : Y \to X$ which takes each x to $c^*\{x\}$ will embed $(Y,\mathcal{V})$ unimorphically into $(X,\mathcal{U})$, and its image will be $\mathrm{Prime}(X)$ since $(Y,\mathcal{V})$ is sober with the topology $\mathcal{T}(\mathcal{V}^{-1})$.

If D is a Cauchy upward-directed subset of X then $\{U[d] \mid U \in \mathcal{V}, d \in D\}$ will be a round, stable filter on Y. By supercompleteness, this filter will the set of $\mathcal{V}$-uniform neighborhoods of $\cap D$, since the members of D are $\mathcal{V}^{-1}$-closed. So (4) is satisfied.

The other properties follow routinely.

Sufficiency. Let $(X,\mathcal{U})$ have the stated properties. To show that $(Y,\mathcal{T}(\mathcal{V}^{-1}))$ is sober, suppose that $A \subseteq \mathrm{Prime}(X,\leq)$ is irreducible for the topology induced on $\mathrm{Prime}(X,\leq)$ by $\mathcal{U}^{-1}$. Let $a = \inf A$.

We show a is prime. Suppose b, $c \in X$ and $\inf\{b,c\} \leq a$. Then for each $p \in A$ we have either $b \leq p$ or $a \leq p$. Hence $A \subseteq c^*\{b\} \cup c^*\{c\}$, and, by irreducibility, either $A \subseteq c^*\{b\}$ or $A \subseteq c^*\{c\}$. It follows that either $b \leq a$ or $c \leq a$.

By (1), the topology induced on $\mathrm{Prime}(X,\leq)$ by $\mathcal{U}^{-1}$ is the hull-kernel topology. It follows that $a \in A$. Since $a = \inf A$, we have that the $\mathcal{U}^{-1}$-closure of $\{a\}$ within $\mathrm{Prime}(X,\leq)$ is A.

It remains to show that $\mathcal{U}$ makes $\mathrm{Prime}(X,\leq)$ supercomplete.

Given a stable, round filter $\mathcal{F}$ on Prime$(X, \leq)$, let $D = \{\inf F \mid \mid F \in \mathcal{F}\}$. D will be Cauchy, as in the proof of Prop. 1.2. By (4), $\sup D$ exists and D is $\mathcal{U}$-convergent to $\sup D$.

Let $A = (\uparrow \sup D) \cap \text{Prime}(X, \leq)$. Given a $U \in \mathcal{U}$ choose V as in (6). Then choose $F \in \mathcal{F}$ so that $\inf F \in V[\sup D]$. Since

$$F \subseteq V[\sup D] \cap \text{Prime}(X, \leq) \subseteq U[(\uparrow \sup D) \cap \text{Prime}(X, \leq)] = U[A],$$

we have shown that every uniform neighborhood of A in Prime$(X, \leq)$ is a member of $\mathcal{F}$. On the other hand, (4) and the fact that $\mathcal{F}$ is stable imply that every $F \in \mathcal{F}$ is a uniform neighborhood of A. $\Diamond$

Property (6) in Prop. 1.5 stands out as stronger than what we needed in Prop. 1.2. It is equivalent to the statement that the mapping from $(X, \mathcal{U})$ to the hyperspace of Prime$(X, \leq)$, which takes each x to $(\uparrow x) \cap \text{Prime}(X, \leq)$, is uniformly continuous. We needed something stronger because our definition of supercomplete did not merely assert the existence of a cluster point, but asserted a property of the set of cluster points.

It is not at all clear that the property can be weakened. It may be appropriate to propose the following problem.

Problem. Characterize the quasi-uniform spaces that can be represented as the prime set of a T_0 quasi-uniform space satisfying properties (1) through (5) of Prop. 1.5 plus the property that every element is bounded above by a prime. Note that uniform spaces with this property will be supercomplete.

It may be that supercompleteness as we have defined it for quasi-uniform spaces does not have a characterization asserting that some class of filters has some kind of cluster point, unlike the case for supercomplete uniform spaces, and unlike the generalization of supercompleteness in [11].

Since we have introduced a property for quasi-uniform spaces, we will try to offer further motivation for it by finding an even stronger property that arises naturally.

Definition. A *pair cover* of a set X is a set $\mathcal{C}$ of pairs $(A, B) \in \mathcal{P}(X) \times \times \mathcal{P}(X)$ such that the set $\{A \cap B \mid (A, B) \in \mathcal{C}\}$ is a cover of X. For a bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, an *open pair cover* is a pair cover of X such that for every $(A, B) \in \mathcal{C}$, $A \in \mathcal{T}$ and $B \in \mathcal{T}^*$. A pair cover $\mathcal{C}_1$ *refines* a pair cover $\mathcal{C}_2$ if for any $(A, B) \in \mathcal{C}_1$ there is a $(C, D) \in \mathcal{C}_2$ with

$A \subseteq C$ and $B \subseteq D$. For a quasi-uniform space $(X, \mathcal{U})$, a pair cover is said to be a *uniform pair cover* if it is refined by the pair cover $\{(U[x], U^{-1}[x]) \mid \mid x \in X\}$ for some $U \in \mathcal{U}$. In such a quasi-uniform space, a pair cover is *open* if it is open for the bitopological space $(X, \mathcal{T}(\mathcal{U}), \mathcal{T}(\mathcal{U}^{-1}))$.

Proposition 1.6. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Suppose that every open pair-cover of X is uniform. Given a round, stable filter $\mathcal{F}$ on X, if $\cap\mathcal{F} \subseteq O \in \mathcal{T}(\mathcal{U})$ we then have $O \in \mathcal{F}$. Consequently, $(X, \mathcal{U})$ is supercomplete.*

Proof. Since $\mathcal{F}$ is round, $\cap\mathcal{F} = \cap\{c^*F \mid F \in \mathcal{F}\}$, where c^* is closure for $\mathcal{U}^{-1}$. So $\{(O, X)\} \cup \{(X, X - c^*F) \mid F \in \mathcal{F}\}$ is an open cover, since $\cap\{c^*F \mid F \in \mathcal{F}\} \subseteq O \in \mathcal{T}(\mathcal{U})$. Choose $U \in \mathcal{U}$ with $\{(U[x], U^{-1}[x]) \mid x \in \in X\}$ refining this cover. Since $\mathcal{F}$ is stable, there is an $F' \in \mathcal{F}$ such that for any $F \in \mathcal{F}$ we have $F' \subseteq U[F]$. For any $x \in X$, either $U[x] \subseteq O$ or $x \notin U[F]$ for some $F \in \mathcal{F}$. But the latter condition is impossible for $x \in F'$, by the choice of F'. So we must have $F' \subseteq O$. Thus $O \in \mathcal{F}$. $\Diamond$

A special case of Prop. 1.6 is the well-known fact that a paracompact uniform space, with its fine uniformity, is supercomplete.

We now turn to the question of what can be done with the bitopological hyperspace.

Definition. A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is *quasisober* if for any $\mathcal{T}$-irreducible set $A \subseteq X$ there is an element $x \in A$ such that for any $O \in \mathcal{T}^*$ with $x \in O$ we have $A \subseteq O$. A bitopological space is *sober* if it is T_0, R_0^*, and quasisober. [1, 2]

Sober was defined in [1] in this way in order to set up the *sobrification*, i.e., any T_0, R_0^* bispace can be embedded in a sober bispace.

Note that if $(X, \mathcal{T})$ is quasisober and $(X, \mathcal{T}, \mathcal{T}^*)$ is R_0^*, then $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober. If, on the other hand, $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober and R_0, then $(X, \mathcal{T})$ is quasisober.

Definition. For any topological space, a set is *saturated* if it is the intersection of its neighborhoods. For a topology $\mathcal{T}$ on X, the *dual topology* $\mathcal{T}^k$ is the topology on X generated by the complements of the saturated compact sets for $\mathcal{T}$.

Note that the specialization order of $\mathcal{T}^k$ is always the reverse of the specialization order for $\mathcal{T}$.

Definition. When $(X, \leq)$ is a semilattice, the meaning of the term *distributive* takes on a new form, which reduces to the one already given for the case of a lattice. A semilattice $(X, \leq)$ is *distributive* if whenever

$\inf\{a,b\} \leq x$ then there are c, $d \in X$ with $a \leq c$, $b \leq d$, and $x = \inf\{c,d\}$ (see [6]).

Our next theorem is a sort of bitopological version of the first part of Th. 0.1.

Proposition 1.7. *For a bitopological space to be sober** *and* R_0^* *it is necessary and sufficient for it to be homeomorphic to the set of primes of a sober***,* R_0^* *bitopological space* $(X,\mathcal{T},\mathcal{T}^*)$*, with the order given by the specialization order of* $\mathcal{T}$*.*

Furthermore, any combination of the following properties of $(X,\mathcal{T},\mathcal{T}^*)$ *may be added above and the equivalence will still hold.*

(1) $(X,\mathcal{T},\mathcal{T}^*)$ *is a bounded-complete distributive semilattice.*

(2) $\mathcal{T}^*$ *induces the hull-kernel topology on* Prime(X)*.*

(3) $\mathcal{T}^* \subseteq \mathcal{T}^k$*.*

Proof. Necessity. Given a sober*, R_0^* bispace $(Y,\mathcal{S},\mathcal{S}^*)$, let $(X,\mathcal{T},\mathcal{T}^*)$ be $(\exp(Y,\mathcal{S}^*),U(\mathcal{S}),L(\mathcal{S}^*))$. Since $(Y,\mathcal{S},\mathcal{S}^*)$ is R_0^* and points of X are $\mathcal{S}^*$-closed, then the $\mathcal{T}$-specialization order is reverse inclusion. By R_0, the mapping $f: Y \to X$ which takes each x to $c^*\{x\}$ will embed $(Y,\mathcal{S},\mathcal{S}^*)$ homeomorphically into $(X,\mathcal{T},\mathcal{T}^*)$, and its image will be Prime(X) since $(Y,\mathcal{S}^*)$ is quasisober.

The hyperspace $(X,\mathcal{T},\mathcal{T}^*)$ will always be T_0, R_0, and quasisober* (Props. 2 and 4 in [1]). The fact that it is R_0^* when $(Y,\mathcal{S},\mathcal{S}^*)$ is R_0 and R_0^* was Cor. 1 in [1].

(1) follows from the consideration that the infimum of a collection (under reverse inclusion) is just the $\mathcal{S}^*$-closure of the union.

(2) follows from the fact that the subbassis element $\langle Y,O\rangle$ is the same set as $X - \uparrow(Y-O)$.

(3) follows from the fact that $L(\mathcal{S}^*)$ is the weakest topology on $\exp(Y,\mathcal{S}^*)$ whose specialization order is inclusion, so it is coarser than any other such topology, including $U(\mathcal{S})^k$.

Sufficiency. Given that $(X,\mathcal{T},\mathcal{T}^*)$ has the stated properties, note that T_0, R_0, and R_0^* will all be inherited by the subspace Prime(X).

To show that Prime(X) is sober with the topology induced by $\mathcal{T}^*$, suppose that A is an irreducible set in Prime(X) under that topology. c^*A will be irreducible in $(X,\mathcal{T}^*)$, so $c^*A = c^*\{a\}$ for some $a \in X$.

We show a is prime. Suppose b, $c \in X$ and $\inf\{b,c\} \leq a$. Then for each $p \in A$ we have either $b \leq p$ or $a \leq p$. Hence $A \subseteq c^*\{b\} \cup c^*\{c\}$, and, by irreducibility, either $A \subseteq c^*\{b\}$ or $A \subseteq c^*\{c\}$. It follows that either $b \leq a$ or $c \leq a$.

So $cA = c\{a\}$, i.e., $A = c\{a\}$, stills holds when restricted to Prime(X). $\Diamond$

Definition. For a partial order $(X, \leq)$, consider the convergence structure given by saying that any upward-directed $D \subseteq X$, for which $\sup D$ exists, converges to any member of $\downarrow \sup D$. The topology generated by this convergence is the *Scott topology.*

Definition. A bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is T_3 if it is T_0 and whenever $x \in O \in \mathcal{T}$ there is an $O' \in \mathcal{T}$ such that $x \in O' \subseteq c^*O' \subseteq O$.

Proposition 1.8. *For a bitopological space to be sober* and T_3^*, it is necessary and sufficient for it to be homeomorphic to the set of primes of a sober*, T_3^* bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, with the order given by the specialization order of $\mathcal{T}$.*

Furthermore, any combination of the following properties of $(X, \mathcal{T}, \mathcal{T}^*)$ may be added above and the equivalence will still hold.

(1) $(X, \leq)$ *is a bounded-complete distributive semilattice.*

(2) $\mathcal{T}^*$ *induces the hull-kernel topology on* Prime(X)*.*

(3) *The Scott topology on* $(X, \leq)$ *is coarser than* $\mathcal{S}$*.*

(4) $\mathcal{T}^* \subseteq \mathcal{T}^k$*.*

(5) $\mathcal{T}^{*k} \subseteq \mathcal{T}$*.*

Proof. Compared to the first paragraph of the last theorem, the additional property of T_3^* has been added to both spaces. Necessity for this follows from Cor. 2 of [1] and sufficiency is because T_3^* is hereditary.

For (3), suppose that $\mathcal{O}$ is a Scott open subset of X and that $A \in \in \mathcal{O}$. By regular*, A is the intersection of a filterbase $\mathcal{F}$ of $\mathcal{S}^*$-closed sets such that for each $F \in \mathcal{F}$, there is an $O \in \mathcal{S}$ such that $A \subseteq O \subseteq F$. Since $\mathcal{O}$ is Scott open, there is some $F' \in \mathcal{F}$ with $F' \in \mathcal{O}$. Then choose $O' \in \mathcal{S}$ with $A \subseteq O' \subseteq F'$. It follows that $A \in \langle O' \rangle \subseteq \langle F' \rangle \subseteq \mathcal{O}$.

(5) follows from R_1^*, which follows from T_3^*. $\Diamond$

2. Some applications of a modified hyperspace

In Sec. 1 the ordered structures were bounded-complete, and this property played an important role in the proofs. In this section, a different kind of hyperspace allows a different property, that of directed-completeness, to hold in place of bounded-completeness.

The hyperspace in question for this section will have points which are closed in one topology and compact in the other.

Definition. For a bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, let $K(X, \mathcal{T}^*, \mathcal{T})$ be the collection of non-empty, $\mathcal{T}^*$-closed, $\mathcal{T}$-compact subsets of X. In this section we will let $\langle A_1, A_2, \ldots, A_n \rangle$ stand for the collection of all members K of $K(X, \mathcal{T}^*, \mathcal{T})$ such that $K \subseteq \cup_{i=1}^n A_i$ and for each $i = 1, 2, \ldots, n$, $K \cap \cap A_i \neq \phi$. The topology $U(\mathcal{T})$ on $(K(X, \mathcal{T}^*, \mathcal{T})$ is generated by the basis consisting of all $\langle O \rangle$ for $O \in \mathcal{T}$, and the topology $L(\mathcal{T}^*)$ on $K(X, \mathcal{T}^*, \mathcal{T})$ is generated by the subbasis consisting of all $\langle X, O \rangle$ for $O \in \mathcal{T}^*$.

Definition. A bitopological space is *dually sober* if it and its dual are both sober.

Proposition 2.1. *For a bitopological space to be dually sober it is necessary and sufficient for it to be homeomorphic to the set of primes of an R_0, sober bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, with the order given by the specialization order of $\mathcal{T}$, such that*

(1) *For any $x \in X$, if O is a $\mathcal{T}$-neighborhood of $(\uparrow x) \cap \mathrm{Prime}(X)$ then there is an $O' \in \mathcal{T}$ with $x \in O'$ and $O' \cap \mathrm{Prime}(X) \subseteq O$.*

(2) *Any $\mathcal{T}^*|_{\mathrm{Prime}(X)}$-irreducible subset A of $\mathrm{Prime}(X)$ has an infimum in c^*A.*

Furthermore, any combination of the following properties of $(X, \mathcal{T}, \mathcal{T}^*)$ may be added above and the equivalence will still hold.

(3) *Any upward-directed set in X is $\mathcal{T}$-convergent to its supremum, i.e., $\mathcal{T}$ is coarser than the Scott topology for $(X, \leq)$. (Note that $(X, \leq)$ directed-complete follows from $(X, \mathcal{T})$ sober.)*

(4) *For any $x \in X$, $(\uparrow x) \cap \mathrm{Prime}(X)$ is $\mathcal{T}$-compact.*

(5) *$(X, \leq)$ is a distributive semilattice.*

(6) *$\mathcal{T}^* \subseteq \mathcal{T}^k$.*

Proof. Necessity. Given a dually sober bispace $(Y, \mathcal{S}, \mathcal{S}^*)$, let $(X, \mathcal{T}, \mathcal{T}^*)$ be $(K(Y, \mathcal{S}^*, \mathcal{S}), U(\mathcal{S}), L(\mathcal{S}^*))$. Many parts of the proof are identical to the corresponding proof in the last section.

Showing that when $(Y, \mathcal{S}, \mathcal{S}^*)$ is quasisober then $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober is an easy modification of the proof of Prop. 2.5 in [2].

To show that (1) holds, assume that $x = A \subseteq Y$. $(\uparrow x) \cap \mathrm{Prime}(X)$ is just $\{c^*\{y\} \mid y \in A\}$ and a $\mathcal{T}$-open neighborhood of it can be given by $\cup_{i \in J} \langle O_i \rangle$. Let $O = \cup_{i \in J} O_i$. Then $A \subseteq O$, so $x \in \langle O \rangle$, any $c^*\{y\}$ that is a member of $\langle O \rangle$ will be a member of $\langle O_i \rangle$ for some i, by R_0.

To show (2), suppose that $\mathcal{A}$ is a collection of $\mathcal{S}^*$-closures of singletons which is irreducible in the space of all $\mathcal{S}^*$-closures of singletons with the topology induced by $L(\mathcal{S}^*)$. Then $c^* \cup \mathcal{A}$ is $\mathcal{S}^*$-irreducible, and so it is itself the $\mathcal{S}^*$-closure of a singleton, by quasisobriety of $(Y, \mathcal{S}^*)$. This makes $c^* \cup \mathcal{A}$ $\mathcal{S}$-compact, by R_0. So $c^* \cup \mathcal{A}$ is in X and it clearly is an infimum for $\mathcal{A}$. It is in the $L(\mathcal{S}^*)$-closure of $\mathcal{A}$ because $\mathcal{A}$ is irreducible.

(3) follows from the bitopological Hofmann–Mislove Theorem, Prop. 1.5 in [2].

To show (4), we note that any $x \in X$ is a $\mathcal{T}$-compact subset of Y, and, from the discussion above, $(\uparrow x) \cap \mathrm{Prime}(X)$ is homeomorphic to x itself.

(5) is obvious from the fact that the collection of $\mathcal{S}^*$-closed, $\mathcal{S}$-compact sets is closed under finite unions.

Sufficiency. Given that $(X, \mathcal{T}, \mathcal{T}^*)$ has the stated properties, note that T_0, R_0, and R_0^* will all be inherited by the subspace $\mathrm{Prime}(X)$.

To show that $\mathrm{Prime}(X)$ is sober with the topology induced by $\mathcal{T}$, suppose that A is an irreducible set in $\mathrm{Prime}(X)$ under that topology. cA will be irreducible for $(X, \mathcal{T})$, so $cA = c\{a\}$ for some $a \in cA$.

We show a is prime. Suppose b, $c \in X$ and $\inf\{b, c\} \leq a$. Suppose that no primes above $\inf\{b, c\}$ are in cA. Then, by (1), there is an $O \in \mathcal{T}$ such that $\inf\{b, c\} \in O$ and none of the prime members of O are in A. But since $a \in O$ this contradicts $a \in cA$. So there is a prime $p \in cA$ with $\inf\{b, c\} \leq p$. Since $p \leq a$, we have either $b \leq a$ or $c \leq a$.

So $cA = c\{a\}$, i.e., $A = c\{a\}$, stills holds when restricted to $\mathrm{Prime}(X)$.

To show that $\mathrm{Prime}(X)$ is sober with the topology induced by $\mathcal{T}^*$, suppose that A is an irreducible set in $\mathrm{Prime}(X)$ under that topology. By (2), A has an infimum $a \in c^*A$.

We show a is prime. Suppose b, $c \in X$ and $\inf\{b, c\} \leq a$. Then for each $p \in A$ we have either $b \leq p$ or $a \leq p$. Hence $A \subseteq c^*\{b\} \cup c^*\{c\}$, and, by irreducibility, either $A \subseteq c^*\{b\}$ or $A \subseteq c^*\{c\}$. It follows that either $b \leq a$ or $c \leq a$.

Since $a \in c^*A$ and $A \subseteq c^*\{a\}$ we have $c^*A = c^*\{a\}$, and within $\mathrm{Prime}(X)$ that becomes $A = c^*\{a\}$. $\Diamond$

Definition. A bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is R_1 if whenever $x \in O \in$ $\in \mathcal{T}$ and $y \notin O$ there is an $O' \in \mathcal{T}$ such that $x \in O' \subseteq c^*O' \subseteq X - \{y\}$.

Proposition 2.2. *For a bitopological space to be dually sober and R_1^*, it is necessary and sufficient for it to be homeomorphic to the set of primes of an R_0, R_1^*, sober bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, with the order given by the specialization order of $\mathcal{T}$, such that*

(1) *For any $x \in X$, if O is a $\mathcal{T}$-neighborhood of $(\uparrow x) \cap \mathrm{Prime}(X)$ then there is an $O' \in \mathcal{T}$ with $x \in O'$ and $O' \cap \mathrm{Prime}(X) \subseteq O$.*

(2) *Any* $\mathcal{T}^*|_{\mathrm{Prime}(X)}$*-irreducible subset* A *of* $\mathrm{Prime}(X)$ *has an infimum in* c^*A.

Furthermore, any combination of the following properties of $(X, \mathcal{T}, \mathcal{T}^*)$ *may be added above and the equivalence will still hold.*

(3) *Any upward-directed set in* X *is* $\mathcal{T}$*-convergent to its supremum.*

(4) *For any* $x \in X$, $(\uparrow x) \cap \mathrm{Prime}(X)$ *is* $\mathcal{T}$*-compact.*

(5) *For any* $K \subseteq X$, K *non-empty and* $\mathcal{T}$*-compact implies that* $\inf(K)$ *exists.*

(6) $(X, \leq)$ *is a distributive semilattice.*

(7) $\mathcal{T}^* \subseteq \mathcal{T}^k$.

(8) $\mathcal{T}^{*k} \subseteq \mathcal{T}$.

Proof. Compared to the first paragraph of the last theorem, the additional property of R_1^* has been added to both spaces. For necessity, let J, $K \in K(Y, \mathcal{S}^*, \mathcal{S})$ be given and suppose there is a basic $L(\mathcal{S}^*)$-open set which contains J but not K. This amounts to saying that there is an $O \in \mathcal{S}^*$ with $J \cap O \neq \phi$ and $K \cap O = \phi$. By R_1^*, using $\mathcal{S}$ compactness of K, we can find an $O' \in \mathcal{S}^*$ and an $O'' \in \mathcal{S}$ with $J \cap O' \neq \phi$, $K \subseteq O''$, and $O' \cap O'' = \phi$. Then $J \in \langle X, O' \rangle$, $K \in \langle O'' \rangle$, and $\langle X, O' \rangle \cap \langle O'' \rangle = \phi$.

Sufficiency for this is because R_1^* is hereditary.

For (5), it is sufficient to show that the $\mathcal{S}^*$-closure of the union of a $U(\mathcal{S})$-compact collection of $\mathcal{S}$-compact, $\mathcal{S}^*$-closed sets is $\mathcal{S}$-compact.

Suppose $\mathcal{K}$ is such a collection. First we show that $c^*(\cup\mathcal{K}) = \cup\mathcal{K}$. Given $y \in c^*(\cup\mathcal{K})$, every $\mathcal{S}^*$ neighborhood of y intersects some $K \in \mathcal{K}$. Define a net $s : D \to \mathcal{K}$, where D is the set of $\mathcal{S}^*$ neighborhoods of y ordered by reverse inclusion, by choosing, for each $O \in D$, $s(O) \in \mathcal{K}$ with $O \cap s(O) \neq \phi$. Since $\mathcal{K}$ is $U(\mathcal{S})$-compact, this net has a $U(\mathcal{S})$-cluster point $K' \in \mathcal{K}$. Then it follows that every $\mathcal{S}^*$ neighborhood of y intersects every $\mathcal{S}$ neighborhood of K'.

Suppose $y \notin K'$. Then, since $(Y, \mathcal{S}^*, \mathcal{S})$ is regular and K' is $\mathcal{S}^*$-closed, there is an $\mathcal{S}^*$-neighborhood of y whose $\mathcal{S}$-closure is disjoint from K'. But this contradicts our conclusion above. So $y \in K' \subseteq \cup\mathcal{K}$. This shows that $\cup\mathcal{K}$ is $\mathcal{S}^*$-closed.

Given $\mathcal{C}$, an $\mathcal{S}$-open cover of $\cup\mathcal{K}$, let $\mathcal{C}'$ be the collection of finite unions of members of $\mathcal{C}$. Then $\{\langle O \rangle \mid O \in \mathcal{C}'\}$ is a cover of $\mathcal{K}$; let $\{\langle O \rangle \mid \mid O \in \mathcal{C}''\}$ be a subcover with $\mathcal{C}''$ finite. So $\mathcal{C}''$ covers $\cup\mathcal{K}$, and from it a finite subcover of $\mathcal{C}$ may be constructed. This completes the proof of (5).

(8) follows from R_1^*. $\Diamond$

Proposition 2.3. *For a bitopological space to be regular, dually sober it is necessary and sufficient for it to be homeomorphic to the set of primes of a regular, sober bitopological space* $(X, \mathcal{T}, \mathcal{T}^*)$*, with the order given by the specialization order of* $\mathcal{T}$*, such that*

(1) *For any* $x \in X$*, if* O *is a* $\mathcal{T}$*-neighborhood of* $(\uparrow x) \cap \mathrm{Prime}(X)$ *then there is an* $O' \in \mathcal{T}$ *with* $x \in O'$ *and* $O' \cap \mathrm{Prime}(X) \subseteq O$*.*

(2) *Any* $\mathcal{T}^*|_{\mathrm{Prime}(X)}$*-irreducible subset* A *of* $\mathrm{Prime}(X)$ *has an infimum in* c^*A*.*

Furthermore, any combination of the following properties of $(X, \mathcal{T}, \mathcal{T}^*)$ *may be added above and the equivalence will still hold.*

(3) *Any upward-directed set in* X *is* $\mathcal{T}$*-convergent to its supremum.*

(4) *For any* $x \in X$*,* $(\uparrow x) \cap \mathrm{Prime}(X)$ *is* $\mathcal{T}$*-compact.*

(5) $(X, \leq)$ *is a distributive semilattice.*

(6) $\mathcal{T}^* = \mathcal{T}^k$*.*

Proof. To show that regular carries from the base space to this hyperspace, suppose $K \in K(Y, \mathcal{S}^*, \mathcal{S})$ and $K \in \langle O \rangle$ for some $O \in \mathcal{S}$. Then by regularity of $(Y, \mathcal{S}, \mathcal{S}^*)$, and $\mathcal{S}$-compactness of K, we can find an $O' \in \mathcal{S}$ with $K \subseteq O' \subseteq c^*O' \subseteq O$. Then $K \in \langle O' \rangle \subseteq \langle c^*O' \rangle \subseteq \langle O \rangle$, and $\langle c^*O' \rangle$ is a $L(\mathcal{S}^*)$-closed set.

(6) has been strengthened because regular implies R_1, which implies $\mathcal{T}^k \subseteq \mathcal{T}^*$. $\Diamond$

3. Further considerations

In Sec. 1 we saw ordered structures that were bounded-complete. In Sec. 2, directed-complete was allowed, but bounded-complete was generally ruled out. Can we have both?

We can if the results in the last two sections are combined into one, which will obviously apply to a more restricted class of spaces. For spaces $(Y, \mathcal{S}, \mathcal{S}^*)$ in which the $\mathcal{S}^*$-closed sets are $\mathcal{S}$-compact, the last two types of hyperspaces we have employed become the same.

Definition. A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is *symmetrically* T_2 if it is T_0, R_1, and R_1^*. It is *sup-compact* if $\mathcal{T} \vee \mathcal{T}^*$ is a compact topology for X. In [2] we introduced the term *compact* for bispaces where every $\mathcal{T}^*$-closed set is $\mathcal{T}$-compact.

Definition. A bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is *regularly locally compact* if for any $x \in X$ and any $O \in \mathcal{T}$ with $x \in O$, there is an $O' \in \mathcal{T}$ and a

$\mathcal{T}^*$-closed, $\mathcal{T}$-compact set K with $x \in O' \subseteq K \subseteq O$.

Definition. A space $(X, \mathcal{T})$ is *coherent* if the intersection of any finitely many saturated compact sets is compact. A space $(X, \mathcal{T})$ is *stably compact* if it is compact, locally compact, coherent, and sober.

A space $(X, \mathcal{T})$ is stably compact iff there is another topology $\mathcal{T}^*$ such that $(X, \mathcal{T}, \mathcal{T}^*)$ is symmetrically T_2 and sup-compact. When this occurs, $\mathcal{T}$ and $\mathcal{T}^*$ are duals to each other. These facts are given as an exercise in [6], and they cite the paper [10] as the source.

Proposition 3.1. *For a bitopological space to be symmetrically T_2 and sup-compact it is necessary and sufficient for it to be homeomorphic to the set of primes of a symmetrically T_2, sup-compact bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$, with the order given by the specialization order of $\mathcal{T}$.*

Furthermore, any combination of the following properties of $(X, \mathcal{T}, \mathcal{T}^*)$ may be added above and the equivalence will still hold.

(1) $(X, \leq)$ *is a bounded complete domain, i.e., a bounded-complete, directed-complete, continuous semilattice.*

(2) $\mathcal{T}$ *is the Scott topology for* $(X, \leq)$.

(3) $\mathcal{T}^{*k} = \mathcal{T}$.

(4) $\mathcal{T}^* = \mathcal{T}^k$.

(5) $(X, \mathcal{T}, \mathcal{T}^*)$ *and its dual are regularly locally compact.*

Proof. Ingredients for most of this are in the other proofs. That sup-compactness of the hyperspace $(\exp(Y, \mathcal{S}^*), U(\mathcal{S}), L(\mathcal{S}^*))$ is equivalent to compactness of $(Y, \mathcal{S}^*, \mathcal{S})$ was Prop. 2.1 in [2]. To get equivalence with sup-compactness of $(Y, \mathcal{S}, \mathcal{S}^*)$ we note that compact and quasisober implies sup-compact (Cor. 1.1 in [2]). We get $(Y, \mathcal{S}^*, \mathcal{S})$ quasisober as in Sec. 1.

(1) and (2) have been covered in other proofs. (3) through (5) are consequences of the combination of symmetrically T_2, sup-compact. $\diamondsuit$

We obviously cannot ask for too many properties to hold in $(X, \mathcal{U})$ without severely restricting the spaces that could be represented as $\mathrm{Prime}(X, \leq)$. The following theorem shows that asking for bounded-complete, directed-complete, and just a few other natural properties already starts to make the spaces resemble those of the last theorem.

Proposition 3.2. *Suppose that $(X, \mathcal{T}, \mathcal{T}^*)$ is a bitopological space, with the order given by the specialization order of $\mathcal{T}$, satisfying the following:*

(1) $(X, \leq)$ *is bounded-complete.*

(2) $(X, \leq)$ *is directed-complete.*

(3) *Maximal elements of* $(X, \leq)$ *are prime.*

(4) *For any* $S \subseteq X$ *and any* $p \in \mathrm{Prime}(X)$*, if* $\inf S \leq p$ *then* $p \in c^*S$*. Then* $\mathrm{Prime}(X, \leq)$ *is* $\mathcal{T}^*$*-compact.*

Proof. Let $\alpha : D \to \mathrm{Prime}(X, \leq)$ be a net. Since $(X, \leq)$ is both bounded-complete and directed-complete, we can find $\underline{\lim}\alpha$. By directed-completeness, there is a maximal element x with $\underline{\lim}\alpha \leq x$. By (3), $x \in$ $\in \mathrm{Prime}(X, \leq)$. By (4), $x \in c^*\alpha[\uparrow d]$ for any $d \in D$. So x is a $\mathcal{T}^*$-cluster point of α. $\Diamond$

Note that (1), (3), and (4) in the theorem are properties of hyperspaces, as defined in this paper and used in Sec. 1, but (2) generally isn't. In Sec. 2, our modified hyperspaces satisfied (2), (3), and (4) but generally not (1). In a future paper, we will introduce a quasi-uniform construction that contains our hyperspace as a subspace, and satisfies (1), (2), and (3) but, in general, fails to satisfy (4).

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